Lie algebroids over infinite jet spaces

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Lie algebroids

Let $\xi\colon \Sigma^{d+n} \to M^n$ be a vector bundle; $[\cdot,\cdot]_A$ be a Lie algebra structure on $\Gamma(\xi)$ such that

$$[f\cdot X, Y]_A = f\cdot[X,Y]_A - (A(Y)f)\cdot X, \quad \forall X, Y \in \Gamma(\xi),\ f \in C^\infty(M),$$

where $A\colon \Sigma \to TM$ is a bundle morphism (the anchor).

Definition

$(\xi, [\cdot,\cdot]_A, A)$ is called a Lie algebroid.

Lemma

The anchor A is a homomorphism of Lie algebras.

Examples:

- tangent bundle;
- cotangent bundle over Poisson manifold;
- Lie algebra (over $\{\text{pt}\}$, $A := 0$).

Lie algebroids and homological vector fields

(x^j) is a local coordinates at $u \in M^n$;
(b^k) is a local coordinates along $\Pi\xi$;
$[e_i, e_j] = c_{ij}^k(x) e_k$
$A(e_i) = A_i^k(x) \frac{\partial}{\partial x^k}$

Theorem (Vaintrob)

A Lie algebroid $(\xi, [\cdot,\cdot]_A, A)$ structure can be equivalently given as homological vector field Q on $\Pi\xi$:

$$Q = A_i^k(x) b^i \frac{\partial}{\partial x^k} - \frac{1}{2} b^i c_{ij}^k(x) b^j \frac{\partial}{\partial b^k}, \quad [Q, Q] = 0, \iff 2Q^2 = 0.$$

Q is differential in $C^\infty(\Pi\xi) \simeq \Gamma(\bigwedge^\bullet \Sigma^*)$

Jets

$\pi\colon E \to M$, $\dim M = n$, $\dim E = m + n$,
$\Gamma(\pi)$ — local section of π

$$\mu_{x_0}\Gamma(\pi) = \{s \in \Gamma(\pi) \mid \exists r \in \Gamma(\pi), \exists \mu \in C^\infty(M)\colon \mu(x_0) = 0, s = \mu{\cdot}r\}$$

$$[s]^k_{x_0} = \Gamma(\pi)/\big(\mu^{k+1}_{x_0}\Gamma(\pi)\big), \qquad J^k(\pi) = \Big\{[s]^k_{x_0} \mid x_0 \in M, s \in \Gamma(\pi)\Big\}$$

$J^\infty(\pi)$ is the inverse limit of the chain

$$\ldots \xrightarrow{\pi_{k+1,k}} J^k(\pi) \xrightarrow{\pi_{k,k-1}} \ldots \xrightarrow{\pi_{2,1}} J^1(\pi) \xrightarrow{\pi_{1,0}} E \xrightarrow{\pi} M$$

Due to projection there exist monomorphism of function algebras

$$C^\infty(M) \subset C^\infty(J^0(\pi)) \subset \ldots \subset C^\infty(J^k(\pi)) \subset \ldots$$

$$\mathcal{F}(\pi) = C^\infty(J^\infty(\pi)) := \bigcup_{k \in \mathbb{N} \cup \{0\}} C^\infty(J^k(\pi))$$

Differential forms on jets

Let us consider embeddings $\Lambda^r(J^k(\pi)) \subset \Lambda^r(J^{k+1}(\pi))$ and set

$$\Lambda^r(\pi) = \bigcup_k \Lambda^r(J^k(\pi)), \qquad \Lambda(\pi) = \bigoplus_{r\geq 0} \Lambda^r(\pi).$$

Locally the differential forms of degree r are finite sums

$$\omega = b^{\sigma_{c+1},\ldots,\sigma_r}_{i_1,\ldots,i_c,j_{c+1},\ldots,j_r}\, \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_c} \wedge \mathrm{d}u^{j_{c+1}}_{\sigma_{c+1}} \wedge \ldots \mathrm{d}u^{j_r}_{\sigma_r},$$

where $\sigma_{c+1}, \ldots, \sigma_r$ are multi-indexes and $b^{\sigma_{c+1},\ldots,\sigma_r}_{i_1,\ldots,i_c,j_{c+1},\ldots,j_r} \in \mathcal{F}(\pi)$.

Horizontal forms and Cartan forms

Any vector field $X \in \operatorname{Der} \mathcal{F}(\pi)$ on can uniquely decomposed as $X = X^v + X^h$ Thus, one has

$$\operatorname{Der} \mathcal{F}(\pi) = \operatorname{Der}^v \mathcal{F}(\pi) \oplus \mathcal{C} \operatorname{Der} \mathcal{F}(\pi).$$

Duality

$$\Lambda^1(\pi) = \mathcal{C}\Lambda^1(\pi) \oplus \bar{\Lambda}^1(\pi),$$

where Cartan forms (or higher contact forms)

$$\mathcal{C}\Lambda^1(\pi) = \langle \mathrm{d}_{\mathcal{C}} u^j_\sigma = \mathrm{d}u^j_\sigma - u^j_{\sigma i}\, \mathrm{d}x^i \rangle,$$

horizontal forms

$$a_i\, \mathrm{d}x^i \in \bar{\Lambda}^1(\pi), \quad \text{where} \quad a_i \in \mathcal{F}(\pi).$$

$$\Lambda^k(\pi) = \bigoplus_{p+q=k} \mathcal{C}\Lambda^p(\pi) \otimes \bar{\Lambda}^q(\pi)$$

Horizontal and Cartan differentials

The de Rham differential $\mathrm{d}\colon \Lambda^k(\pi) \mapsto \Lambda^{k+1}(\pi)$ splits

$$\mathrm{d} = \mathrm{d}_h + \mathrm{d}_{\mathcal{C}}, \qquad \mathrm{d}_{\mathcal{C}} \circ \mathrm{d}_h + \mathrm{d}_h \circ \mathrm{d}_{\mathcal{C}} = 0.$$

In coordinates, for $f \in \mathcal{F}(\pi)$ we have

$$\mathrm{d}_h f = D_{x^i}(f)\,\mathrm{d}x^i, \qquad D_{x^i} = \frac{\partial}{\partial x^i} + u^j_{\sigma i}\frac{\partial}{\partial u^j_\sigma},$$

$$\mathrm{d}_{\mathcal{C}} f = \frac{\partial f}{\partial u^j_\sigma}\,\mathrm{d}_{\mathcal{C}} u^j_\sigma \qquad \mathrm{d}_{\mathcal{C}} u^j_\sigma = \mathrm{d}u^j_\sigma - u^j_{\sigma i}\,\mathrm{d}x^i.$$

The horizontal cohomology

$$\mathcal{F}(\pi) \xrightarrow{\mathrm{d}_h} \bar{\Lambda}^1(\pi) \xrightarrow{\mathrm{d}_h} \dots \xrightarrow{\mathrm{d}_h} \bar{\Lambda}^n(\pi) \xrightarrow{\mathrm{d}_h} 0$$

$$\bar{H}^p(\pi) = \frac{\operatorname{Ker} \mathrm{d}_h\colon \bar{\Lambda}^p(\pi) \to \bar{\Lambda}^{p+1}(\pi)}{\operatorname{Im} \mathrm{d}_h\colon \bar{\Lambda}^{p-1}(\pi) \to \bar{\Lambda}^p(\pi)}$$

$\mathcal{C}$-differential operators

Let ξ and ξ' be two vector bundles over $J^\infty(\pi)$
P and Q — be the $\mathcal{F}(\pi)$-modules of their sections

Definition

An $\mathbb{R}$-linear map $\Delta\colon P \to Q$ is a $\mathcal{C}$-differential operator of order k if for any $\theta \in J^\infty(\pi)$ and a section $p \in P$ the value of $\Delta(p)$ at θ is completely determined by the values of $D_{x^\sigma}(p)$, $|\sigma| \leq k$, at this point.

The space of such operator is denoted by $\mathcal{C}\mathrm{Diff}_k(P, Q)$ and we also set $\mathcal{C}\mathrm{Diff}(P, Q) = \cup_k \mathcal{C}\mathrm{Diff}_k(P, Q)$.

Horizontal jets

Let $p, p' \in P$. The section p and p' are called *horizontally k-equivalent* at point $\theta \in J^\infty(\pi)$ if

$$D_{x^\sigma}(p) = D_{x^\sigma}(p'), \qquad |\sigma| \leq k.$$

The corresponding class of equivalence is denoted by $[p]^k_\theta$.

$$J^k_h(\xi_\pi) = \left\{ [p]^k_\theta \mid \theta \in J^\infty(\pi), p \in P \right\}$$

$$\begin{aligned} \xi_k \colon J^k_h(\xi_\pi) &\to J^\infty(\pi), & [p]^k_\theta &\mapsto \theta, \\ j^h_k(p) \colon J^\infty(\pi) &\to J^k_h(\xi_\pi), & \theta &\mapsto [p]^k_\theta. \end{aligned}$$

If $P = \Gamma(\pi^*_\infty(\xi))$, where ξ is a vector bundle, then

$$J^\infty_h(\xi_\pi) = J^\infty(\pi \times_M \xi).$$

Horizontal jets: examples

- $\xi = \pi$

$$\varkappa(\pi) := \Gamma(\pi^*_\infty(\pi)) \simeq \Gamma(\pi) \otimes_{C^\infty(M)} \mathcal{F}(\pi) \ni \varphi$$

$$E^{(u)}_\varphi = D_{x^\sigma}(\varphi^j)\frac{\partial}{\partial u^j_\sigma}$$

- ξ be q-th exterior power of cotangent bundle $T^*M \to M$

$$\Gamma(\pi^*_\infty(\xi)) \simeq \bar{\Lambda}^q(\pi)$$

Adjoint $\mathcal{C}$-differential operators

Let ξ be a vector bundle over $J^\infty(\pi)$ and $P = \Gamma(\xi)$.
The adjoint module $\hat{P} = \mathrm{Hom}_{\mathcal{F}(\pi)}(P, \bar{\Lambda}^n(\pi))$.
Consider another module of sections Q and $\Delta \in \mathcal{C}\mathrm{Diff}(P, Q)$.
The adjoint operator $\Delta^* \colon \hat{Q} \to \hat{P}$ is defined to satisfy the Green formula

$$\langle \Delta(p), \hat{q} \rangle - \langle p, \Delta^*(q) \rangle = \mathrm{d}_h \omega,$$

for all $p \in P$, $\hat{q} \in \bar{Q}$ and some $\omega(p, \hat{q}) \in \bar{\Lambda}^{n-1}(\pi)$.

- Δ is called self-adjoint if $\Delta^* = \Delta$.
- Δ is called skew-adjoint if $\Delta^* = -\Delta$.

In coordinates, if $\Delta = a_\sigma D_{x^\sigma}$ is a scalar operator then

$$\Delta^* = (-1)^\sigma D_{x^\sigma} \circ a_\sigma.$$

For matrix operator $\Delta = (\Delta_{ij})$ we have

$$\Delta^* = (\Delta^*_{ji}).$$

Hamiltonian operators

Define the operator $\delta\colon \bar{\Lambda}^n(\pi) \to \hat{\varkappa}(\pi)$ by the rule

$$\delta(\omega) = \ell^*_\omega(1), \quad \omega \in \bar{\Lambda}^n(\pi), \quad \ell_\varphi(\psi) = E_\psi(\varphi).$$

In coordinates, for $\omega = L\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ we have

$$\delta\omega = \left(\frac{\delta L}{\delta u^1}, \ldots, \frac{\delta L}{\delta u^m}\right), \qquad \text{where} \qquad \frac{\delta L}{\delta u^j} = (-1)^{|\sigma|} D_{x^\sigma} \frac{\partial L}{\partial u^j_\sigma}.$$

The skew-adjoint operator $A \in \mathcal{C}\mathrm{Diff}(\hat{\varkappa}(\pi), \varkappa(\pi))$ is called Hamiltonian if corresponding Possion bracket on $\bar{H}^n(\pi)$

$$\{\omega_1, \omega_2\}_A = \langle A(\delta\omega_1), \delta\omega_2\rangle, \qquad \omega_1, \omega_2 \in \bar{H}^n(\pi),$$

defines Lie algebra structure on $\bar{H}^n(\pi)$.

Hamiltonian equations

The differential equation is called Hamiltonian (w.r.t Hamiltonian operator A) if its in the form

$$u_t = A(\delta\omega), \qquad \text{for some } \omega \in \bar{H}^n(\pi)$$

Korteweg–de Vries equation

$$u_t = -u_{xxx} + 6uu_x$$

$$u_t = D_x\Big(\delta\big((u^3 - \tfrac{1}{2}u_x^2)\,\mathrm{d}x\big)\Big), \qquad A_1^{\mathsf{KdV}} = D_x,$$

$$u_t = \big(-\tfrac{1}{2}D_x^3 + uD_x + D_x \circ u\big)\Big(\delta\big(u^2\,\mathrm{d}x\big)\Big),$$
$$A_2^{\mathsf{KdV}} = -\tfrac{1}{2}D_x^3 + uD_x + D_x \circ u$$

Lie algebroids over $J^\infty(\pi)$

Consider a $\mathcal{C}$-differential operator $A\colon \Gamma(\pi^*_\infty(\xi)) \to \varkappa(\pi)$ whose image is closed under the commutation

$$[\operatorname{Im} A, \operatorname{Im} A] \subseteq \operatorname{Im} A \Longleftrightarrow [A(p_1), A(p_2)] = A([p_1, p_2]_A),$$

for all $p_1, p_2 \in \Gamma(\pi^*_\infty(\xi))$. The operator A transfers Lie algebra structure $[\ ,\ ]_A$ in the quotient $\Gamma(\pi^*_\infty(\xi))/\ker A$.

Definition (Kiselev, van der Leur(2011))

The traid

$$(\Gamma(\pi^*_\infty(\xi)), [\ ,\ ]_A) \xrightarrow{A} (\varkappa(\pi), [\ ,\ ])$$

is the (variational) Lie algebroid over the infinite jet space $J^\infty(\pi)$, and the Lie algebra homomorphism A is the (variational) anchor

Lie algebroids over $J^\infty(\pi)$: examples

The Korteweg–de Vries equation

$$u_t = -u_{xxx} + 6uu_x.$$

Consider its second Hamiltonian operator

$$A_2^{\mathsf{KdV}} = -\frac{1}{2}D_x^3 + uD_x + D_x \circ u$$

The image of A_2^{KdV} is closed under commutation. We have

$$[p_1, p_2]_{A_2^{\mathsf{KdV}}} = E_{A_2^{\mathsf{KdV}}(p_1)}(p_2) - E_{A_2^{\mathsf{KdV}}(p_2)}(p_1) + D_x(p_1)\cdot p_2 - p_1 \cdot D_x(p_2)$$

Lie algebroids over $J^\infty(\pi)$: examples

The $(2+1)$-dimensional 'heavenly' Toda equation

$$\mathcal{E} = \{u_{xy} = exp(-u_{zz})\}.$$

The generators of point symmetries

$$\varphi^x = \hat{\Box}^x(p(x)) \qquad \text{and} \qquad \varphi^y = \hat{\Box}^y(\bar{p}(y)),$$

where p and $\bar{p}$ are arbitrary smooth functions and

$$\hat{\Box}^x = u_x + \frac{1}{2}z^2 D_x,$$
$$\hat{\Box}^y = u_y + \frac{1}{2}z^2 D_y.$$

The image of each operator is closed under commutation. We have

$$[p_1, p_2]_{\hat{\Box}^x} = E_{\hat{\Box}^x(p_1)}(p_2) - E_{\hat{\Box}^x(p_2)}(p_1) + p_1 \cdot D_x(p_2) - D_x(p_1) \cdot p_2$$

for any $p_1(x)$ and $p_2(x)$.

Structure of the bracket $[\ ,\ ]_A$

Jacobi bracket on $\varkappa(\pi)$ is given by

$$[E_{\varphi_1}, E_{\varphi_2}] = E_{[\varphi_1,\varphi_2]}, \qquad [\varphi_1, \varphi_2] = E_{\varphi_1}(\varphi_2) - E_{\varphi_2}(\varphi_1).$$

Since $[D_{x^i}, E_\varphi] = 0$, we have

$$\begin{aligned}[A(p_1), A(p_2)] = A\left(E_{A(p_1)}(p_2) - E_{A(p_2)}(p_2)\right) \\ + E_{A(p_1)}(A)(p_2) - E_{A(p_2)}(A)(p_1).\end{aligned}$$

Consequently,

$$[p_1, p_2]_A = E_{A(p_1)}(p_2) - E_{A(p_1)}(p_2) + \{\{p_1, p_2\}\}_A,$$

where $\{\{\ ,\ \}\}_A$ bi-differential skew-symmetric $\mathcal{C}$-differential operator.

Homological evolutionary vector fields

ξ be a vector bundle over $J^\infty(\pi)$

- x^i — the coordinates on M;
- u^j_σ — the coordinates in fibres of the bundle $\pi_\infty\colon J^\infty(\pi) \to M$;
- p — the fibre coordinates in the induced bundle $\pi^*_\infty(\xi)$ over $J^\infty(\pi)$;
- p_σ — the coordinates in fibres of $J^\infty_h(\xi_\pi)$;
- b — the odd coordinates in the parity-reserved bundle $\Pi\xi$;
- b_σ — the odd coordinates in fibres of $J^\infty_h(\Pi\xi_\pi)$;
- $[\ ,\ ]_A$ — the bracket on $\Gamma(\pi^*_\infty(\xi))$ defined up to the kernel A.

The odd evolutionary vector field on $J^\infty_h(\Pi\xi_\pi)$

$$Q = E^{(u)}_{A(b)} - \frac{1}{2} E^{(b)}_{\{\{p,q\}\}_A|_{p:=b,q:=b}}, \qquad [Q,Q] = 2Q^2 = 0.$$

is Homological.

Homological evolutionary vector fields: examples

For the second Hamiltonian operator of KdV

$$A_2^{\mathsf{KdV}} = -\frac{1}{2}D_x^3 + uD_x + D_x \circ u$$

we have

$$\{\{p_1, p_2\}\}_{A_2^{\mathsf{KdV}}} = D_x(p_1) \cdot p_2 - p_1 \cdot D_x(p_2).$$

The corresponding Homological vector field is

$$Q = E^{(u)}_{A_2^{\mathsf{KdV}}(b)} + E^{(b)}_{bb_x}.$$

Note:

$$E^{(u)}_{A_2^{\mathsf{KdV}}(b)} = A_2^{\mathsf{KdV}}(b)\frac{\partial}{\partial u} + D_x(A_2^{\mathsf{KdV}}(b))\frac{\partial}{\partial u_x}\ldots,$$

$$E^{(b)}_{bb_x} = bb_x\frac{\partial}{\partial b} + D_x(bb_x)\frac{\partial}{\partial b_x} + \ldots.$$

Homological evolutionary vector fields: examples

Let A be a Hamiltonian operator then

$$Q = E^{(u)}_{A(b)} + E^{(b)}_{bb_x}.$$

is Hamiltonian vector field with respect to

W-charge $\bar{\Omega} = \frac{1}{2}\langle b, A(b)\rangle$, and symplectic sturcture $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$Q = E^{(u)}_{\delta\bar{\Omega}/\delta b} + E^{(b)}_{-\delta\bar{\Omega}/\delta u}.$$

We have

$$Q^2 = 0 \iff [\![\bar{\Omega}, \bar{\Omega}]\!] = 0.$$

$$Q = E^{(u)}_{[\![\bar{\Omega},u]\!]} + E^{(b)}_{[\![\bar{\Omega},b]\!]}, \qquad \text{so that } Q(\Xi) = [\![\bar{\Omega}, \Xi]\!],$$

where $[\![\cdot,\cdot]\!]$ is the variational Schouten bracket

Zero-curvature representations

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra which corresponds to the Lie group G. Consider $\mathfrak{g} \otimes_{\mathbb{R}} \bar{\Lambda}^1(\mathcal{E}^\infty)$

$$[A \otimes \mu, B \otimes \nu] = [A, B] \otimes \mu \wedge \nu, \qquad \mathrm{d}_h(A \otimes \mu) = A \otimes \mathrm{d}_h \mu$$

where $\mu, \nu \in \bar{\Lambda}^1(\mathcal{E}^\infty)$, $A, B \in \mathfrak{g}$.
A horizontal one-form $\alpha \in \mathfrak{g} \otimes \bar{\Lambda}^1(\mathcal{E}^\infty)$ is called a $\mathfrak{g}$-valued zero-curvature representation (ZCR) for $\mathcal{E}$ if

$$\mathrm{d}_h \alpha = \frac{1}{2}[\alpha, \alpha]$$

Let $g \in C^\infty(\mathcal{E}^\infty, G)$. The ZCR α^g is called gauge equivalent to the initial α if

$$\alpha^g = \mathrm{d}_h g \cdot g^{-1} + g \cdot \alpha \cdot g^{-1}$$

ZCRs: removabilty

α_λ — family of $\mathfrak{g}$-valued ZCRs. The $\lambda \in \mathbb{R}$ is removable, i.e. $\exists g \in C^\infty(\mathcal{E}^\infty, G)$ and $\exists \lambda_0 \in \mathbb{R}$ such that

$$\alpha_\lambda = (\alpha_{\lambda_0})^g.$$

Theorem (Marvan(2002))

λ is removable if $\exists Q \in \mathfrak{g} \otimes \bar{\Lambda}^0(\mathcal{E}^\infty)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\alpha_\lambda = \partial_{\alpha_\lambda}(Q) = \mathrm{d}_h Q + [Q, \alpha_\lambda].$$

$$\partial_\alpha \colon \mathfrak{g} \otimes \bar{\Lambda}^0(\mathcal{E}^\infty) \to \mathfrak{g} \otimes \bar{\Lambda}^1(\mathcal{E}^\infty).$$

NB!

$$\partial_\alpha \circ \partial_\alpha = 0$$

Non-Abelian Lie algebroids over infinite jet spaces

- $\chi\colon \mathfrak{g}\otimes\Lambda^1(M)\to M$
- $\xi\colon M\times\mathfrak{g}\to M$

Theorem (Kiselev,AK(2014))

The odd vector field on $J^\infty(\chi\times_M\Pi\xi)$

$$Q=E^{(\alpha)}_{\mathrm{d}_h b+[b,\alpha]}+\frac{1}{2}E^{(b)}_{[b,b]},\qquad [Q,Q]=0,\quad Q^2=0,$$

encode the non-Abelian variational Lie algebroid.

- α is a even coordinates along fibres of χ;
- b is odd coordinates in the bundle $\Pi\xi$;
- the operator $\partial_\alpha=\mathrm{d}_h+[\,\cdot\,,\alpha]$ is the anchor.

Non-Abelian Lie algebroids over infinite jet spaces

- $\chi\colon \mathfrak{g}\otimes\Lambda^1(M)\to M$
- $\xi\colon M\times\mathfrak{g}\to M$
- $\Pi\chi^*\colon D_1(M)\otimes\Pi\mathfrak{g}^*\to M$
- $\xi^*\colon M\times\mathfrak{g}\to M$

odd-parity evolutionary vector field on
$J^\infty(\chi\times_M\Pi\chi^*\times_M\Pi\xi\times_M\xi^*)$

$$\hat{Q}=E^{(\alpha)}_{\mathrm{d}_h b+[b,\alpha]}+E^{(\alpha^*)}_{(\alpha^*)\overleftarrow{\mathrm{ad}}}+\frac{1}{2}E^{(b)}_{[b,b]}+E^{(b^*)}_{-\mathrm{ad}^*_\alpha(\alpha^*)+(\alpha^*)\overleftarrow{\mathrm{d}_h}+\mathrm{ad}^*_b(b^*)},$$

where $\langle(\alpha^*)\overleftarrow{\mathrm{ad}}^*_b,\alpha\rangle=\langle\alpha^*,[b,\alpha]\rangle$ and $\langle\mathrm{ad}^*_b(b^*),p\rangle=\langle b^*,[b,p]\rangle$ for any $\alpha\in\Gamma(\chi)$ and $p\in\Gamma(\xi)$.

The structure of non-Abelian variational Lie algebroid is encoded on the Whitney sum $J^\infty(\chi \times_{M^n} \Pi\chi^* \times_{M^n} \Pi\xi \times_{M^n} \xi^*)$ by

$$\widehat{S} = \int \mathrm{dvol}(M^n) \left\{ \langle \alpha^*, [b,\alpha] + \mathrm{d}_h(b) \rangle + \tfrac{1}{2} \langle b^*, [b,b] \rangle \right\} \in \overline{H}^n$$

which satisfies the classical master-equation

$$[\![\widehat{S}, \widehat{S}]\!] = 0.$$

The functional $\widehat{S}$ is the Hamiltonian of $\widehat{Q}$

$$\widehat{Q}(\mathcal{H}) \cong [\![\widehat{S}, \mathcal{H}]\!] \tag{1}$$

for any $\mathcal{H} \in \overline{H}^n(\chi \times_{M^n} \Pi\chi^* \times_{M^n} \Pi\xi \times_{M^n} \xi^*)$.

$$\widehat{Q} = \overrightarrow{\partial}^{(\alpha)}_{[b,\alpha]+\mathrm{d}_h(b)} + \overrightarrow{\partial}^{(\alpha^*)}_{(\alpha^*)\overleftarrow{\mathrm{ad}}^*_b} + \tfrac{1}{2}\overrightarrow{\partial}^{(b)}_{[b,b]} + \overrightarrow{\partial}^{(b^*)}_{-\mathrm{ad}^*_\alpha(\alpha^*)+(\alpha^*)\overleftarrow{\mathrm{d}}_h+\mathrm{ad}^*_b(b^*)}, \tag{2}$$

where $\langle (\alpha^*)\overleftarrow{\mathrm{ad}}^*_b, \alpha \rangle \stackrel{\text{def}}{=} \langle \alpha^*, [b,\alpha] \rangle$ and $\langle \mathrm{ad}^*_b(b^*), p \rangle = \langle b^*, [b,p] \rangle$ for any $\alpha \in \Gamma(\chi)$ and $p \in \Gamma(\xi)$.

$$\widehat{Q}^2 = 0.$$