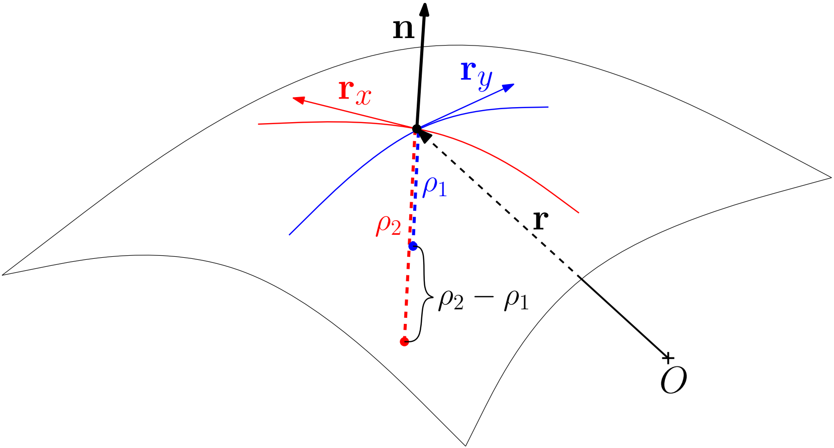
On the constant astigmatism equation (CAE) and surfaces of constant astigmatism

Adam Hlaváč, Michal Marvan

May 15, 2017

Surfaces of constant astigmatism – definition

Definition: A surface is said to be of *constant astigmatism (CA)* if the difference $\rho_2 - \rho_1$ between the principal radii of curvature is a nonzero constant.



Results from 19th century

L. Bianchi, Ricerche sulle superficie elicoidali e sulle superficie a curvatura costante, *Ann. Scuola Norm. Sup. Pisa, I* **2** (1879) 285–341.

- evolutes (focal surfaces) of surfaces of CA are pseudospherical
- involutes corresponding to parabolic geodesic systems on pseudospherical surfaces are of constant astigmatism
- some surfaces of constant astigmatism were obtained explicitly, for example involute corresponding to Dini's pseudospherical helicoid

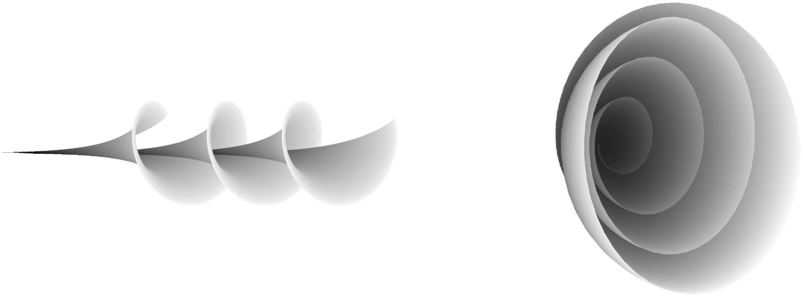


Figure: Dini's pseudospherical surface (left) and its involute (right)

Results from 19th century

R. Lipschitz, Zur Theorie der krummen Oberflächen, *Acta Math.* **10** (1887) 131–136

$$\tilde{\mathbf{r}}(\phi,\theta) = \frac{1}{2}\begin{pmatrix} (2P+M\phi)\cos\theta - 2Q + L\phi \\ \\ (2P+M\phi)\sin\theta\cos\phi - \frac{L\cos\theta+M}{\sin\theta}\sin\phi \\ \\ (2P+M\phi)\sin\theta\sin\phi + \frac{L\cos\theta+M}{\sin\theta}\cos\phi \end{pmatrix},$$

where L, M are real constants and P, Q are defined by formulas

$$P = \int \frac{\sqrt{\sin^4\theta - (L+M\cos\theta)^2}}{\sin^3\theta}\cos\theta\,\mathrm{d}\theta,$$

$$Q = \int \frac{\sqrt{\sin^4\theta - (L+M\cos\theta)^2}}{\sin^3\theta}\,\mathrm{d}\theta.$$

Results from 19th century

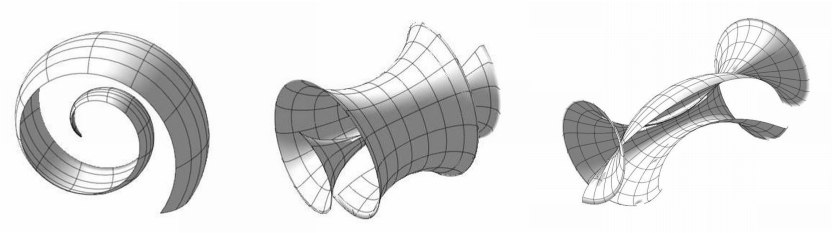


Figure: Lipschitz surfaces of constant astigmatism

Results from 19th century

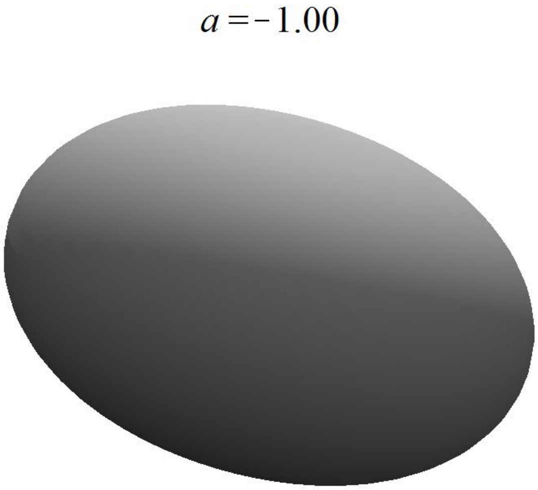
R. von Lilienthal, Bemerkung über diejenigen Flächen bei denen die Differenz der Hauptkrümmungsradien constant ist, *Acta Mathematica* **11** (1887) 391–394.

The one parameter family of von Lilienthal surfaces of revolution (involutes of the pseudosphere) in terms of principal coordinates x, y is given by

$$\mathbf{r}(x,y) = \begin{pmatrix} (x-a+1)\mathrm{e}^{-x}\cos y \\ (x-a+1)\mathrm{e}^{-x}\sin y \\ \operatorname{arccosh}\mathrm{e}^{x} - (x-a+1)\sqrt{1-\mathrm{e}^{-2x}} \end{pmatrix},$$

where a is a real constant.

Example 1 – Gallery of von Lilienthal surfaces



Baran & Marvan 2009, 2010

H. Baran and M. Marvan, On integrability of Weingarten surfaces: a forgotten class, *J. Phys. A: Math. Theor.* **42** (2009)

H. Baran and M. Marvan, Classification of integrable Weingarten surfaces possessing an $\mathfrak{sl}(2)$-valued zero curvature representation, *Nonlinearity* **23** (2010)

- the constant astigmatism equation (CAE)

$$z_{yy} + \left(\frac{1}{z}\right)_{xx} + 2 = 0$$

- transformation to the sine-Gordon equation

$$z_{yy} + \left(\frac{1}{z}\right)_{xx} + 2 = 0 \qquad \longleftrightarrow \qquad u_{\xi\eta} = \sin u$$

Hlaváč & Marvan 2010–present

- A. Hlaváč, More exact solutions of the constant astigmatism equation, in progress...
- A. Hlaváč and M. Marvan, Nonlocal conservation laws of the constant astigmatism equation, *Journal of Geometry and Physics* **113** (2017), p. 117–130
- A. Hlaváč, On multisoliton solutions of the constant astigmatism equation, *J. Phys. A: Math. Theor.* **48** (2015) 365202.
- A. Hlaváč and M. Marvan, A reciprocal transformation for the constant astigmatism equation, *SIGMA* **10** (2014), 091
- A. Hlaváč and M. Marvan, On Lipschitz solutions of the constant astigmatism equation, *Journal of Geometry and Physics* **85** (2014), p. 88–98
- A. Hlaváč and M. Marvan, Another integrable case in two-dimensional plasticity, *J. Phys. A: Math. Theor.* **46** (2013) 045203.

Other papers concerning CAE

M. Pavlov and S. Zykov, Lagrangian and Hamiltonian structures for the constant astigmatism equation, *J. Phys. A: Math. Theor.* **46** (2013) 395203

N. Manganaro and M. Pavlov, The constant astigmatism equation. New exact solution, *J. Phys. A: Math. Theor.* **47** (2014) 075203

1. The constant astigmatism equation (CAE)

Parameterization by lines of curvature

Under parameterization by the lines of curvature (principal coordinates), the fundamental forms of every regular surface can be written as

$$\mathbf{I} = u^2\,\mathrm{d}x^2 + v^2\,\mathrm{d}y^2\,,$$

$$\mathbf{II} = \frac{u^2}{\rho_1}\,\mathrm{d}x^2 + \frac{v^2}{\rho_2}\,\mathrm{d}y^2\,,$$

$$\mathbf{III} = \frac{u^2}{\rho_1^2}\,\mathrm{d}x^2 + \frac{v^2}{\rho_2^2}\,\mathrm{d}y^2\,,$$

where ρ_1 and ρ_2 are the principal radii of curvature of the surface.

We assume the ambient space to be scaled so that $\rho_2 - \rho_1 = \pm 1$.

Adapted parameterization by lines of curvature

Definition: A parameterization by lines of curvature is said to be *adapted* if

$$uv = \pm\rho_1\rho_2 \tag{1}$$

holds.

Adapted parameterization by lines of curvature

Definition: A parameterization by lines of curvature is said to be *adapted* if

$$uv = \pm\rho_1\rho_2 \tag{1}$$

holds.

Every CA surface can be equipped with an adapted parameterization by lines of curvature. Moreover, the nonzero coefficients of the three fundamental forms of a surface of constant astigmatism can be expressed through a single function $z(x, y)$:

$$u = \frac{z^{\frac{1}{2}}(\ln z - 2)}{2}, \qquad v = \frac{\ln z}{2z^{\frac{1}{2}}}, \qquad \rho_1 = \frac{\ln z - 2}{2}, \qquad \rho_2 = \frac{\ln z}{2}.$$

Obviously, $\rho_2 - \rho_1 = 1$ and the condition (1) also holds.

Gauss–Weingarten equations

Let $\mathbf{r}(x,y)$ be the surface of constant astigmatism and let $\mathbf{n}(x,y)$ denote the unit normal vector. Then $\mathbf{r},\mathbf{n}$ satisfy the Gauss–Weingarten system

$$\begin{aligned}
\mathbf{r}_{xx} &= \frac{(\ln z)z_x}{2(\ln z-2)z}\mathbf{r}_x - \frac{(\ln z-2)zz_y}{2\ln z}\mathbf{r}_y + \frac{1}{2}(\ln z-2)z\mathbf{n},\\
\mathbf{r}_{xy} &= \frac{(\ln z)z_y}{2(\ln z-2)z}\mathbf{r}_x - \frac{(\ln z-2)zz_x}{2\ln z}\mathbf{r}_y,\\
\mathbf{r}_{yy} &= \frac{(\ln z)z_x}{2(\ln z-2)z^3}\mathbf{r}_x - \frac{(\ln z-2)z_y}{2z\ln z}\mathbf{r}_y + \frac{\ln z}{2z}\mathbf{n},\\
\mathbf{n}_x &= -\frac{2}{\ln z-2}\mathbf{r}_x, \qquad \mathbf{n}_y = -\frac{2}{\ln z}\mathbf{r}_y.
\end{aligned}$$

Constant astigmatism equation

Compatibility conditions of the Gauss–Weingarten system reduce to the *constant astigmatism equation (CAE)*

$$z_{yy} + \left(\frac{1}{z}\right)_{xx} + 2 = 0\,.$$

Thus, under parameterization by adapted lines of curvature **surfaces of constant astigmatism correspond to solutions of the constant astigmatism equation**.

The simplest example – von Lilienthal solutions

The CAE:

$$z_{yy} + \left(\frac{1}{z}\right)_{xx} + 2 = 0$$

The simplest solutions of the CAE – solutions corresponding to von Lilienthal surfaces:

$$z = -y^2 + c_1, \qquad z = \frac{1}{-x^2 + c_2}$$

2. Construction of the CA surfaces and solutions of the CAE

Construction of the CA surface from the pair of complementary evolutes

Proposition 1: Let $\omega^{(1)}(\xi, \eta, c)$ be a Bäcklund transformation of $\omega(\xi, \eta)$, where c is an integration constant. Let $\mathbf{r}$ and $\mathbf{r}^{(1)}$ be pair of complementary pseudospherical surfaces. Denote

$$\tilde{\mathbf{n}} = \mathbf{r}^{(1)} - \mathbf{r} = \frac{\sin(\omega - \omega^{(1)})}{\sin(2\omega)}\mathbf{r}_\xi + \frac{\sin(\omega + \omega^{(1)})}{\sin(2\omega)}\mathbf{r}_\eta.$$

Then

$$\tilde{\mathbf{r}} = \mathbf{r} - f\tilde{\mathbf{n}}, \quad \text{where} \quad f = \ln \frac{\mathrm{d}\omega^{(1)}}{\mathrm{d}c},$$

is a surface of constant astigmatism having surfaces $\mathbf{r}$ and $\mathbf{r}^{(1)}$ as evolutes.

Proposition 1 shows that the constant astigmatism surfaces can be found by purely algebraic manipulations and differentiation once a one-parameter family of functions $\omega^{(1)}$ is known.

Construction of the corresponding solution of the CAE

Proposition 2: Let $\omega^{(1)}(\xi, \eta, c)$ be a Bäcklund transformation of $\omega(\xi, \eta)$, where c is an integration constant. Let $f = \ln(\mathrm{d}\omega^{(1)}/\,\mathrm{d}c)$ and $x = \mathrm{d}f/\,\mathrm{d}c$. Let $y(\xi, \eta)$ be a solution of the system

$$y_\xi = \mathrm{e}^{-f}\sin(\omega + \omega^{(1)}), \quad y_\eta = \mathrm{e}^{-f}\sin(\omega - \omega^{(1)}).$$

Then x, y are adapted curvature coordinates on the surface $\tilde{\mathbf{r}}$. Moreover, if $z = \mathrm{e}^{-2f}$, then $z(x, y)$ is a solution of the constant astigmatism equation. Finally, $z\,\mathrm{d}x^2 + \mathrm{d}y^2/z$ is an orthogonal equiareal pattern on the unit sphere $\tilde{\mathbf{n}}$, while ξ, η is the associated slip line field.

Proposition 2 allows us to construct one of the curvature coordinates by purely algebraic manipulations and differentiation, while the other curvature coordinate has to be obtained by integration.

3. Superposition principle for the CAE

Associated potentials (solutions of the CAE)

$$g_\xi^{(\lambda)} = g^{(\lambda)}\lambda\cos(\omega^{(\lambda)}+\omega), \qquad g_\eta^{(\lambda)} = g^{(\lambda)}\frac{1}{\lambda}\cos(\omega^{(\lambda)}-\omega),$$

$$x_\xi^{(\lambda)} = \lambda g^{(\lambda)}\sin(\omega^{(\lambda)}+\omega), \qquad x_\eta^{(\lambda)} = \frac{1}{\lambda}g^{(\lambda)}\sin(\omega^{(\lambda)}-\omega),$$

$$y_\xi^{(\lambda)} = \frac{\lambda\sin(\omega^{(\lambda)}+\omega)}{g^{(\lambda)}}, \qquad y_\eta^{(\lambda)} = -\frac{\sin(\omega^{(\lambda)}-\omega)}{\lambda g^{(\lambda)}}.$$

Expressing $z^{(\lambda)} = 1/{g^{(\lambda)}}^2$ in terms of $x^{(\lambda)}$ and $y^{(\lambda)}$ one obtains a solution of the CAE.

Superposition principle for the CAE

Proposition 3: Let $\omega, \omega^{(\lambda_1)}, \omega^{(\lambda_2)}, \omega^{(\lambda_1\lambda_2)}$ be four sine-Gordon solutions related by the Bianchi superposition principle. Then $g^{(\lambda_1\lambda_2)}$, $x^{(\lambda_1\lambda_2)}$, $y^{(\lambda_1\lambda_2)}$ corresponding to the pair $\omega^{(\lambda_1)}, \omega^{(\lambda_1\lambda_2)}$ are related to $g^{(\lambda_2)}$, $x^{(\lambda_2)}$, $y^{(\lambda_2)}$ corresponding to the pair $\omega, \omega^{(\lambda_2)}$ by formulas

$$g^{(\lambda_1\lambda_2)} = \frac{-\lambda_1\lambda_2}{\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2\cos(\omega^{(\lambda_1)}-\omega^{(\lambda_2)})}g^{(\lambda_2)},$$

$$x^{(\lambda_1\lambda_2)} = \frac{\lambda_1\lambda_2}{\lambda_1^2-\lambda_2^2}\left(x^{(\lambda_2)} - \frac{2\lambda_1\lambda_2\sin(\omega^{(\lambda_1)}-\omega^{(\lambda_2)})}{\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2\cos(\omega^{(\lambda_1)}-\omega^{(\lambda_2)})}g^{(\lambda_2)}\right),$$

$$y^{(\lambda_1\lambda_2)} = \frac{\lambda_1^2-\lambda_2^2}{\lambda_1\lambda_2}y^{(\lambda_2)} - \frac{2\sin(\omega^{(\lambda_1)}-\omega^{(\lambda_2)})}{g^{(\lambda_2)}}.$$

up to an additive constant.

$$\begin{tikzcd}
 & \omega^{(\lambda_1)} \arrow[dr, "{f^{(\lambda_1\lambda_2)},\, x^{(\lambda_1\lambda_2)},\, y^{(\lambda_1\lambda_2)}}"] & \\
\omega \arrow[ur] \arrow[dr, "{f^{(\lambda_2)},\, x^{(\lambda_2)},\, y^{(\lambda_2)}}"'] & & \omega^{(\lambda_1\lambda_2)} \\
 & \omega^{(\lambda_2)} \arrow[ur] &
\end{tikzcd}$$

4. Orthogonal equiareal patterns

Orthogonal equiareal patterns

Definition: By an *orthogonal equiareal pattern (OEP)* on a surface S we shall mean a parameterization x, y such that the corresponding first fundamental form is

$$\mathbf{I}_S = z\,\mathrm{d}x^2 + \frac{1}{z}\,\mathrm{d}y^2\,,$$

z being an arbitrary function of x, y.

Geometric meaning of $z(x, y)$

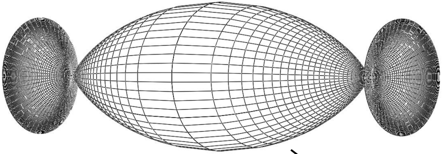
The third fundamental form of the constant astigmatism surface turns out to be

$$\mathbf{III} = z\,\mathrm{d}x^2 + \frac{1}{z}\,\mathrm{d}y^2\,.$$

Hence, one obtains orthogonal equiareal parameterization of the Gaussian sphere.

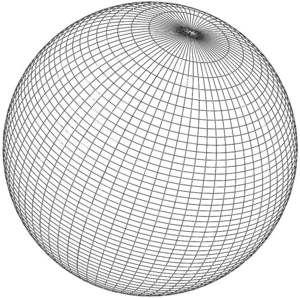
Geometric meaning of $z(x, y)$

constant astigmatism surface $\mathbf{r}(x, y)$
$(x, y \ldots$ principal coordinates)



Gaussian map $\mathbf{n}(x, y)$

orthogonal equiareal pattern
on unit sphere



The Archimedean projection.

Example: *The Archimedean projection.* Consider the parameterization $(x, y) \mapsto \left(\sqrt{1-x^2}\cos y, \sqrt{1-x^2}\sin y, x\right)$. The corresponding first fundamental form is

$$\mathbf{I}_{\mathrm{Arch}} = \frac{\mathrm{d}x^2}{1-x^2} + (1-x^2)\,\mathrm{d}y^2,$$

i.e., $z = 1/(1-x^2)$. This solution of the CAE corresponds to von Lilienthal surfaces.

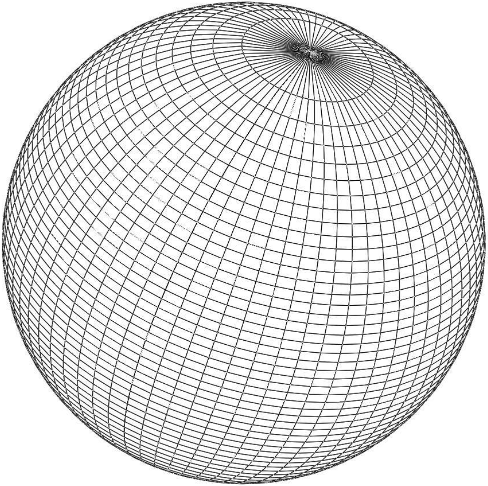


Figure: The Archimedean equiareal parameterization of the unit sphere

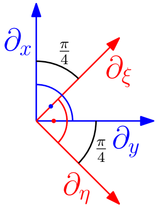
5. Slip line fields

Slip line fields

Definition: By a *slip line field* associated with the OEP

$$\mathbf{I}_S = z\,\mathrm{d}x^2 + \frac{1}{z}\,\mathrm{d}y^2$$

on a surface S we shall mean a parameterization ξ, η such that the angle between ∂_x and ∂_ξ as well as the angle between ∂_y and ∂_η is equal to $\frac{1}{4}\pi$.

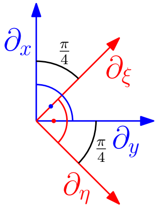


Slip line fields

Definition: By a *slip line field* associated with the OEP

$$\mathbf{I}_S = z\,\mathrm{d}x^2 + \frac{1}{z}\,\mathrm{d}y^2$$

on a surface S we shall mean a parameterization ξ, η such that the angle between ∂_x and ∂_ξ as well as the angle between ∂_y and ∂_η is equal to $\frac{1}{4}\pi$.



Example: The net of slip lines corresponding to the Archimedean equiareal pattern is, by definition, formed by the $\pm 45^\circ$ loxodromes.

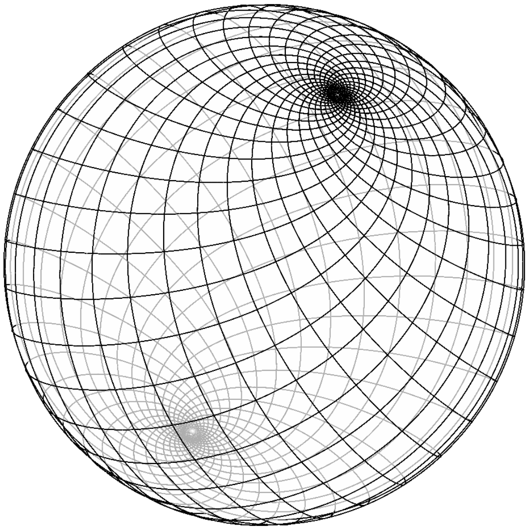
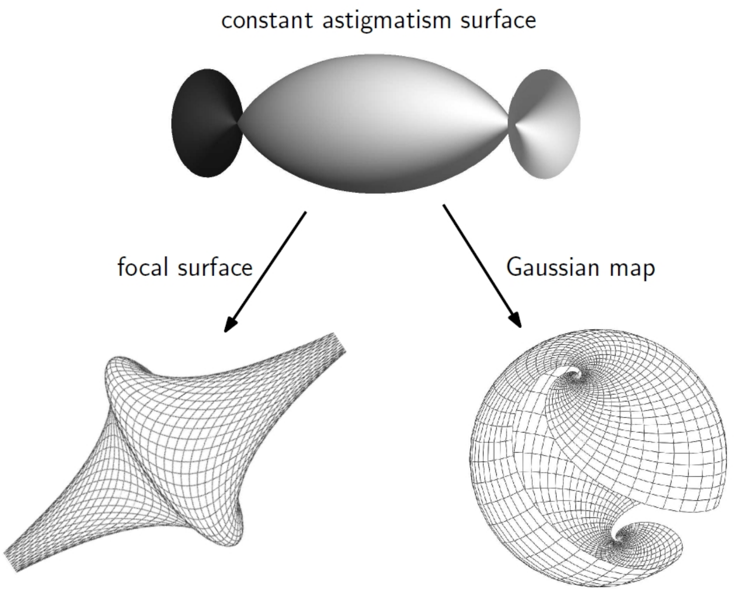


Figure: Sphere's slip line field composed of loxodromes

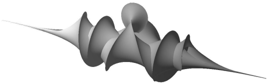


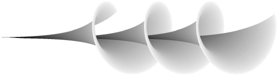
asymptotic lines on
pseudospherical surface

slip line field
on unit sphere

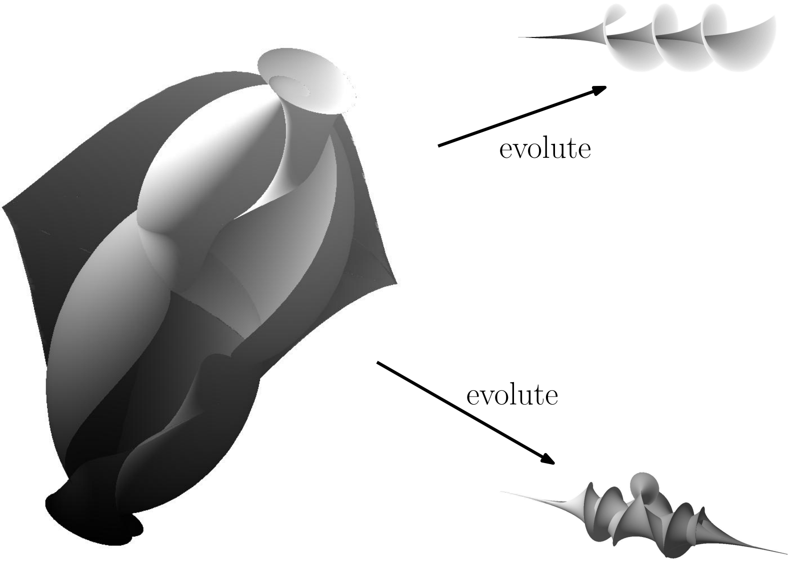
Example of using the superposition principle

$$\begin{tikzcd}
 & \omega^{(1)} \arrow[dr, "\lambda"] & \\
0 \arrow[ur, "1"] \arrow[dr, "\lambda"'] & & \omega^{(\lambda 1)} \\
 & \omega^{(\lambda)} \arrow[ur, red, "1"'] &
\end{tikzcd}$$

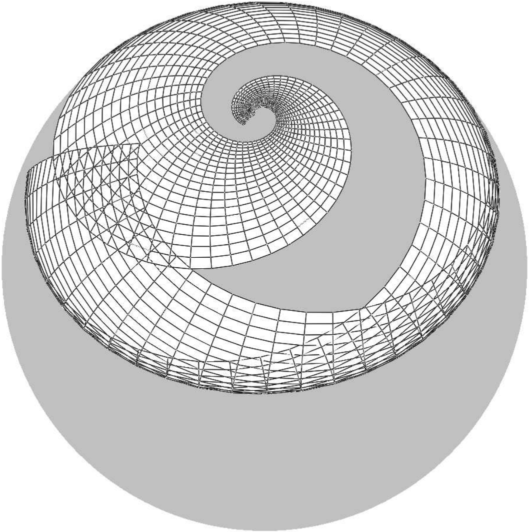




The corresponding constant astigmatism surface ($\lambda = 0.9, c_i = 0$):



The associated slip line field on the Gaussian sphere:



6. Reciprocal transformations

Reciprocal transformations

We introduce two (interrelated) auto-transformations $\mathcal{X}$ and $\mathcal{Y}$ that, in geometric terms, correspond to taking the involute of the evolute.

Formulas for transformations

Let us introduce functions η, ξ satisfying

$$\eta_x = x z_y, \qquad \eta_y = x\frac{z_x}{z^2} + \frac{1}{z} - x^2,$$
$$\xi_x = -y z_y + z - y^2, \qquad \xi_y = -y\frac{z_x}{z^2}.$$

Compatibility of these equations is equivalent to the CAE.

Proposition: Let $z(x,y)$ be a solution of the CAE. Denote $\mathcal{X}(x,y,z) = (x',y',z')$ and $\mathcal{Y}(x,y,z) = (x^*,y^*,z^*)$, where

$$x' = -\frac{xz}{x^2z+1}, \quad y' = \eta, \quad z' = \frac{(x^2z+1)^2}{z}$$

and

$$x^* = \xi, \quad y^* = -\frac{y}{z+y^2}, \quad z^* = \frac{z}{(z+y^2)^2}.$$

Then $z'(x',y')$ and $z^*(x^*,y^*)$ are solutions of the CAE.

Properties of reciprocal transformations

- The following identities hold:

$$\mathcal{X} \circ \mathcal{X} = \mathrm{Id}, \qquad \mathcal{Y} \circ \mathcal{Y} = \mathrm{Id}.$$

Properties of reciprocal transformations

- The following identities hold:

$$\mathcal{X} \circ \mathcal{X} = \mathrm{Id}, \qquad \mathcal{Y} \circ \mathcal{Y} = \mathrm{Id}.$$

- The connection between $\mathcal{X}$ and $\mathcal{Y}$ can be expressed using the involution $\mathcal{I}(x, y, z) = (y, x, 1/z)$:

$$\mathcal{X} \circ \mathcal{I} = \mathcal{I} \circ \mathcal{Y}.$$

Properties of reciprocal transformations

- The following identities hold:

$$\mathcal{X} \circ \mathcal{X} = \mathrm{Id}, \qquad \mathcal{Y} \circ \mathcal{Y} = \mathrm{Id}.$$

- The connection between $\mathcal{X}$ and $\mathcal{Y}$ can be expressed using the involution $\mathcal{I}(x, y, z) = (y, x, 1/z)$:

$$\mathcal{X} \circ \mathcal{I} = \mathcal{I} \circ \mathcal{Y}.$$

- On the level of sine-Gordon solutions:

$$\begin{tikzcd}
\bullet \arrow[rr, "\mathcal{X}"'] \arrow[dr, "\mathcal{F}_1"'] & & \bullet \arrow[rr, "\mathcal{Y}"'] \arrow[dl, "\mathcal{F}_1"] \arrow[dr, "\mathcal{F}_2"'] & & \bullet \arrow[rr, "\mathcal{X}"'] \arrow[dl, "\mathcal{F}_2"] \arrow[dr, "\mathcal{F}_1"'] & & \bullet \arrow[rr, "\mathcal{Y}"'] \arrow[dl, "\mathcal{F}_1"] \arrow[dr, "\mathcal{F}_2"'] & & \bullet \arrow[dl, "\mathcal{F}_2"] \arrow[dr, "\mathcal{F}_1"'] & \\
& \bullet \arrow[rr, "\mathcal{B}^{(1)}"'] & & \bullet \arrow[rr, "\mathcal{B}^{(-1)}"'] & & \bullet \arrow[rr, "\mathcal{B}^{(1)}"'] & & \bullet \arrow[rr, "\mathcal{B}^{(-1)}"'] & & \bullet
\end{tikzcd}$$

Example – von Lilienthal solution

Let us apply the transformations $\mathcal{X}$ and $\mathcal{Y}$ to the von Lilienthal solution

$$z = -y^2 + l\,,$$

where $l > 0$.

Example – von Lilienthal solution

Then $\mathcal{X}(x, y, z) = (x', y', z')$, where

$$x' = \frac{x(l - y^2)}{x^2(l - y^2) + 1}\,, \quad y' = \frac{1}{\sqrt{l}}\operatorname{arctanh}\frac{y}{\sqrt{l}} - x^2 y + c_1\,,$$

$$z' = \frac{(x^2(l - y^2) + 1)^2}{l - y^2}$$

and $\mathcal{Y}(x, y, z) = (x^*, y^*, z^*)$, where

$$x^* = lx + c_2\,, \quad y^* = -\frac{y}{l}\,, \quad z^* = \frac{l - y^2}{l^2},$$

c_1, c_2 being the integration constants.

Example – von Lilienthal solution

Then $\mathcal{X}(x, y, z) = (x', y', z')$, where

$$x' = \frac{x(l - y^2)}{x^2(l - y^2) + 1}\,, \quad y' = \frac{1}{\sqrt{l}}\mathrm{arctanh}\frac{y}{\sqrt{l}} - x^2 y + c_1\,,$$

$$z' = \frac{(x^2(l - y^2) + 1)^2}{l - y^2}$$

and $\mathcal{Y}(x, y, z) = (x^*, y^*, z^*)$, where

$$x^* = lx + c_2\,, \quad y^* = -\frac{y}{l}\,, \quad z^* = \frac{l - y^2}{l^2},$$

c_1, c_2 being the integration constants.

- $z^* = -{y^*}^2 + 1/l$ is another von Lilienthal solution

Example – von Lilienthal solution

Then $\mathcal{X}(x,y,z) = (x',y',z')$, where

$$x' = \frac{x(l-y^2)}{x^2(l-y^2)+1}\,, \quad y' = \frac{1}{\sqrt{l}}\operatorname{arctanh}\frac{y}{\sqrt{l}} - x^2y + c_1\,,$$

$$z' = \frac{(x^2(l-y^2)+1)^2}{l-y^2}$$

and $\mathcal{Y}(x,y,z) = (x^*,y^*,z^*)$, where

$$x^* = lx + c_2\,, \quad y^* = -\frac{y}{l}\,, \quad z^* = \frac{l-y^2}{l^2},$$

c_1, c_2 being the integration constants.

- $z^* = -{y^*}^2 + 1/l$ is another von Lilienthal solution
- $z'(x',y')$ is a substantially new solution of the CAE, which cannot be expressed explicitly using elementary functions

Example – von Lilienthal solution

An implicit formula for the solution $z'(x', y')$ is

$$y' = \frac{1}{\sqrt{l}} \operatorname{arctanh} \sqrt{\frac{lz' - (x'^2 z' + 1)^2}{lz'}} - \frac{x'^2 z'^{\frac{3}{2}} \sqrt{lz' - (x'^2 z' + 1)^2}}{(x'^2 z' + 1)^2} + c_1 \, .$$

Example – von Lilienthal solution

Continuing the previous example we provide a picture of the surface of constant astigmatism generated from the von Lilienthal seed:

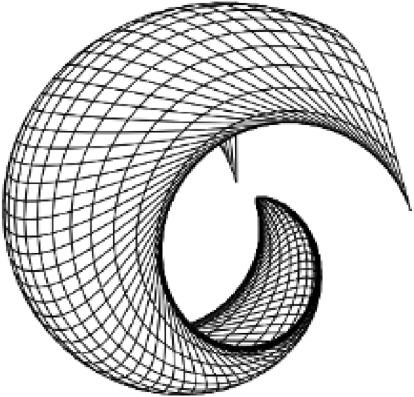
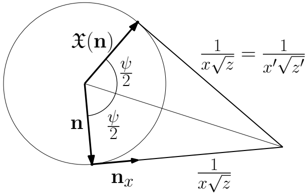


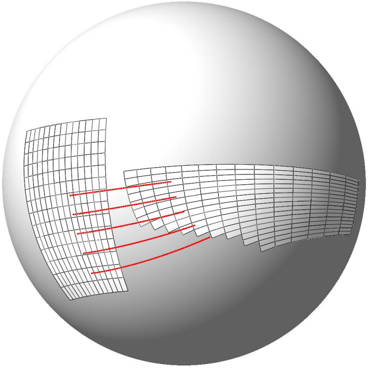
Figure: A transformed von Lilienthal surface.

Acting of reciprocal transformations on the OEP

The construction:



Example:



7. Lipschitz solutions

Theorem: The general Lipschitz solution of the CAE depends on four real parameters $h_{11}, h_{10}, h_{01}, h_{00}$ and is a nonzero root of the quadratic polynomial

$$h_y^2 z^2 + (h^2 - 1)z + h_x^2,$$

where

$$h = h_{11}xy + h_{10}x + h_{01}y + h_{00},$$

$$h_y = h_{11}x + h_{01}, \quad h_x = h_{11}y + h_{10},$$

under the condition that h is not a constant (i.e., at least one of the coefficients h_{11}, h_{10}, h_{01} is not zero).

Proposition: The class of Lipschitz solutions coincides with the class of solutions invariant under linear combinations of the Lie symmetries $\mathcal{T}^x, \mathcal{T}^y, \mathcal{S}$.

Proposition: Denote

$$E_{a,b} = \int_{h_0}^{h} \frac{\sqrt{(1-\chi^2)^2 - 4(a\chi - b)^2}}{2(a\chi - b)(1-\chi^2)}\,\mathrm{d}\chi,$$

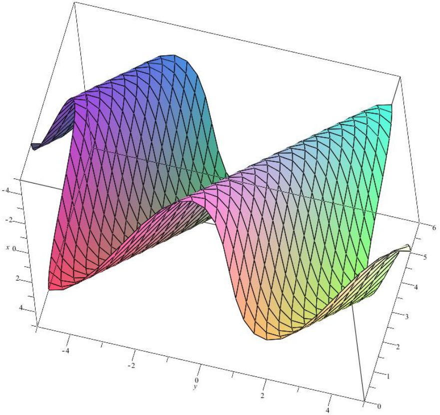
choosing the lower integration limit h_0 so that $E_{a,b}$ is real. Then the orthogonal equiareal pattern corresponding to the general Lipschitz solution is given by the unit vector $\mathbf{n} = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$, where $\theta = \arccos h$ and

$$\phi = \frac{1}{2a} \ln \frac{h_x}{h_y} \pm E_{a,b} \quad \text{if } a \neq 0,$$

$$\phi = \frac{h_{01}y - h_{10}x + h_{00}}{2b} \pm E_{0,b} \quad \text{if } a = 0,\ b \neq 0.$$

The corresponding sine–Gordon solutions turn out to be well known travelling wave solution also known as a “fluxon chain”. The simplest analytic expression for it through the Jacobi amplitude is

$$q = 2\,\mathrm{am}\,(k\xi + \eta, k^{-1/2}) + \pi.$$



OEP corresponding to Lipschitz surfaces

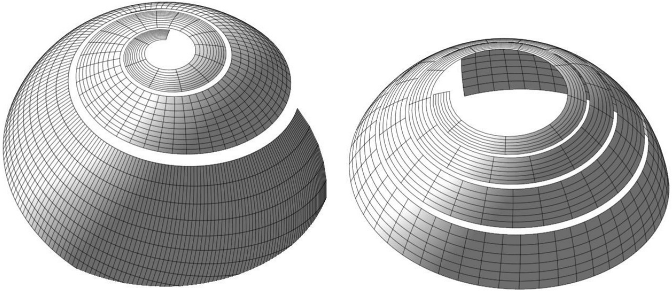


Figure: Orthogonal equiareal patterns on the sphere corresponding to Lipschitz solutions

8. Algebraic formula producing infinitely many exact solutions

How to iterate the superposition for the CAE?

Let $\omega^{[0]} = \bar{\omega}^{[0]}$ be some seed solution of the sine-Gordon equation. Fix Bäcklund parameters $\lambda_1, \ldots, \lambda_{k+1}$ and let us denote

$$\omega^{[k]} = \omega^{(\lambda_1 \lambda_2 \ldots \lambda_k)}, \qquad \bar{\omega}^{[k]} = \omega^{(\lambda_2 \lambda_3 \ldots \lambda_{k+1})}$$

$$\begin{tikzcd}
\omega^{[0]} \arrow[r, "\lambda_2"] \arrow[d, "\lambda_1"'] & \bar{\omega}^{[1]} \arrow[r, "\lambda_3"] \arrow[d, "\lambda_1"'] & \bar{\omega}^{[2]} \arrow[r, "\lambda_4"] \arrow[d, "\lambda_1"'] & \bar{\omega}^{[3]} \arrow[r, "\lambda_5"] \arrow[d, "\lambda_1"'] & \bar{\omega}^{[4]} \arrow[r, "\lambda_6"] \arrow[d, "\lambda_1"'] & \cdots \\
\omega^{[1]} \arrow[r, "\lambda_2"] & \omega^{[2]} \arrow[r, "\lambda_3"] & \omega^{[3]} \arrow[r, "\lambda_4"] & \omega^{[4]} \arrow[r, "\lambda_5"] & \omega^{[5]} \arrow[r, "\lambda_6"] & \cdots
\end{tikzcd}$$

How to iterate the superposition for the CAE?

Let $g^{[j]}, x^{[j]}, y^{[j]}$ denote the associated potentials corresponding to the pair $\bar{\omega}^{[j-1]}, \omega^{[j]}$. In this notation, the superposition formulas turn out to be

$$x^{[j+1]} = \frac{\lambda_{j+1}\lambda_1}{\lambda_{j+1}^2 - \lambda_1^2}\left(x^{[j]} - \frac{2\lambda_{j+1}\lambda_1 \sin(\bar{\omega}^{[j]} - \omega^{[j]})}{\lambda_{j+1}^2 + \lambda_1^2 - 2\lambda_{j+1}\lambda_1 \cos(\bar{\omega}^{[j]} - \omega^{[j]})} g^{[j]}\right),$$

$$y^{[j+1]} = \frac{\lambda_{j+1}^2 - \lambda_1^2}{\lambda_{j+1}\lambda_1} y^{[j]} - \frac{2\sin(\bar{\omega}^{[j]} - \omega^{[j]})}{g^{[j]}},$$

$$g^{[j+1]} = \frac{-\lambda_{j+1}\lambda_1}{\lambda_{j+1}^2 + \lambda_1^2 - 2\lambda_{j+1}\lambda_1 \cos(\bar{\omega}^{[j]} - \omega^{[j]})} g^{[j]}.$$

The above formulas constitute recurrence relations for the quantities $x^{[n]}, y^{[n]}, g^{[n]}$ with the initial conditions

$$x^{[1]} = x_1, \quad y^{[1]} = y_1, \quad g^{[1]} = g_1.$$

Solving the recurrence relations

Proposition:

$$\begin{pmatrix} x^{[n]} \\ y^{[n]} \\ g^{[n]} \\ 1/g^{[n]} \end{pmatrix} = \left(\prod_{i=1}^{n-1} S^{[i]}\right) \begin{pmatrix} x_1 \\ y_1 \\ g_1 \\ 1/g_1 \end{pmatrix},$$

where $S^{[j]}$ are 4×4 matrices with entries defined by formulas

$$S_{11}^{[j]} = \frac{\lambda_{j+1}\lambda_1}{\lambda_{j+1}^2 - \lambda_1^2}, \quad S_{13}^{[j]} = -\frac{\lambda_{j+1}^2\lambda_1^2}{\lambda_{j+1}^2 - \lambda_1^2} \cdot \frac{2\sin(\bar{\omega}^{[j]} - \omega^{[j]})}{\lambda_{j+1}^2 + \lambda_1^2 - 2\lambda_{j+1}\lambda_1 \cos(\bar{\omega}^{[j]} - \omega^{[j]})},$$

$$S_{22}^{[j]} = \frac{\lambda_{j+1}^2 - \lambda_1^2}{\lambda_{j+1}\lambda_1}, \quad S_{24}^{[j]} = -2\sin(\bar{\omega}^{[j]} - \omega^{[j]}),$$

$$S_{33}^{[j]} = \frac{1}{S_{44}^{[j]}} = \frac{-\lambda_{j+1}\lambda_1}{\lambda_{j+1}^2 + \lambda_1^2 - 2\lambda_{j+1}\lambda_1 \cos(\bar{\omega}^{[j]} - \omega^{[j]})}$$

all the other entries being zero. Moreover, if $z^{[n]} = 1/{g^{[n]}}^2$, then $z^{[n]}(x^{[n]}, y^{[n]})$ is a solution of the CAE.

9. Multisoliton solutions of the CAE

Multisoliton solutions of the sine-Gordon equation

Let $\omega^{[0]} = 0$. Let us define

$$a_i := \mathrm{e}^{\lambda_i \xi + \frac{\eta}{\lambda_i} + c_i}.$$

The Bäcklund transformations of zero solution are one-soliton solutions of the sine-Gordon equation

$$\omega^{[1]} = 2 \arctan a_1, \qquad \bar{\omega}^{[1]} = 2 \arctan a_2$$

and, applying the superposition principle we easily obtain the two-soliton solutions

$$\omega^{[2]} = 2 \arctan \frac{(\lambda_1 + \lambda_2)(a_1 - a_2)}{(\lambda_1 - \lambda_2)(1 + a_1 a_2)},$$
$$\bar{\omega}^{[2]} = 2 \arctan \frac{(\lambda_2 + \lambda_3)(a_2 - a_3)}{(\lambda_2 - \lambda_3)(1 + a_2 a_3)}.$$

Multisoliton solutions of the sine-Gordon equation

An exact analytic n-soliton solution, in our notation $\omega^{[n]}$, is of the form

$$\omega^{[n]} = \frac{1}{2} \arccos \varphi^{[n]},$$

where

$$\varphi^{[n]} = 1 - 2 \frac{\partial^2}{\partial\xi \, \partial\eta} \ln \det M$$

M being the $n \times n$ matrix with entries

$$M_{ij} = \frac{1}{\lambda_i + \lambda_j} \left(\sqrt{a_i a_j} + \frac{1}{\sqrt{a_i a_j}} \right).$$

Multisoliton solutions of the CAE

Definition: By a *j-soliton solution of the constant astigmatism equation* we shall mean a triple $(x^{[j]}, y^{[j]}, g^{[j]})$ formed by associated potentials corresponding to the j-soliton solution $\omega^{[j]}$ and the $(j-1)$-soliton solution $\bar{\omega}^{[j-1]}$ of the sine-Gordon equation.

$$\begin{tikzcd}
\omega^{[0]} \arrow[r, "\lambda_2"] \arrow[d, "\lambda_1"'] & \bar{\omega}^{[1]} \arrow[r, "\lambda_3"] \arrow[d, "\lambda_1"'] & \bar{\omega}^{[2]} \arrow[r, "\lambda_4"] \arrow[d, "\lambda_1"'] & \bar{\omega}^{[3]} \arrow[r, "\lambda_5"] \arrow[d, "\lambda_1"'] & \bar{\omega}^{[4]} \arrow[r, "\lambda_6"] \arrow[d, "\lambda_1"'] & \cdots \\
\omega^{[1]} \arrow[r, "\lambda_2"] & \omega^{[2]} \arrow[r, "\lambda_3"] & \omega^{[3]} \arrow[r, "\lambda_4"] & \omega^{[4]} \arrow[r, "\lambda_5"] & \omega^{[5]} \arrow[r, "\lambda_6"] & \cdots
\end{tikzcd}$$

One-soliton solution of the CAE

A one-soliton solution of the CAE is easy to construct:

$$g_1 = \frac{2\mathrm{e}^{\lambda_1\xi + \frac{\eta}{\lambda_1} + c_1}}{\mathrm{e}^{2(\lambda_1\xi + \frac{\eta}{\lambda_1} + c_1)} + 1},$$

$$x_1 = \frac{\mathrm{e}^{2(\lambda_1\xi + \frac{\eta}{\lambda_1} + c_1)} - 1}{\mathrm{e}^{2(\lambda_1\xi + \frac{\eta}{\lambda_1} + c_1)} + 1},$$

$$y_1 = \lambda_1\xi - \frac{\eta}{\lambda_1} + k_1.$$

Setting $z_1 = 1/g_1^2$ and eliminating ξ, η one reveals the *von Lilienthal solution*

$$z = \frac{1}{1 - x^2}.$$

One-soliton solutions

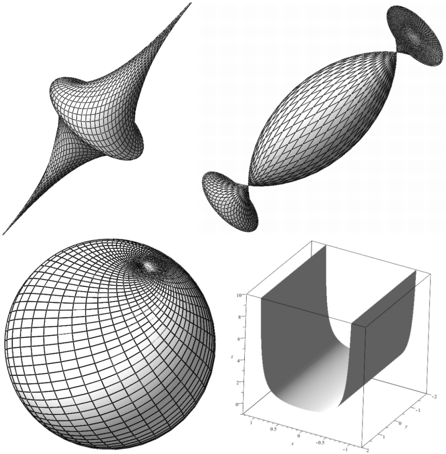


Figure: Pseudosphere (upper left); von Lilienthal surface (upper right); Gaussian map (lower left); solution $z = 1/(1 - x^2)$ of the CAE (lower right).

Multisoliton solutions of the CAE

Proposition: Let us denote $A^{[j]} = 2\bar{\varphi}^{[j]}\varphi^{[j]}$ and $B^{[j]} = 2\sqrt{(\bar{\varphi}^{[j]2} - 1)({\varphi^{[j]}}^2 - 1)}$. Then the n-soliton solution of the CAE is given by the formula

$$\begin{pmatrix} x^{[n]} \\ y^{[n]} \\ g^{[n]} \\ 1/g^{[n]} \end{pmatrix} = \left(\prod_{i=1}^{n-1} S^{[i]}\right) \begin{pmatrix} x_1 \\ y_1 \\ g_1 \\ 1/g_1 \end{pmatrix},$$

where the only nonzero entries of matrices $S^{[j]}$ are given by

$$S_{11}^{[j]} = \frac{\lambda_{j+1}\lambda_1}{\lambda_{j+1}^2 - \lambda_1^2}, \qquad S_{13}^{[j]} = \frac{\lambda_{j+1}^2\lambda_1^2}{\lambda_1^2 - \lambda_{j+1}^2} \cdot \frac{\sqrt{2 - A^{[j]} - B^{[j]}}}{\lambda_1^2 + \lambda_{j+1}^2 - \lambda_{j+1}\lambda_1\sqrt{2 + A^{[j]} + B^{[j]}}}$$

$$S_{22}^{[j]} = \frac{\lambda_{j+1}^2 - \lambda_1^2}{\lambda_{j+1}\lambda_1}, \qquad S_{24}^{[j]} = -\sqrt{2 - A^{[j]} - B^{[j]}},$$

$$S_{33}^{[j]} = \frac{1}{S_{44}^{[j]}} = \frac{-\lambda_{j+1}\lambda_1}{\lambda_1^2 + \lambda_{j+1}^2 - \lambda_{j+1}\lambda_1\sqrt{2 + A^{[j]} + B^{[j]}}}.$$

Examples

Two-soliton solutions

$$\omega^{[2]} = 2\arctan\frac{(\lambda_1+\lambda_2)(a_1-a_2)}{(\lambda_1-\lambda_2)(1+a_1a_2)}$$

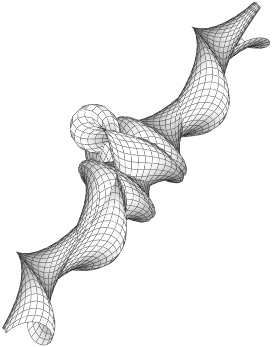


Figure: Two-soliton pseudospherical surface

Two-soliton solution of the CAE

$$x^{[2]} = \frac{\lambda_1\lambda_2}{\lambda_2^2-\lambda_1^2}\cdot\frac{(\lambda_1+\lambda_2)^2(a_1^2-a_2^2)+(\lambda_1-\lambda_2)^2(a_1^2a_2^2-1)}{(\lambda_1+\lambda_2)^2(a_1^2+a_2^2)+(\lambda_1-\lambda_2)^2(a_1^2a_2^2+1)-8\lambda_1\lambda_2a_1a_2},$$

$$y^{[2]} = \frac{\lambda_2^2-\lambda_1^2}{\lambda_1^2\lambda_2}(\lambda_1^2\xi-\eta)+\frac{2(1+a_1a_2)(a_1-a_2)}{a_1(1+a_2^2)},$$

$$g^{[2]} = \frac{-2\lambda_1\lambda_2a_1(1+a_2^2)}{(\lambda_1+\lambda_2)^2(a_1^2+a_2^2)+(\lambda_1-\lambda_2)^2(a_1^2a_2^2+1)-8\lambda_1\lambda_2a_1a_2},$$

$$z^{[2]} = \frac{1}{{g^{[2]}}^2} = \left(\frac{(\lambda_1+\lambda_2)^2(a_1^2+a_2^2)+(\lambda_1-\lambda_2)^2(a_1^2a_2^2+1)-8\lambda_1\lambda_2a_1a_2}{2\lambda_1\lambda_2a_1(1+a_2^2)}\right)^2.$$

Two-soliton solution of the CAE

Eliminating ξ, η one obtains an implicit formula for the function $z(x,y) = z^{[2]}(x^{[2]}, y^{[2]})$, namely

$$y = 2\ln a_2 - \frac{(\lambda_1^2+\lambda_2^2)\ln a_1}{\lambda_1\lambda_2} + \frac{2(1+a_1a_2)(a_1-a_2)}{a_1(1+a_2^2)},$$

where

$$a_1 = \frac{-(x^2{\lambda_{12}^{\ominus}}^2 - \lambda_1^2\lambda_2^2)^2 z^2 - 2{\lambda_{12}^{+}}^4(x^2{\lambda_{12}^{-}}^4 - \lambda_1^2\lambda_2^2)z + 2K\lambda_1\lambda_2{\lambda_{12}^{+}}^2\sqrt{z} - {\lambda_{12}^{\ominus}}^4}{(x\lambda_{12}^{\ominus}+\lambda_1\lambda_2)^2(4\lambda_1^2\lambda_2^2 z^{\frac{3}{2}} + Kz) + 4\lambda_1^2\lambda_2^2{\lambda_{12}^{\ominus}}^2\sqrt{z} + K{\lambda_{12}^{\ominus}}^2},$$

$$a_2 = \frac{4\lambda_2^2\lambda_1^2\sqrt{z}+K}{{\lambda_{12}^{\ominus}}^2 + (x^2{\lambda_{12}^{\ominus}}^2 - \lambda_1^2\lambda_2^2)z}, \qquad K = 16\lambda_2^4\lambda_1^4 z - [{\lambda_{12}^{\ominus}}^2 + (x^2{\lambda_{12}^{\ominus}}^2 - \lambda_1^2\lambda_2^2)z]^2.$$

Here we have denoted

$$\lambda_{12}^{+} := \lambda_1 + \lambda_2, \qquad \lambda_{12}^{-} := \lambda_1 - \lambda_2, \qquad \lambda_{12}^{\ominus} := \lambda_1^2 - \lambda_2^2.$$

Two-soliton solutions

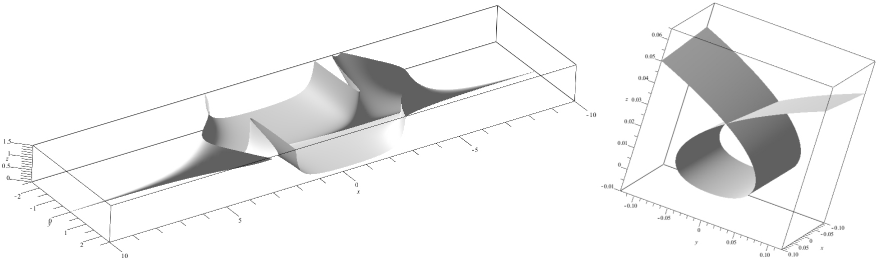


Figure: Two soliton solution of the CAE, $\lambda_1 = 1.2$, $\lambda_2 = 1.5$, $c_i = 0$. Right part of the figure shows the behavior around the origin.

Two-soliton solutions

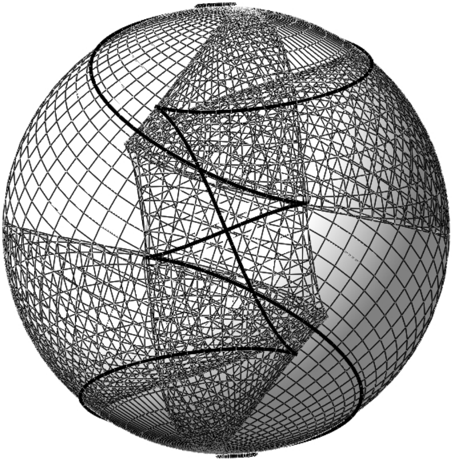


Figure: Slip line field $\tilde{\mathbf{n}}^{[2]}$, $\lambda_2 = 1.5$, $c_i = 0$ with coordinate lines $\xi = 0$ and $\eta = 0$ highlighted (thick black curves).

Two-soliton solutions

To obtain an associated orthogonal equiareal pattern one needs to invert the transformation $(x, y) \leftrightarrow (\xi, \eta)$ given by

$$x = \frac{\lambda_1\lambda_2}{\lambda_2^2 - \lambda_1^2} \cdot \frac{(\lambda_1 + \lambda_2)^2(a_1^2 - a_2^2) + (\lambda_1 - \lambda_2)^2(a_1^2 a_2^2 - 1)}{(\lambda_1 + \lambda_2)^2(a_1^2 + a_2^2) + (\lambda_1 - \lambda_2)^2(a_1^2 a_2^2 + 1) - 8\lambda_1\lambda_2 a_1 a_2},$$

$$y = \frac{\lambda_2^2 - \lambda_1^2}{\lambda_1^2\lambda_2}(\lambda_1^2\xi - \eta) + \frac{2(1 + a_1 a_2)(a_1 - a_2)}{a_1(1 + a_2^2)},$$

which is not possible in terms of elementary functions. The parameterisation $\tilde{\mathbf{n}}^{[2]}(x, y)$ then would provide orthogonal equiareal net sought.

Two-soliton solutions

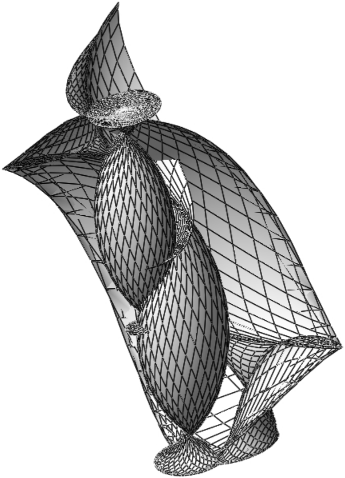


Figure: Two soliton surface $\tilde{\mathbf{r}}^{[2]}$ of constant astigmatism, $\lambda_2 = 1.5$, $c_i = 0$, $k = 1$.

Three-soliton solutions

$$\omega^{[3]} = 2\arctan\left(\frac{\lambda_{12}^{+}\lambda_{13}^{+}\lambda_{23}^{-}a_1 - \lambda_{12}^{+}\lambda_{13}^{-}\lambda_{23}^{+}a_2 + \lambda_{12}^{-}\lambda_{13}^{+}\lambda_{23}^{+}a_3 + \lambda_{12}^{-}\lambda_{13}^{-}\lambda_{23}^{-}a_1a_2a_3}{\lambda_{12}^{-}\lambda_{13}^{+}\lambda_{23}^{+}a_1a_2 - \lambda_{12}^{+}\lambda_{13}^{-}\lambda_{23}^{+}a_1a_3 + \lambda_{12}^{+}\lambda_{13}^{+}\lambda_{23}^{-}a_2a_3 + \lambda_{12}^{-}\lambda_{13}^{-}\lambda_{23}^{-}}\right)$$

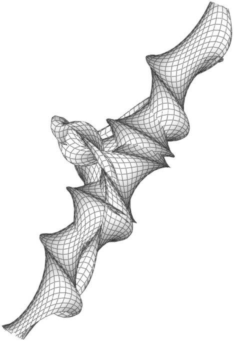


Figure: Three-soliton pseudospherical surface $r^{[3]}$.

Three-soliton solutions

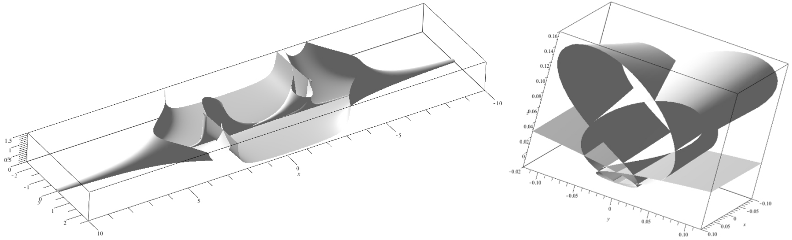


Figure: Three soliton solution of the CAE, $\lambda_1 = 1.2$, $\lambda_2 = 1.5$, $\lambda_2 = 1.8$, $c_i = 0$. Right part of the figure shows the behavior around the origin.

Three-soliton solutions

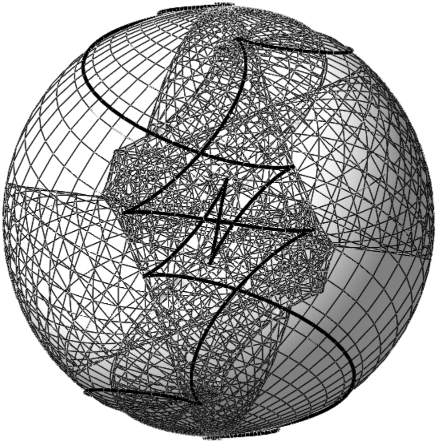


Figure: Slip line field $\tilde{\mathbf{n}}^{[3]}$, $\lambda_2 = 1.5$, $\lambda_3 = 1.8$, $c_i = 0$ with coordinate lines $\xi = 0$ and $\eta = 0$ highlighted (thick black curves).

Three-soliton solutions

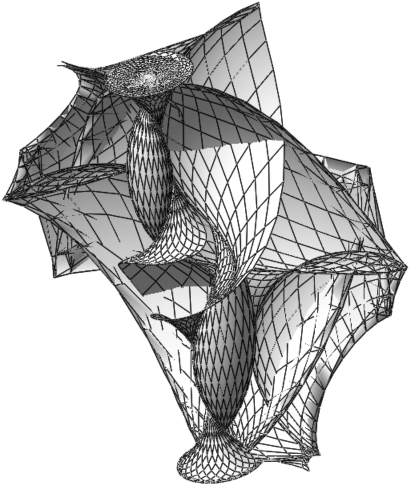


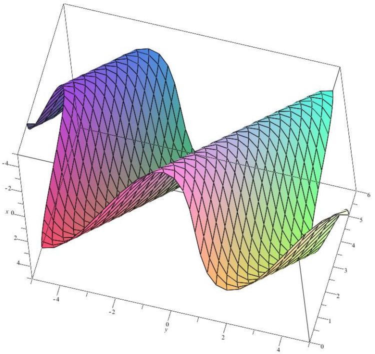
Figure: Three soliton surface $\tilde{\mathbf{r}}^{[3]}$ of constant astigmatism, $\lambda_2 = 1.5$, $\lambda_3 = 1.8$, $c_i = 0$, $k = 1$.

Unfinished work – simplest case when $\omega^{[0]} \neq 0$

Solution of the s-G equation corresponding to Lipchitz's solution of the CAE satisfies

$$\omega_\xi = k\omega_\eta,$$

where k is a nonzero constant. Thus, it is of the form $\omega(k\xi + \eta + C)$, where C is a constant.



In coordinates $\alpha = k\xi + \eta$ and $\beta = k\xi - \eta$ the sine-Gordon equation turns out to be

$$\omega_{\alpha\alpha} - \omega_{\beta\beta} = \frac{1}{k} \sin \omega.$$

The condition $\omega_\xi = k\omega_\eta$ reduces to

$$\omega_\beta = 0$$

and, therefore, the sine-Gordon equation transforms to the ODE

$$k\omega_{\alpha\alpha} = \sin \omega.$$

In coordinates $\alpha = k\xi + \eta$ and $\beta = k\xi - \eta$ the sine-Gordon equation turns out to be

$$\omega_{\alpha\alpha} - \omega_{\beta\beta} = \frac{1}{k} \sin\omega.$$

The condition $\omega_\xi = k\omega_\eta$ reduces to

$$\omega_\beta = 0$$

and, therefore, the sine-Gordon equation transforms to the ODE

$$k\omega_{\alpha\alpha} = \sin\omega.$$

Moreover, one can reduce the order of the equation which becomes

$$k\omega_\alpha^2 = -2\cos\omega + 2l,$$

l being a constant. Solving for ω_α one obtains

$$\omega_\alpha = \pm\sqrt{\frac{2l - 2\cos\omega}{k}}.$$

Explicit form of sine-Gordon seed ω_0

The results of integration can be written in the form

(a)

$$\omega_0^{(a)} = 2\arccos\left[\mathrm{sn}\left(\frac{-\alpha p}{\sqrt{k}}; \frac{1}{p}\right)\right] \quad \text{for } l/k > 1/k,$$

(b)

$$\omega_0^{(b)} = 2\arcsin\left[\mathrm{dn}\left(\frac{\alpha}{\sqrt{k}}; p\right)\right] \quad \text{for } |l/k| < 1/k,$$

(c)

$$\omega_0^{(c)} = 4\arctan\left(\exp\frac{\alpha}{\sqrt{k}}\right) \quad \text{for } l = 1,$$

where $p = \sqrt{(1+l)/2}$.

A Bäcklund transformation of the travelling wave solution of the s-G eq.

Let ω be a solution of $k\omega_{\alpha\alpha} = \sin\omega$. Its Bäcklund transformation with parameter λ can be conveniently written as[1]

$$\omega^{(\lambda)} = 4\arctan\delta^{(\lambda)} - \omega,$$

where $\delta^{(\lambda)}$ satisfies the system

$$\delta^{(\lambda)}_\alpha = \frac{\left({\delta^{(\lambda)}}^2 - 1\right)\sin\omega + 2\delta^{(\lambda)}\left(\cos\omega + \frac{\lambda^2}{k}\right) + \lambda\left({\delta^{(\lambda)}}^2 + 1\right)\omega_\alpha}{4\lambda},$$

$$\delta^{(\lambda)}_\beta = \frac{\left(-{\delta^{(\lambda)}}^2 + 1\right)\sin\omega - 2\delta^{(\lambda)}\left(\cos\omega - \frac{\lambda^2}{k}\right) + \lambda\left({\delta^{(\lambda)}}^2 + 1\right)\omega_\alpha}{4\lambda}.$$

[1] C.A. Hoenselaers and S. Micciché, Transcendental solutions of the sine-Gordon equation, in: A. Coley, D. Levi, R. Milson, C. Rogers and P. Winternitz, eds., *Bäcklund and Darboux transformations. The geometry of solitons*, AARMS-CRM Workshop, Halifax, 1999, CRM Proc. Lecture Notes 29 (Amer. Math. Soc., Providence, RI, 2001) 261–271.

A Bäcklund transformation of the travelling wave solution of the s-G eq.

The equation for $\delta_\beta^{(\lambda)}$ is separable and it has a solution

$$\delta^{(\lambda)} = \frac{f}{ak^2} + \frac{c}{ak} \tanh c(\beta + b(\alpha) + K),$$

where

$$a = \frac{\sin\omega}{4k\lambda} - \frac{\omega_\alpha}{4k}, \qquad f = \frac{\lambda}{4} - \frac{k\cos\omega}{4\lambda},$$

$$c = \frac{\sqrt{\lambda^4 - 2k\lambda^2\cos\omega - k^2(\lambda^2\omega_\alpha^2 - 1)}}{4k\lambda}$$

and $b(\alpha)$ is yet unknown function of α.

A Bäcklund transformation of the travelling wave solution of the s-G eq.

Substituting into the equation for $\delta_\alpha^{(\lambda)}$ one obtains the equation for b, namely

$$b_\alpha = \frac{\lambda\omega_\alpha + \sin\omega}{\lambda\omega_\alpha - \sin\omega}.$$

Associated potentials $g^{(\lambda)}, x^{(\lambda)}, y^{(\lambda)}$

Proposition: Let ω be a solution of the sine-Gordon equation $k\omega_{\alpha\alpha} = \sin\omega$ and let $\omega^{(\lambda)} = 4\arctan\delta^{(\lambda)} - \omega$ be its Bäcklund transformation with parameter λ. Then the associated potentials $x^{(\lambda)}, y^{(\lambda)}, g^{(\lambda)}$ corresponding to the pair $\omega, \omega^{(\lambda)}$ are given by formulas

Associated potentials $g^{(\lambda)}, x^{(\lambda)}, y^{(\lambda)}$

$$x^{(\lambda)} = \frac{8c^2kf\cosh 2B + 4c(f^2+k^4a^2+c^2k^2)\sinh 2B}{(f^2+k^4a^2+c^2k^2)\cosh 2B + 2ckf\sinh 2B + f^2+k^4a^2-c^2k^2},$$

$$\begin{aligned} y^{(\lambda)} = {} & \left(\frac{f\sin\omega}{16\lambda c^2k^2a} - \frac{2f-\lambda}{8c^2k}\right)\cosh 2B \\ & - \left(\frac{(k^4a^2-c^2k^2-f^2)\sin\omega}{32\lambda c^3k^3a} + \frac{f(2f-\lambda)}{8c^3k^2}\right)\sinh 2B \\ & - \left(\frac{(k^4a^2+c^2k^2-f^2)\sin\omega}{16\lambda c^2k^3a} + \frac{f(2f-\lambda)}{4c^2k^2}\right)\beta \\ & + \frac{\lambda^4+k^2}{2(\lambda^4-2kl\lambda^2+k^2)}\alpha - \frac{k\lambda^2}{\lambda^4-2kl\lambda^2+k^2}\int\cos\omega\,\mathrm{d}\alpha, \end{aligned}$$

$$g^{(\lambda)} = \frac{4c^2k^3a(1-\tanh^2 B)}{k^4a^2+f^2+2ckf\tanh B + c^2k^2\tanh^2 B},$$

where $B = c(\beta+b+K)$ and l is a constant. Moreover, if $z^{(\lambda)} = 1/{g^{(\lambda)}}^2$, then $z^{(\lambda)}(x^{(\lambda)}, y^{(\lambda)})$ is a solution of the constant astigmatism equation.

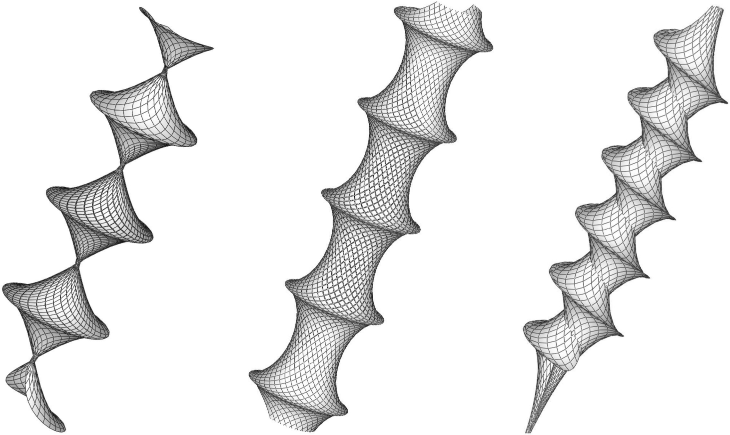


Figure: From the left: Pseudospherical surfaces corresponding to the case (a), (b) and (c) respectively.

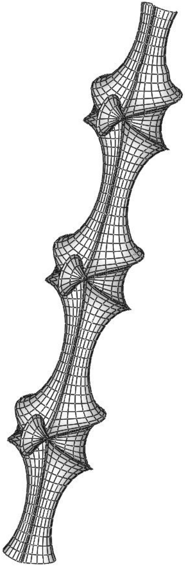


Figure: Transformed case (b).