Ostrava Seminar on Mathematical Physics
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Laplace operator in domains with many holes: an overview

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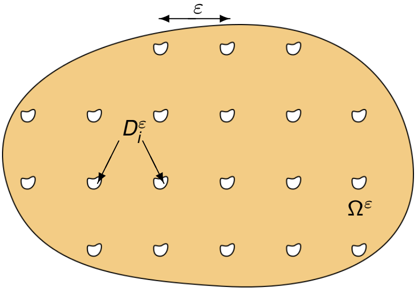
Domain with holes



$$\Omega^\varepsilon = \Omega \setminus D^\varepsilon, \quad D^\varepsilon = \bigcup_i D_i^\varepsilon$$

- $\Omega \subset \mathbb{R}^n$ is a bounded domain
- $D_i^\varepsilon = \eta^\varepsilon D + \varepsilon i$, where $D \subset \mathbb{R}^n$, $\varepsilon, \eta^\varepsilon > 0$, $i \in \mathbb{Z}^n$

The problem

We consider the following problem:

$$\begin{aligned} -\Delta u + u = f, &\quad \text{in } \Omega^\varepsilon, \\ u = 0, &\quad \text{on } \partial\Omega^\varepsilon. \end{aligned}$$

Here $f \in L_2(\Omega)$ is a given function.

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The goal: to describe the behaviour of u_f^ε as $\varepsilon \to 0$

Critical regime is determined by $\alpha := \lim_{\varepsilon\to 0} \varepsilon^{-n} \begin{cases} (\eta^\varepsilon)^{n-2}, & n > 2 \\ |\ln \eta^\varepsilon|^{-1}, & n = 2 \end{cases}$

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Theorem 1: $\alpha < \infty$

For each $f \in L_2(\Omega)$

$$\|u_f^\varepsilon - u_f\|_{L_2(\Omega^\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0.$$

Here u_f is the solution to the problem

$$\begin{aligned} -\Delta u + u + Wu &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned}$$

where $W = \alpha C_D$.

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Theorem 2: $\alpha = \infty$

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- V.A. Marchenko, E.Ya. Khruslov, Math. Sbornik 65 (1964)
- J. Rauch, M. Taylor, J. Funct. Anal. 18 (1975)
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Remark: non-periodic perforations

[Khruslov, 1971], [Buttazzo–Dal Maso–Mosco, 1987]

In the case of Dirichlet boundary conditions the form of the limiting operator is independent of the removed domain D^ε: it is always of the form

$$-\Delta + W,$$

where W is a certain distribution.

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- G. DAL MASO, Boll. Un. Mat. Ital. A 11 (1997)

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The question: what about **uniform** resolvent convergence?

Theorem, A.K., O. Post (2017)

- $\|R^\varepsilon - R\|_{L_2(\Omega^\varepsilon)} \le C(\varepsilon)\|f\|_{L_2(\Omega)}$, where

$$C(\varepsilon) = \begin{cases} C_{n,D}\,\varepsilon, & n = 2,3 \\ C_{n,D,\delta}\,\varepsilon^{1-\delta}, & n = 4, \\ C_{n,D}\,\varepsilon^{1-\frac{n-4}{n-2}}, & n > 4 \end{cases}$$

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- $\|R^\varepsilon - R\|_{L_2(\Omega^\varepsilon)} \le C_{n,D}\,\varepsilon\|(-\Delta_\Omega + I)^{m/2}\|_{L_2(\Omega)}$, where

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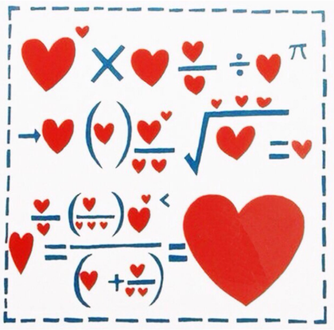
Then

$$|\lambda_k^\varepsilon - \lambda_k| \leq c_k\,\varepsilon$$



Thank you for the attention...

...and Happy Valentine's Day!