Spectral optimization for the Robin Laplacian on exterior domains

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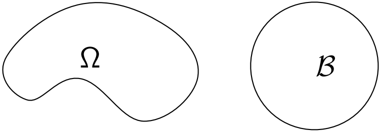
Outline

1. Motivation & background
   - Optimization in bounded domains
   - The Robin Laplacian on an exterior domain
2. Spectral optimization in exterior domains
   - Two dimensions and the role of connectedness
   - Higher dimensions and the Willmore energy
   - Planes with cuts
3. Summary and open questions

Classical geometric isoperimetric inequality

Geometric setting

Bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, with C^∞-boundary $\partial\Omega$; ball $\mathcal{B} = \mathcal{B}_R \subset \mathbb{R}^d$.



Classical isoperimetric inequality

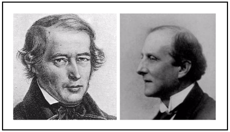
$$\begin{cases} |\Omega| = |\mathcal{B}| \\ \Omega \not\cong \mathcal{B} \end{cases} \quad \Longrightarrow \quad \boxed{|\partial\Omega| > |\partial\mathcal{B}|}$$

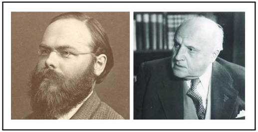
$\Longleftarrow$ It was known to ancient Greeks.

$d = 2$: J. Steiner (1882),
completed by C. Caratheodory.

$d = 3$: H. Schwarz (1890).

$d > 3$: E. Schmidt (1939).





The Faber-Krahn inequality

$\Omega, \mathcal{B} \subset \mathbb{R}^d$, $d \geq 2$

Dirichlet eigenvalues of the Laplacian on Ω

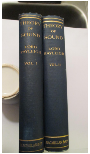
$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad \Longrightarrow \quad 0 < \lambda_1^{\mathrm{D}}(\Omega) \leq \lambda_2^{\mathrm{D}}(\Omega) \leq \lambda_3^{\mathrm{D}}(\Omega) \leq \ldots$$

The Faber-Krahn inequality

$$\begin{cases} |\Omega| = |\mathcal{B}| \\ \Omega \not\cong \mathcal{B} \end{cases} \quad \Longrightarrow \quad \boxed{\lambda_1^{\mathrm{D}}(\Omega) > \lambda_1^{\mathrm{D}}(\mathcal{B})}$$

Conjecture: `Lord Rayleigh (1877).`

Proofs: $\begin{cases} d = 2 : \texttt{G. Faber (1923)}, \\ d \geq 3 : \texttt{E. Krahn (1926)}. \end{cases}$



Faber-Krahn inequality for other boundary conditions?

Dirichlet BC: $u = 0$ on $\partial\Omega$ (*quantum mechanics,...*)

One of many that give well-posed spectral problem for $-\Delta$ in Ω.

Could one generalise the Faber-Krahn inequality for other BC?

$\partial_n u$ – normal derivative with the outer normal n to Ω.

Neumann BC: $\partial_n u = 0$ on $\partial\Omega$ (*heat insulators,..*)

Trivial setting: `the lowest eigenvalue` $= 0$.

Robin BC: $\partial_n u + \alpha u = 0$ on $\partial\Omega$, $\alpha \in \mathbb{R}$ (*elasticity, superconductivity*)

Non-trivial! In physics, searching for the shape minimizing the critical temperature of the superconductivity (`Giorgi-Smits-07`).

$\boxed{\alpha > 0 :}$ complete

$d = 2$: `M. Bossel (1986)`

$d \geq 3$: `D. Daners (2006)`

$\boxed{\alpha < 0 :}$ partial results

`Freitas-Krejčiřík-15`

`Antunes-Freitas-Krejčiřík-17`

The Robin Laplacian on a bounded domain

Robin eigenvalues of the Laplacian on Ω

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ \partial_n u + \alpha u = 0, & \text{on } \partial\Omega, \end{cases} \quad \Longrightarrow \quad \lambda_1^\alpha(\Omega) \le \lambda_2^\alpha(\Omega) \le \lambda_3^\alpha(\Omega) \le \dots$$

$\lambda_k^\alpha(\Omega)$ are eigenvalues of the self-adjoint operator in $L^2(\Omega)$:

$$\begin{aligned} -\Delta_\alpha^\Omega u &:= -\Delta u, \\ \mathrm{dom}\,(-\Delta_\alpha^\Omega) &:= \{u \colon u, \nabla u, \Delta u \in L^2(\Omega), \partial_n u + \alpha u = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

$\alpha \mapsto \lambda_1^\alpha(\Omega)$ is increasing with the properties

$\alpha > 0$: $\lambda_1^\alpha(\Omega) \in (0, \lambda_1^{\mathrm{D}}(\Omega))$,

$\alpha < 0$: $\lambda_1^\alpha(\Omega) < 0$,

$\alpha \to +\infty$: $\lambda_1^\alpha(\Omega) \to \lambda_1^{\mathrm{D}}(\Omega)$,

$\alpha \to -\infty$: $\lambda_1^\alpha(\Omega) \to -\infty$.

FK-inequality for Robin Laplacian on a bounded domain

The original Faber-Krahn technique fails!

The Bossel-Daners inequality ($\boxed{\alpha > 0}$, Bossel-86, Daners-06)

$$\begin{cases} |\Omega| = |\mathcal{B}| \\ \Omega \not\cong \mathcal{B} \end{cases} \qquad \Longrightarrow \qquad \boxed{\lambda_1^\alpha(\Omega) > \lambda_1^\alpha(\mathcal{B})}$$

Flipped inequality ($d=2$, $\boxed{\alpha < 0}$, Antunes-Freitas-Krejčiřík-17)

$$|\partial\Omega| = |\partial\mathcal{B}| \qquad \Longrightarrow \qquad \boxed{\lambda_1^\alpha(\Omega) \le \lambda_1^\alpha(\mathcal{B})}$$

Many open questions left for $\alpha < 0$:

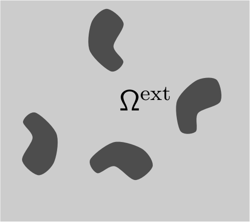
- $|\Omega| = |\mathcal{B}|$: the inequality is wrong for $d \ge 2$, might be true for simply connected domains in $\mathbb{R}^2$.
- $|\partial\Omega| = |\partial\mathcal{B}|$: the inequality is wrong for $d \ge 3$, might be true for convex domains in $\mathbb{R}^d$, $d \ge 3$.

For $\alpha < 0$ spectral optimization is also meaningful for unbounded Ω.

The Robin Laplacian on an exterior domain

Exterior domain

$\Omega^{\rm ext} := \mathbb{R}^d \setminus \overline{\Omega}$, where $\Omega \subset \mathbb{R}^d$ is a bounded domain, having $N_\Omega < \infty$ simply connected components.



$\Omega^{\rm ext}$ is connected, unbounded and with compact boundary.

$$-\Delta_\alpha^{\Omega^{\rm ext}} u := -\Delta u,$$

$$\mathrm{dom}\,(-\Delta_\alpha^{\Omega^{\rm ext}}) := \{u \colon u, \nabla u, \Delta u \in L^2(\Omega^{\rm ext}), \partial_n u - \alpha u = 0 \text{ on } \partial\Omega\}.$$

Proposition

The Robin Laplacian $-\Delta_\alpha^{\Omega^{\rm ext}}$ is self-adjoint in $L^2(\Omega^{\rm ext})$.

Spectral portrait of $-\Delta_\alpha^{\Omega^{\rm ext}}$

$\lambda_1^\alpha(\Omega^{\rm ext}) := \inf \sigma(-\Delta_\alpha^{\Omega^{\rm ext}})$.

- $\sigma_{\rm cont}(-\Delta_\alpha^{\Omega^{\rm ext}}) = [0,\infty)$.
- $\lambda_1^\alpha(\Omega^{\rm ext}) \to -\infty$ as $\alpha \to -\infty$.

Proposition

(i) $d = 2$: $\lambda_1^\alpha(\Omega^{\rm ext}) < 0$ *if, and only if,* $\alpha < 0$.

(ii) $d \geq 3$: $\lambda_1^\alpha(\Omega^{\rm ext}) < 0$ *if, and only if,* $\alpha < \alpha_\star(\Omega^{\rm ext}) < 0$.

Why spectral shape optimization for $-\Delta_\alpha^{\Omega^{\rm ext}}$

- New geometric setting: optimization in unbounded domains.
- Robin BC is crucial: for Dirichlet BC the problem is meaningless.
- Interplay with continuous spectrum: optimization of novel spectral quantities like $\alpha_\star(\Omega^{\rm ext})$.

Spectral isoperimetric inequality for exterior planar domains

Theorem (Krejčiřík-VL-17, $d = 2$, $\boxed{\alpha < 0}$)

$$\frac{|\partial\Omega|}{N_\Omega} = |\partial\mathcal{B}| \quad \Longrightarrow \quad \boxed{\lambda_1^\alpha(\Omega^{\rm ext}) \le \lambda_1^\alpha(\mathcal{B}^{\rm ext})}$$

$\lambda_1^\alpha(\Omega^{\rm ext}) < \lambda_1^\alpha(\mathcal{B}^{\rm ext})$ for convex $\Omega \not\cong \mathcal{B}$ (convexity might be redundant).

For all $u_\star \in L^2(\Omega^{\rm ext})$, $u_\star \ne 0$, with $\nabla u_\star \in L^2(\Omega^{\rm ext})$

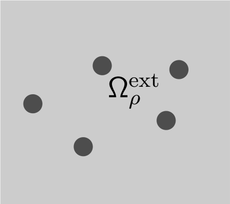
$$\lambda_1^\alpha(\Omega^{\rm ext}) \le \frac{\displaystyle\int_{\Omega^{\rm ext}} |\nabla u_\star|^2 + \alpha \int_{\partial\Omega^{\rm ext}} |u_\star|^2}{\displaystyle\int_{\Omega^{\rm ext}} |u_\star|^2} \qquad \left(\begin{array}{c}\text{The min-max}\\ \text{principle}\end{array}\right)$$

How to find $u_\star$ such that the RHS in the min-max $\le \lambda_1^\alpha(\mathcal{B}^{\rm ext})$?

- Pick the ground-state $v\colon \mathcal{B}^{\rm ext} \to (0,\infty)$ of $-\Delta_\alpha^{\mathcal{B}^{\rm ext}}$.
- $u_\star :=$ transplantation of v onto $\Omega^{\rm ext}$ via generalized polar coordinates (variable r replaced by distance from $\partial\Omega$) (Payne-Weinberger-61).

Necessity of N_Ω in the constraint

It is impossible to replace $\frac{|\partial\Omega|}{N_\Omega} = |\partial\mathcal{B}_R|$ by $|\partial\Omega| = |\partial\mathcal{B}_R|$.



Union of $N \geq 2$ disjoint disks

$\Omega_\rho = \cup_{n=1}^N \mathcal{B}_\rho(x_n)$; $|x_n - x_m| > 2\rho$, $n \neq m$

$$|\partial\Omega_\rho| = |\partial\mathcal{B}_R| \implies \rho = \frac{R}{N}$$

Strong coupling $\boxed{\alpha \to -\infty}$ (Pankrashkin-Popoff-16)

$$\lambda_1^\alpha(\Omega_\rho^{\rm ext}) - \lambda_1^\alpha(\mathcal{B}_R^{\rm ext}) = |\alpha|\left(\frac{1}{\rho} - \frac{1}{R}\right) + o(\alpha) = |\alpha|\frac{N-1}{R} + o(\alpha).$$

For sufficiently large $|\alpha|$

The inequality flips $\lambda_1^\alpha(\Omega_\rho^{\rm ext}) > \lambda_1^\alpha(\mathcal{B}_R^{\rm ext})$.

Spectral isochoric inequality for exterior planar domains

Proposition (Krejčiřík-VL-17, $d = 2$, $\boxed{\alpha < 0}$)

$$\begin{cases} |\Omega| = |\mathcal{B}| \\ N_\Omega = 1, \Omega \not\cong \mathcal{B} \end{cases} \quad \Longrightarrow \quad \boxed{\lambda_1^\alpha(\Omega^{\rm ext}) < \lambda_1^\alpha(\mathcal{B}^{\rm ext})}$$

Proof.

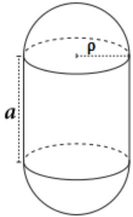
⋆ Let $\widehat{\mathcal{B}}$ be a disk such that $|\partial\Omega| = |\partial\widehat{\mathcal{B}}|$.

⋆ Then $|\widehat{\mathcal{B}}| > |\mathcal{B}|$ and $\lambda_1^\alpha(\Omega^{\rm ext}) \le \lambda_1^\alpha(\widehat{\mathcal{B}}^{\rm ext})$.

⋆ Explicit computations give $\lambda_1^\alpha(\widehat{\mathcal{B}}^{\rm ext}) < \lambda_1^\alpha(\mathcal{B}^{\rm ext})$. □

Trick fails for bounded domains: reverse monotonicity $\lambda_1^\alpha(\widehat{\mathcal{B}}) > \lambda_1^\alpha(\mathcal{B})$.

The constraint $|\partial\Omega| = |\partial\mathcal{B}|$ is "wrong" for $d \geq 3$



Long cylinder with 2 hemispherical caps

$\Omega_{\rho,a} = \mathrm{Conv}(\mathcal{B}_\rho(x_0) \cup \mathcal{B}_\rho(x_1))$, where $|x_0 - x_1| = a$.

For any $\rho < R$ exists $a > 0$ such that $|\partial\Omega_{\rho,a}| = |\partial\mathcal{B}_R|$.

Strong coupling $\boxed{\alpha \to -\infty}$

$$\lambda_1^\alpha(\Omega_{\rho,a}^{\mathrm{ext}}) - \lambda_1^\alpha(\mathcal{B}_R^{\mathrm{ext}}) = |\alpha| \left(\frac{d-2}{\rho} - \frac{d-1}{R} \right) + o(\alpha)\,.$$

$\rho < \frac{d-2}{d-1} R$ and $|\alpha|$ sufficiently large: $\lambda_1^\alpha(\Omega_{\rho,a}^{\mathrm{ext}}) > \lambda_1^\alpha(\mathcal{B}_R^{\mathrm{ext}})$.

$\Omega_\star \subset \mathbb{R}^d$, $|\partial\Omega_\star| = |\partial\mathcal{B}_R|$, exists s.t. $\lambda_1^\alpha(\Omega_\star^{\mathrm{ext}}) < \lambda_1^\alpha(\mathcal{B}_R^{\mathrm{ext}})$ for large $|\alpha|$.

For bounded case, $|\partial\Omega| = |\partial\mathcal{B}|$ is expected to be suitable under convexity.

Curvatures

$\Omega \subset \mathbb{R}^d$, $d \geq 3$ – bounded domain.

Principal curvatures of $\partial\Omega$

$\kappa_1, \kappa_2, \ldots, \kappa_{d-1} \colon \partial\Omega \to \mathbb{R}$ – non-negative for convex Ω.

The mean curvature of $\partial\Omega$

$$M := \frac{\kappa_1 + \kappa_2 + \cdots + \kappa_{d-1}}{d-1} \colon \partial\Omega \to \mathbb{R}\,.$$

Averaged $(d-1)^{\mathrm{st}}$-power of the mean curvature

$$\mathcal{M}(\partial\Omega) = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} M^{d-1}(s) \mathrm{d}\sigma(s)\,.$$

$\mathcal{M}(\partial\mathcal{B}_R) = \left(\frac{1}{R}\right)^{d-1}$.

Spectral shape optimization for $d \geq 3$

Theorem (Krejčiřík-VL-17, $d \geq 3$, $\alpha < 0$)

$$\begin{cases} \mathcal{M}(\partial\Omega) = \mathcal{M}(\partial\mathcal{B}) \\ \Omega \text{ convex} \end{cases} \quad \Longrightarrow \quad \begin{cases} \lambda_1^\alpha(\Omega^{\rm ext}) \leq \lambda_1^\alpha(\mathcal{B}^{\rm ext}) \\ \alpha_\star(\Omega^{\rm ext}) \geq \alpha_\star(\mathcal{B}^{\rm ext}) \end{cases}$$

Key points in the proof

- Common ideas with the two-dimensional case.
- Higher dimension complicates, but convexity simplifies.
- Gauss-Bonnet formula, Steiner polynomials,...
- Properties of convex bodies: Alexandrov-Fenchel inequality,...

Intermezzo: the Willmore energy

Let $\Omega \subset \mathbb{R}^3$.

$$\mathcal{W}(\partial\Omega) := \int_{\partial\Omega} M^2(s)\mathrm{d}\sigma(s) \qquad \text{(the Willmore energy)}$$

$\mathcal{M}(\partial\Omega) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} M^2(s)\mathrm{d}\sigma(s) = \frac{\mathcal{W}(\partial\Omega)}{|\partial\Omega|}$ in $\mathbb{R}^3$.

$\mathcal{W}(\partial\Omega)$ – dimensionless and measures the discrepancy from the sphere.

In elasticity: cell membrane positions itself so as to minimize $\mathcal{W}(\partial\Omega)$

Isoperimetric inequality for the Willmore energy

$\mathcal{W}(\partial\Omega) \geq \mathcal{W}(\partial\mathcal{B}) = 4\pi$.

The Willmore conjecture: F. Marques & A. Neves, *Ann. Math.* 2014

$\mathcal{W}(\partial\Omega) \geq 2\pi^2$ for $\partial\Omega$ diffeomorphic to a torus.

The Robin Laplacian on a plane with a cut

$\Sigma \subset \mathbb{R}^2$ – smooth open arc. $\mathcal{S} \subset \mathbb{R}^2$ – a line segment.

$$-\Delta_\alpha^{\mathbb{R}^2 \setminus \Sigma} u := -\Delta u,$$

$$\mathrm{dom}\left(-\Delta_\alpha^{\mathbb{R}^2 \setminus \Sigma}\right) := \left\{u \colon u, \nabla u, \Delta u \in L^2(\mathbb{R}^2 \setminus \Sigma), \partial_{n_\pm} u = \alpha u \text{ on } \Sigma_\pm\right\}.$$

Basic spectral properties

$\sigma_{\mathrm{cont}}(-\Delta_\alpha^{\mathbb{R}^2 \setminus \Sigma}) = [0, \infty)$ and $\lambda_1^\alpha(\mathbb{R}^2 \setminus \Sigma) := \inf \sigma(-\Delta_\alpha^{\mathbb{R}^2 \setminus \Sigma}) < 0$, $\forall \alpha < 0$.

Spectral isoperimetric inequality for the plane with a cut

Theorem (VL-16, $d = 2$, $\boxed{\alpha < 0}$)

$$\begin{cases} |\Sigma| = |\mathcal{S}| \\ \Sigma \not\cong \mathcal{S} \end{cases} \qquad \Longrightarrow \qquad \boxed{\lambda_1^\alpha(\mathbb{R}^2 \setminus \Sigma) < \lambda_1^\alpha(\mathbb{R}^2 \setminus \mathcal{S})}$$

Key tools for the proof

- Min-max principle.
- Reduction to integral operators in $L^2(\Sigma)$ and $L^2(\mathcal{S})$.
- Line segment is the shortest path connecting two endpoints.

In the two-dimensional setting ($d = 2$, $\alpha < 0$)

For connected Ω, the inequality $\lambda_1^\alpha(\Omega^{\rm ext}) \le \lambda_1^\alpha(\mathcal{B}^{\rm ext})$ holds if

$$\texttt{length of } \partial\Omega = \texttt{length of } \partial\mathcal{B} \quad \text{or} \quad \texttt{area of } \Omega = \texttt{area of } \mathcal{B}$$

For possibly disconnected Ω, $\lambda_1^\alpha(\Omega^{\rm ext}) \le \lambda_1^\alpha(\mathcal{B}^{\rm ext})$ holds if

$$\frac{\texttt{length of } \partial\Omega}{\texttt{number of components in } \Omega} = \texttt{length of } \partial\mathcal{B}$$

For an arc Σ & a line segment $\mathcal{S}$, $\lambda_1^\alpha(\mathbb{R}^2 \setminus \Sigma) \le \lambda_1^\alpha(\mathbb{R}^2 \setminus \mathcal{S})$ holds if

$$\texttt{length of } \Sigma = \texttt{length of } \mathcal{S}$$

Open direction

Results for higher eigenvalues are missing.

Higher dimensions ($d \geq 3$, $\alpha < 0$)

The constraint $|\partial\Omega| = |\partial\mathcal{B}|$ is "wrong" as a counterexample shows.

For convex Ω, $\lambda_1^\alpha(\Omega^{\rm ext}) \leq \lambda_1^\alpha(\mathcal{B}^{\rm ext})$ & $\alpha_\star(\Omega^{\rm ext}) \geq \alpha_\star(\mathcal{B}^{\rm ext})$ hold if

$$\frac{\texttt{Willmore-type energy of } \partial\Omega}{\texttt{the area of } \partial\Omega} = \frac{\texttt{Willmore-type energy of } \partial\mathcal{B}}{\texttt{the area of } \partial\mathcal{B}}$$

Open problem

Is the result still true for (a class of) non-convex Ω?

Thank you

- D. Krejčiřík and V. L., Optimisation of the lowest Robin eigenvalue in the exterior of a compact set, to appear in J. Convex Anal., arXiv:1608.04896.
- D. Krejčiřík and V. L., Optimisation of the lowest Robin eigenvalue in the exterior of a compact set, II: non-convex domains and higher dimensions, arXiv:1707.02269.
- V. L., Spectral isoperimetric inequalities for δ-interactions on open arcs and for the Robin Laplacian on planes with slits, arXiv:1609.07598.

Thank you for your attention!