

Singularities, self-similar solutions and their generalizations

@ Ostrava Seminar on Mathematical Physics

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Part I - PDEs and singularities

The following equation was investigated in [Bizoń, Chmaj, Tabor],[Kycia 2012]:

Semilinear wave equation

$$U_{tt} - \Delta U - U^p = 0, \quad (1)$$

$U = U(x, t)$, $x \in \mathbf{R}^n$, $p > 1$ -integer;

There is a dichotomy:

- 'Small' initial data disperse to infinity as for linear equations.
- 'Large' initial data blows up in finite time.
- Movies - wave and nonlinear wave equations...

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Self-similar solutions

Transformation

$$U(t, r) = (T - t)^{-\alpha} u(\rho), \quad \rho := \frac{r}{T - t}, \quad \alpha := \frac{2}{p - 1}, \quad (2)$$

Equation for self-similar profiles

$$(1 - \rho^2)u'' + \left(\frac{n - 1}{\rho} - \frac{2(p + 1)}{p - 1} \rho \right) u' - \frac{2(p + 1)}{(p - 1)^2} u + u^p = 0 \quad (3)$$

- The Huygens principle - blowup appears for $t = T$ which corresponds to the inner part of the cone.
- The inner part of the cone corresponds to $\rho \in [0; 1]$.

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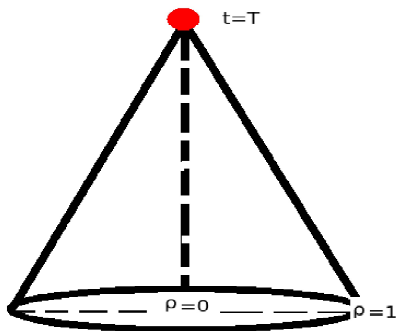


Figure : Inner part of the cone corresponds to $\rho \in [0; 1]$.

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- Equation has fixed singularities at $\rho = 0$, $\rho = \pm 1$ and $\rho = \infty$.
- Task: Construct (global) analytic solution on the unit interval.
- It is not a trivial task as
 - $\rho = 0$ and $\rho = 1$ are fixed singularities of the equation.
 - There can be movable singularities of the solutions inside $[0; 1]$.

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Matching conditions

Construction of global analytic solutions

The method of construction of solutions was provided in [Bizoń, Maison, Wasserman] for $n = 3$ and was extended for $n > 3$ in [Kycia2011].

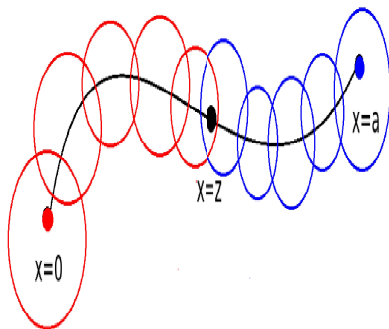


Figure : Analytic continuation along the curve.

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- Construct local analytic solution at $\rho = 0$.
- Construct local analytic solution at $\rho = 1$.
- Prove that local analytic solution $\rho = 0$ can be extended toward $\rho = 1$.
- Prove that local analytic solution $\rho = 1$ can be extended toward $\rho = 0$.
- Match these local solution at some point $x_0 \in (0; 1)$.

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Construction of global analytic solutions cont.

- Local analytic solution extended towards x_0 is parametrized by initial data.
- When initial data are varied then on the plane $(u(x_0), u'(x_0))$ the curves are prescribed:
 - The curve C_0 is a local analytic solution extended from $p = 0$.
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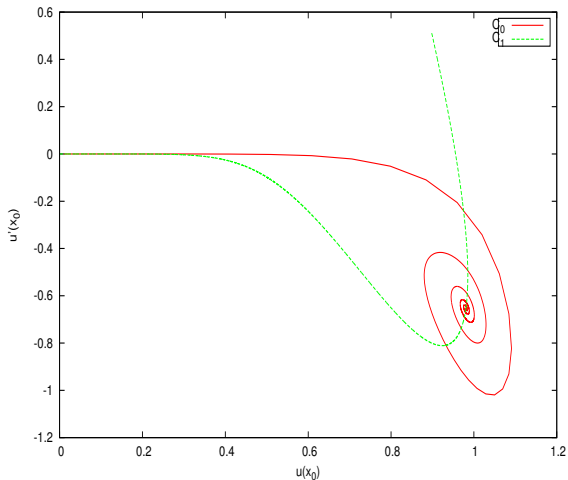


Figure : Exact matching

Construction of global analytic solutions cont.

- Every intersection gives global analytic solution on $[0; 1]$.
- There is countable many intersections, i.e., countable family of global solutions.

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The Lane-Emden equation - a digression

The Lane-Emden equation

The problem of movable singularities for local solutions at $\rho = 0$ has close connection with the singularities of the Lane-Emden equation:

Equation for self similar profiles

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...and its generalization [Kycia, Filipuk 1, Kycia, Filipuk 2]

The Generalized Emden-Fowler equation

$$\frac{d^2 u(x)}{dx^2} + \frac{\alpha}{x} \frac{du(x)}{dx} + x^n u(x)^p = 0 \quad (7)$$

with solutions analytic at $x = 0$.

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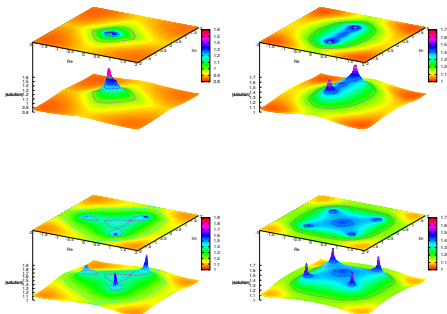
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The Lane-Emden equation

Location of singularities [Kycia, Filipuk 1, Kycia, Filipuk 2]

A nonzero analytic solutions of the Generalized Emden-Fowler equation have $n + 2$ singularities located symmetrically with respect to the origin on the rays connecting the origin with all $(n + 2)$ roots of -1 in the complex plane.

An example - $p = 5$ and $u(0) = 1.5$:



Generalization

Construction of global analytic solutions cont.

Natural questions arise:

- Is an exact matching a coincidence?
- How to significantly generalize these results?

Some hints: Consider the simple equations of this kind [MAAR].

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A simplified equation

$$(1-x)u''(x) + \left(\frac{\alpha}{x} + \beta\right)u'(x) + \delta u(x)^p = 0, \quad (8)$$

where $\alpha > 0$, β , $\delta \neq 0$ are real numbers and $p > 1$ is odd.

Find solution on $x \in [0; 1]$.

A simplified equation

It occurs that depending on the values of parameter:

- There exists a countable family of solutions as in semilinear wave profiles equations.
- There is a finite family of global solutions.
- There is only one solution - the trivial one.

The essential condition for the matching condition is the existence of some special singular solution:

$$u(x) = b_{\infty} x^a, \quad b_{\infty} = \left(\frac{a(1-a-\alpha)}{\delta} \right)^{\frac{1}{p-1}}, \quad a = \frac{2}{1-p}. \quad (9)$$

...work in progress...

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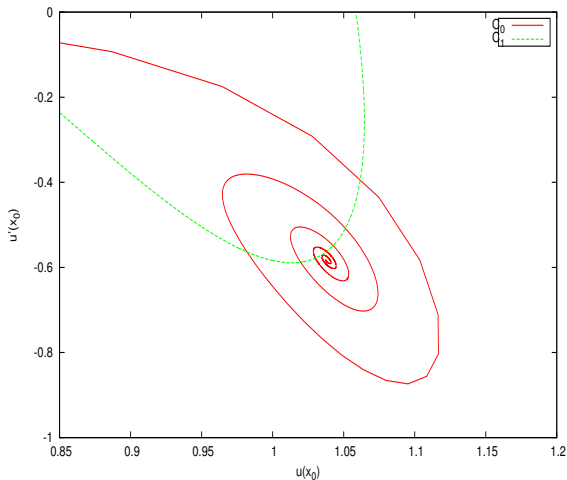


Figure : Slightly violated matching condition

A simplified equation

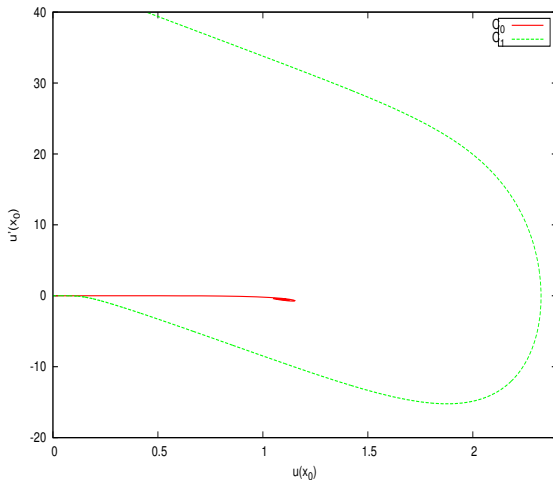


Figure : Only the trivial solution

Part II - General Context

Painlevé property

Function

A function is an application of a set of objects into a set of images which applies given object onto one **and only one** image.

- Function is single valued prescription - there is no 'multivalued functions',
- $f : x \rightarrow e^x$ - a function,
- $f : x \rightarrow \sqrt{x}$ - not a function if defined on the whole $\mathbb{C}$.

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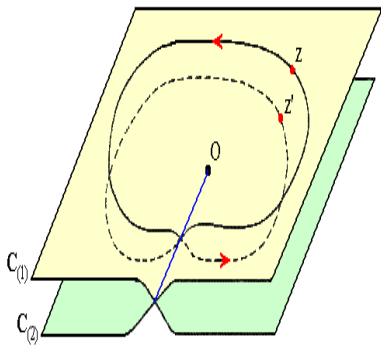
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Critical point

Critical point

A critical point of an application of the Riemann sphere $\mathbb{CP}^1$ onto itself is any singular point, isolated or not, around which at least two determinations are permuted. Such a point is an obstacle for an application to be a function.



Singular points($x = 0$):

- Pole $\frac{1}{x^a}$, $a \in \mathbb{N} \setminus \{0\}$ (noncritical),
- Branch point x^a , a - noninteger; (critical),
- Essential singularity:
 - $e^{1/x}$ (noncritical),
 - $\operatorname{tg}(\log(x))$ (critical and nonisolated).

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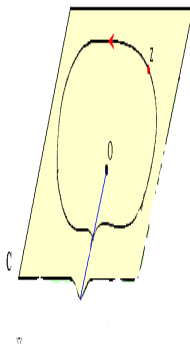
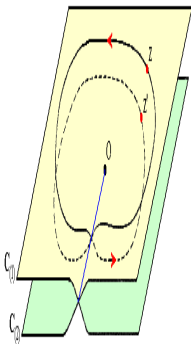
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Uniformization

Uniformization

Consider a multivalued application of the Riemann sphere onto itself. There exists two classical methods, called uniformizations, to define from it a single valued application, i.e., a function.

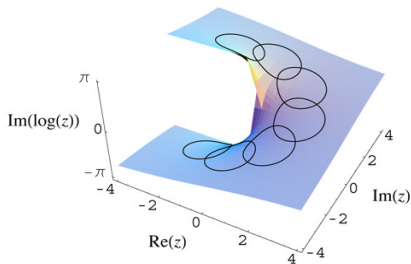
Uniformization is possible only when the location of singularities is known!



Solution of ODEs

'Classical'/Cauchy version:

Find local solution in terms of a power series in some neighbourhood of an expansion point and extend it by analytic continuation.



Painlevé version

To integrate an ODE is to find for the general solution a **finite expression**, possibly multivalued, in a **finite** number of functions, valid in the whole domain of definition.

For $u' + u^2 = 0$:

- $u = u_0 \sum_{j=0}^{\infty} [-(x - x_0)u_0]^j$ - nonintegrated (no finite form, locally defined),
- if radius of convergence is known, summation performed and result analytically continued with the meromorphic function $(x - x_1)^{-1}$ it is solved.

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To integrate an ODE is to find for the general solution a **finite expression**, possibly multivalued, in a **finite** number of functions, valid in the whole domain of definition.

- **Painlevé Property (PP)** is of an ODE is the uniformizability of its solution.
- 'Double interest' in differential equations:
 - Source of new functions (since 1614, Lord Napier ($u' = u$ Galileo) 'Mirifici Logarithmorum Canonis Descriptio' ¹,
 - Class of equations that can be integrated with existing functions available.

¹A Description of the Wonderful Law of Logarithms

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Linear equations

Linear ODE

The general solution of a linear ODE is uniformizable.

$$\sum_{k=0}^N a_k(x) \frac{d^k u}{dx^k} = 0, a_N(x) \neq 0. \quad (10)$$

The only solutions of these equations are singularities of the coefficients $a_k(x)$ (Frobenius method).

Fixed singularities

The singularities of the equation coefficients are called **fixed singularities**.

Linear equations have only fixed singularities.

- Linear ODEs define functions (Airy, Bessel, etc.),
- To extend a list of known functions it is necessary to consider nonlinear ODEs.

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Nonlinear equations

Nonlinear ODEs possess two types of singularities:

- fixed - singularities of the coefficients of ODE,
- movable - the singularities of solutions; position depends on initial data; not present in linear ODEs.

Up to now no general methods exist that allow to determine the positions of movable singularities.

(Trivial) example [Goriely]

The equation

$$\dot{x} = x^3, \quad x(t_0) = x_0$$

has the solutions

$$x(t) = (2(t_0 - t) + x_0^{-2})^{-1/2}.$$

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$$\dot{x} = x^3, \quad x(t_0) = x_0$$

has the solutions

$$x(t) = (2(t_0 - t) + x_0^{-2})^{-1/2}.$$

Nonlinear equations

Nonlinear ODEs possess two types of singularities:

- fixed - singularities of the coefficients of ODE,
- movable - the singularities of solutions; position depends on initial data; not present in linear ODEs.

Up to now no general methods exist that allow to determine the positions of movable singularities.

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- French Prime Minister (1917), (1925),
- ODEs: P. property, P. transcendents,
- ODEs: In 1908, he became Wilbur Wright's first airplane passenger in France and in 1909 created the first university course in aeronautics,
- Painlevé conjecture: Among the solutions to the n -body problem: there are noncollision singularities for $n \geq 4$,
- General Relativity: Gullstrand–Painlevé coordinates for Schwarzschild metric.

Painlevé property

Painlevé property (reformulated)

One calls Painlevé property of an ODE the absence of movable critical singularities in its general solution.

General (i.e. not singular) solutions are considered.

- Movable critical singularities are obstacles in uniformization.
- One has to know where start cuts and how to do it.
- Only noncritical movable singularities are tractable.

Class of abstractions of the equations with PP

The PP of an ODE is invariant under an arbitrary homographic transformation of the dependent variable and an arbitrary holomorphic change of independent variable:

$$(x, u) \rightarrow (X, U) : u = \frac{a(x)U + b(x)}{c(x)U + d(x)}, \quad X = e(x), \quad ad - bc \neq 0 \quad (11)$$

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Painlevé program

Determine all the **algebraic** differential equations of first order, then second order, then third order, etc., whose general solution has no movable critical points.

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- Current state of classification:
 - 1st order - Riccati and Weierstrass equation,
 - 2ns order - 53 canonical equations, 47 integrable in terms of known functions,
 - 3rd order - nothing interesting...
 - 4th order - in progress...

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Painlevé transcendents

Painlevé transcendents

6 algebraic ODEs which solutions has only noncritical movable singularities. They cannot be integrated in terms of know transcendet functions, they define new functions.

- (PI) $x'' = 6x^2 + \lambda t$,
- (PII) $x'' = 2x^3 + tx + \mu$,
- (PIII) $txx'' = tx'^2 - xx' + at + bx + cx^3 + dtx^4$,
- (PIV) $txx'' = tx'^2 - xx' + at + bx + cx^3 + dtx^4$,
- (PV) very long equation,
- (PVI) very long equation.

(PIII) can be used to construct correlation function for 2D Ising model: T.T. Wu, B.M McCoy, C.A.Tracy and E. Barouch (1976), *Spin-spin correlation functions for the two dimensional Ising model: exact theory in the scaling region*, Physical Review B13, 316-374

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Painlevé test

The Painlevé property/Painlevé test

- Deduce global structure of solution (types of singularities) from the local behaviour around some points in the complex plane. Only sufficient conditions $\rightarrow$ by the contraposition - it gives a result when it fails.
- Consists of two steps:
 - (a local study) necessary conditions for absence of critical points,
 - (a global study) proof of sufficiency (difficult).

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Substitute formal power series into an equation and do the following steps:

- Expansion around 'singular solutions' - finding all dominant balances.
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Painlevé test - step 1

Scale-invariant solution

Vector field $\vec{f}$ is scale-invariant if for weight vector $\vec{w} = \{w_1, \dots, w_n\}$ the weight of $\deg(f_i, \vec{w}) = w_i$.

$$t \rightarrow \epsilon t, \quad x_i \rightarrow \epsilon^{w_i} x_i$$

If there exist non-trivial solution/solutions for

$$\alpha_i w_i = f_i(\vec{\alpha})$$

then the solution $\vec{x} = \vec{\alpha} t^{\vec{w}}$.

Simple Mathematica notebook to search for scale-invariant terms available on [Software] - for description see: [Kycia 2014].

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Painlevé test - step 1, dominant balance

Weight-homogeneous decomposition

Decompose (if it is possible) the vector field into weight-homogenous components

$$\vec{f} = \vec{f}^{(0)} + \vec{f}^{(1)} + \dots,$$

where $\vec{f}^{(0)}$ is scale invariant with the solution $\vec{x}^{(0)} = \vec{\alpha} t^{\vec{p}}$ and

$$f_i^{(j)}(t^{\vec{p}} \vec{x}) = t^{p_i + q^{(j)} - 1} f_i^{(j)}(\vec{x}),$$

where

$$0 < q^{(i)} < q^{(j)}, \quad \forall i < j, \quad q^{(i)} \in \mathbb{Q}.$$

Dominant balance

$F = \{\vec{p}, \vec{\alpha}\}$ and decomposition which fulfils above conditions.

Painlevé test - step 1, dominant balance

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Painlevé test - step 2, the Kovalevskaya exponents

The Kovalevskaya exponents

For given dominant balance F of the system $\dot{\vec{x}} = \vec{f}(\vec{x})$ the Kovalevskaya exponents $R = \{-1, \rho_2, \dots, \rho_n\}$ are the eigenvalues of the matrix $K := D\vec{f}^{(0)}(\vec{\alpha}) - \text{diag}(\vec{p})$.

Painlevé test - step 3, formal solution

Puiseux series

- Fix balance F .
- R^+ - set of Kovalevskaya exponents with positive real part.
- $1/s$ - the least common denominator of $\{q^{(1)}, \dots, q^{(m)}\} \cup R^+$.

We assume the expansion

$$\vec{x}(t) = t^{\vec{p}} \left(\vec{\alpha} + \sum_{i=0}^{\infty} \vec{c}_i t^{i/s} \right)$$

The recursion for the coefficients

$$K \vec{c}_j = \frac{j}{s} \vec{c}_j - \vec{P}_j(\vec{c}_1, \dots, \vec{c}_{j-1}).$$

If $\rho = \frac{j}{s}$ is a Kovalevskaya exponents then compatibility conditions have to be introduced

- K is semi-simple.
- The Fredholm alternative must holds $\bar{\beta} \vec{P} = 0, \forall \rho \in R^+$.

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Otherwise:

Ψ -series

General solution, when Kovalevskaya exponents are irrational:

$$\vec{x}(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \vec{c}_{ij} t^{\vec{p} + i/s} (t^{\rho} \log(t))^{j/s}$$

Integrability

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The existence of **sufficiently many** first integrals.

Local integrability

Integrability of linearized system. There is always local integrability.

Arnold-Liouville integrability (Hamiltonian systems)

The Hamiltonian $H(p, q) = 0$ is Liouville integrable if there exist n independent analytic first integrals $I_1 = H, I_2, \dots, I_n$ in involution ($\{I_i, I_j\} = 0$). Moreover, if $\bigcap_{i=1}^n \{I_i = a_i, (p, q) \in \mathbb{R}^{2n}\}$ is compact and connected then it is topologically a real tori.

Algebraic integrability

A vector field $\dot{x} = f(x)$ with $x \in \mathbb{K}^n$ is algebraically integrable if there exist $(n - 1)$ independent algebraic first integrals I_i .

Some integrals can appear from the second and higher integrals.

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Some integrals can appear from the second and higher integrals.

- Only algebraic integrability is closely connected with PP.
- Arnold-Liouville integrability requires only a half of the first integrals to exist and the remaining part (angle variables integrals) are generally multivalued.
- PP probably is not connected with chaos but no explicit proof exists.

Theorem [Goriely]

Let $I(x)$ be an algebraic first integral of vector field $\delta_f := \sum f_i \partial_i$. Assume that there is a weight-homogeneous decomposition of $\delta_f = \delta_0 + \delta_1 + \dots + \delta_p$ and a decomposition of $I = I^{(0)} + \dots + I^{(q)}$. Then, $I^{(0)}$ is a first integral of δ_0 and $I^{(q)}$ is a first integral of δ_p . Conclusion: Let I be an algebraic first integral of δ_f , a weight-homogeneous vector field of weight w . Then, every weight-homogeneous component of I is a first integral of δ_f .

Yoshida's theorem (1983)

If there is at least one irrational or imaginary Kovalevskaya exponent, the system is not algebraically integrable.

(Non)integrability-like theorems

If there is a dominant balance such that the Kovalevskaya matrix is semi-simple and the Kovalevskaya exponents are $\mathbb{Z}$ ($\mathbb{N}$) independent, then there is no rational (polynomial) first integral.

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Numerical scanning of the complex plane

Based on [Kycia 2014].

Problem statement

- ODE as a system of first order DE

$$\frac{d\vec{y}(x)}{dx} = \vec{f}(\vec{y}; x), \quad \vec{y}(x) : x \in \mathbb{C} \rightarrow \mathbb{C}^n. \quad (12)$$

- Initial value $\vec{y}(x_0) = \vec{y}_0$.
- Path, e.g., $(t \in \mathbb{R}^+)$
 - Semiline $x(t) = x_0 + (t + shift) \cdot e^{i\phi}$
 - Spiral $x(t) = (x_0 + (at + b)e^{i \cdot dir \cdot t})e^{i\phi}$
- Domain - path connected region (ideally connected by paths along which integration is performed).
- Condition for singularity proximity - the crude estimation $\|\vec{y}\| < \text{Large const.}$ Not the state of art, but it can be improved.

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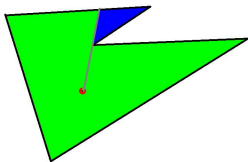
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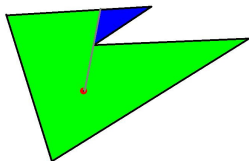


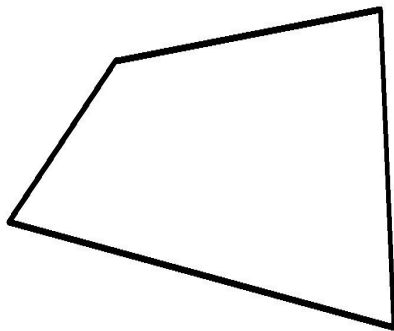
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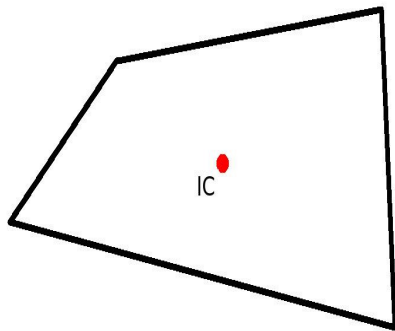
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 - Spiral $x(t) = (x_0 + (at + b)e^{i \cdot dir \cdot t})e^{i\phi}$
- Domain - path connected region (ideally connected by paths along which integration is performed).
- Condition for singularity proximity - the crude estimation $\|\vec{y}\| < \text{Large const.}$ Not the state of art, but it can be improved.

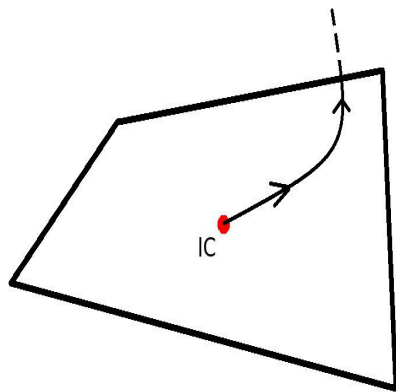




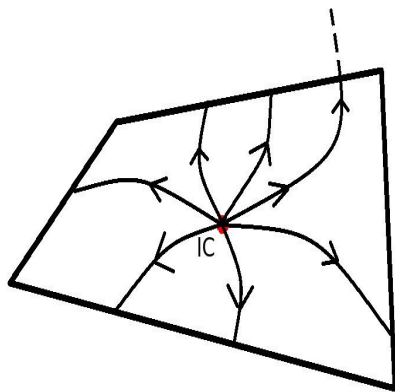
Initial Conditions



Integration along path



Full integration



Examples

Equations [Kycia, Filipuk 1]

The Emden-Fowler equation

$$\frac{d^2 u(x)}{dx^2} + \frac{\alpha}{x} \frac{du(x)}{dx} + x^n u(x)^p = 0 \quad (13)$$

Generalized Isothermal Sphere equation

$$\frac{d^2 u(x)}{dx^2} + \frac{\alpha}{x} \frac{du(x)}{dx} - x^n e^{-u(x)} = 0, \quad (14)$$

$$u(0) = 0$$

Location of singularities [Kycia, Filipuk 1]

A nonzero analytic solutions of the Generalized Emden-Fowler and Isothermal Sphere equations have $n + 2$ singularities located symmetrically with respect to the origin on the rays connecting the origin with all $(n + 2)$ roots of -1 in the complex plane.

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The Emden-Fowler equations [Kycia, Filipuk 1]

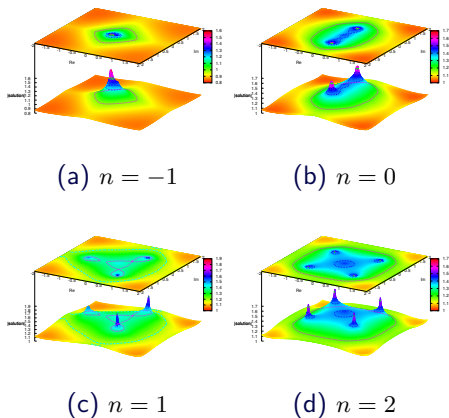


Figure : $p = 5$ and $u(0) = 1.5$, the Generalized Emden-Fowler solution.

Generalized isothermal sphere equations

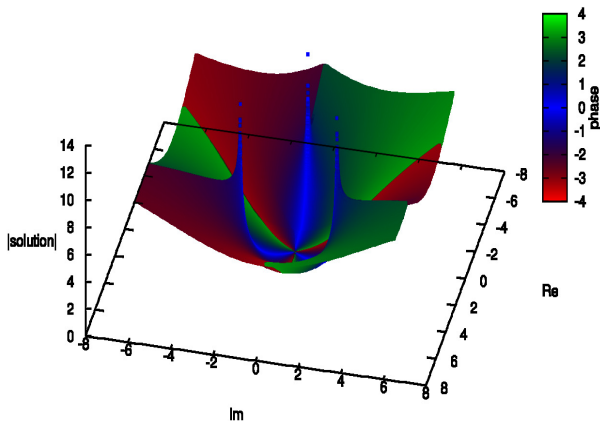


Figure : $u(0) = 0$, $n = 1$

Conclusions

- Painlevé property is not well known.
- It has intriguing connections with the existence of algebraic first integrals and integrability.
- No global Painlevé tests exist. Case by case approach.
- Every ODE obtained by a similarity reduction of an integrable PDE in the sense of Inverse Scattering Transform has PP - ARS (Ablowitz-Ramani-Segur) conjecture.

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-  My homepage <http://www.mimuw.edu.pl/~rkycia/>

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