On the CLT for spectral statistics of Wigner and sample covariance random matrices

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Why random matrices?

Mathematics:

- Statistics
- Combinatorics
- Topology
- Probability
- Functional Analysis
- Integrable systems

Physics:

- Nuclear Physics
- Quantum Chaology
- Quantum Field Theory
- Condensed Matter
- Statistical Physics
- Wave propagation

- Structural Mechanics
- Telecommunications
- Quantitative Finances
- Quantum Information Theory

Content

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- An analog of the Law of Large Numbers. The Wigner semicircle law.

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- Linear eigenvalue statistics.
- An analog of the Law of Large Numbers. The Wigner semicircle law.
- An analog of the Central Limit Theorem.
- An example contradicting universality.

Wigner real symmetric matrices

$$M_n = n^{-1/2} W_n$$

- $W_n = \{W_{jk}\}_{j,k=1}^{n}$, $W_{jk} = W_{kj} \in \mathbb{R}$,
- W_{jk}, $1 \le j \le k \le n$, are independent,
- $\mathbf{E} W_{jk} = 0$, $\mathbf{E} W_{jk}^2 = w^2(1 + a^2 \delta_{jk})$.

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In particular, if all entries of M_n are independent Gaussian random variables,

$$W_{jk} \sim N(0, 1 + \delta_{jk}), \quad 1 \le j \le k \le n,$$

then we call M_n the Gaussian Orthogonal Ensemble (GOE).

Sample Covariance Matrices

Consider m independent samples of n observables

$$X_1 = \begin{pmatrix} X_{11} \\ \vdots \\ X_{n1} \end{pmatrix}, \quad \ldots, X_m = \begin{pmatrix} X_{1m} \\ \vdots \\ X_{nm} \end{pmatrix}$$

and construct an $n \times m$ matrix

$$X = \begin{bmatrix} X_1 & X_2 & \ldots & X_m \end{bmatrix}.$$

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We suppose that

- $\mathbf{E}\{X_{j\alpha}\} = 0,\ \mathbf{E}\{X_{j\alpha}^2\} = 1,\ j \leq n,\ \alpha \leq m,$
- $m = m_n :\ m_n/n \to c \in (0, \infty), \quad n \to \infty.$

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$$\mathcal{N}_n[\varphi] := \sum_{j=1}^n \varphi(\lambda_j) = \operatorname{Tr} \varphi(M_n).$$

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Important examples of LES:

- Counting Measure of eigenvalues $\mathcal{N}_n(\Delta)$ corresponds to

$$\varphi(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \Delta, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathcal{N}_n[\varphi] = \int_{\mathbb{R}} \varphi(\lambda) \mathcal{N}_n(d\lambda)$.

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- **Stieltjes transform** of $\mathcal{N}_n(\Delta)$ corresponds to $\varphi(\lambda) = (\lambda - z)^{-1}$:

$$\operatorname{Tr}(M_n - zI)^{-1} = \int_{\mathbb{R}} \frac{\mathcal{N}_n(d\lambda)}{\lambda - z}, \quad \operatorname{Im} z \neq 0.$$

Stieltjes transform of a non-negative finite measure *m*:

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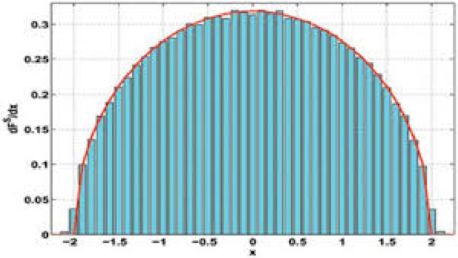
- $$s_n(z) := \int_{\mathbb{R}} \frac{\mathcal{N}_n(d\lambda)}{\lambda - z} = \mathrm{Tr}(M_n - zI)^{-1}, \quad \mathrm{Im}\, z \neq 0.$$

Wigner's Semicircle Law

For any bounded continuous function φ, with probability 1,

$$\lim_{n\to\infty} n^{-1}\sum_{\ell=1}^{n}\varphi(\lambda_\ell) = \lim_{n\to\infty}\int_{\mathbb{R}}\varphi(\lambda)dN_n(\lambda) = \int_{-2w}^{2w}\varphi(\lambda)\rho_{scl}(\lambda)d\lambda,$$

$$\rho_{scl}(\lambda) = \frac{1}{2\pi w^2}\sqrt{(4w^2-\lambda^2)_+}.$$



Marchenko-Pastur distribution

Let $\quad M_n = n^{-1} B_n B_n^T, \quad B_n = \{X_{j\alpha}\}_{j,\alpha=1}^{n,m},$

$\{X_{j\alpha}\}_{j,\alpha}$ are independent,

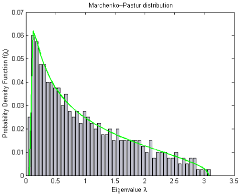
$\mathbf{E} X_{j\alpha} = 0, \quad \mathbf{E} X_{j\alpha}^2 = a^2,$

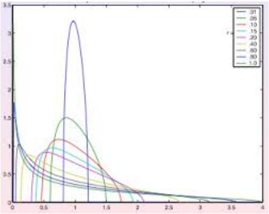
$m, n \to \infty, \; m/n \to c \geq 1.$

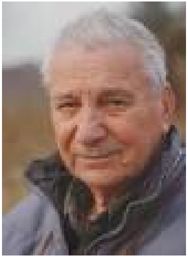
Then $N_n(d\lambda) \to \rho_{MP}(\lambda) d\lambda$ a.s.,

$$\rho_{MP}(\lambda) = \frac{\sqrt{((\lambda - a_-)(a_+ - \lambda))_+}}{2\pi a^2 \lambda},$$

$$a_\pm = a^2(\sqrt{c} \pm 1)^2.$$







Vladimir Marchenko

Leonid Pastur

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What can be said about fluctuations?

$$? : \ \nu_n(\mathcal{N}_n[\varphi] - \mathbb{E}\mathcal{N}_n[\varphi]) \underset{n\to\infty}{\longrightarrow} \mathcal{N}(0, V) \ \textit{in distribution}$$

Variance of linear eigenvalue statistic $\mathcal{N}_n[\varphi] = \sum_{j=1}^n \varphi(\lambda_j)$

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} = \mathbf{E}\{(\mathcal{N}_n^\circ[\varphi])^2\}, \quad \mathcal{N}_n^\circ[\varphi] = \mathcal{N}_n[\varphi] - \mathbf{E}\{\mathcal{N}_n[\varphi]\},$$

For $M \in$GOE / Wigner ensemble / Sample Covariance matrices

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} = O(1), \quad n \to \infty,$$

provided that φ is smooth enough.

The typical size of fluctuations depends on the smoothness of the test-function!

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Example. If

$$\varphi(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \Delta, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$\mathbf{Var}\{\mathcal{N}_n(\Delta)\} = \frac{1}{\pi^2} \ln n + O(1), \quad n \to \infty.$$

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So, for smooth functions φ CLT, if any, is valid for the centered linear eigenvalue statistic $\mathcal{N}_n^\circ[\varphi]$ itself without any normalization constant in front.

The CLT for Linear Eigenvalue Statistics for GOE

Theorem.
Let $\widehat{M}_n = n^{-1/2}\widehat{W}_n$ be the GOE,

$$\widehat{W}_{jk} \sim N(0, 1+\delta_{jk}), \quad j \le k, \qquad \text{are independent.}$$

Let $\mathcal{N}_n[\varphi]$ be the linear eigenvalue statistic corresponding to a bounded test function φ with bounded derivative. Then $\mathcal{N}_n^\circ[\varphi]$ converges in distribution to the Gaussian random variable with zero mean and the variance

$$V_{GOE}[\varphi] = \frac{1}{2\pi^2}\int_{-2}^{2}\int_{-2}^{2}\left(\frac{\varphi(\lambda_1)-\varphi(\lambda_2)}{\lambda_1-\lambda_2}\right)^2 \frac{(4-\lambda_1\lambda_2)d\lambda_1 d\lambda_2}{\sqrt{4-\lambda_1^2}\sqrt{4-\lambda_2^2}}.$$

The CLT for Linear Eigenvalue Statistics for Wigner Random Matrices

Theorem (AL, Pastur'09). Let $M_n = n^{-1/2} W_n$ be a Wigner matrix:

- $W_{jk} = W_{kj} \in \mathbb{R}$,
- W_{jk}, $j \le k$, are independent,
- $\mathbf{E}\{W_{jk}\} = 0, \quad \mathbf{E}\{W_{jk}^2\} = (1 + \delta_{jk})$,
- the fifth absolute moments of matrix entries are uniformly bounded, the third and the fourth moments, $\mu_{3,4} = \mathbf{E}\{W_{jk}^{3,4}\}$, do not depend on j, k, n when $j \neq k$.

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Suppose also that

$$\varphi : \mathbb{R} \to \mathbb{R} : \quad \int (1 + |t|^5) |\mathcal{F}[\varphi](t)| dt < \infty.$$

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Then $\mathcal{N}_n^\circ[\varphi]$ converges in distribution to the Gaussian random variable with zero mean and the variance

$$V_{Wig}[\varphi] = V_{GOE}[\varphi] + \frac{\kappa_4}{2\pi^2}\left(\int_{-2}^{2} \varphi(\mu) \frac{2 - \mu^2}{\sqrt{4 - \mu^2}} d\mu\right)^2,$$

where $\kappa_4 = \mu_4 - 3$.

Limiting probability law of fluctuations of $\sqrt{n}\varphi_{jj}(M_n)$

E. Borel (1906): *Let $X_{1,n}$ denote the first coordinate of X_n, an n-dimensional random vector that is uniformly distributed on the unit sphere S^{n-1}; then, as $n \to \infty$ the random variables $\sqrt{n}X_{1,n}$ converge in distribution to a standard normal random variable.*

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We have

$$\varphi_{jj}(M_n) = \sum_{\ell=1}^{n} \varphi(\lambda_\ell)|\psi_\ell \cdot e_j|^2,$$

where $\{\psi_\ell\}_\ell$ are unit eigenvectors and $\{e_j\}_j$ are unit coordinate vectors.

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$$\varphi_{jj}(M_n) \approx n^{-1} \sum_{\ell=1}^{n} \varphi(\lambda_\ell) = n^{-1}\mathcal{N}_n[\varphi].$$

$\Rightarrow$ one could expect that the asymptotic behaviors of $\varphi_{jj}(M_n)$ and $n^{-1}\mathcal{N}_n[\varphi]$ are the same.

Limiting probability law of fluctuations of $\sqrt{n}\varphi_{jj}(M_n)$

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$$V^{GOE}_{m.el.}[\varphi] = \int_{-2}^{2}\int_{-2}^{2}(\varphi(\lambda_1)-\varphi(\lambda_2))^2\rho_{scl}(\lambda_1)\rho_{scl}(\lambda_2)d\lambda_1 d\lambda_2.$$

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Assume

- $M_n = n^{-1/2}\{W_{jk}\}_{j,k=1}^n$, $W_{jk} = W_{kj} \in \mathbb{R}$ are i.i.d.,
- $\mathbf{E}\{W_{11}\} = 0, \quad \mathbf{E}\{W_{11}^2\} = 1$,
- $f(x) := \mathbf{E}\{e^{ixW_{11}}\}$: $\ln f(z)$ is an entire function,
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Then $\sqrt{n}\varphi_{jj}^{\circ}(M_n)$ converges in distribution to a random variable ξ such that $\forall x \in \mathbb{R}$

$$\mathbf{E}\{e^{ix\xi}\} = \exp\{-x^2 V_{m.el.}^W[\varphi]/2 + x^{*2}\} \cdot f(x^*),$$

where $x^* = x\int_{-2}^2 \varphi(\mu)\mu\rho_{scl}(\mu)d\mu$, and

$$V_{m.el.}^W[\varphi] = V_{m.el.}^{GOE}[\varphi] + \kappa_4\left|\int_{-2}^2 \varphi(\mu)(1-\mu^2)\rho_{scl}(\mu)d\mu\right|^2.$$

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- $f(x) := \mathbf{E}\{e^{ixW_{11}}\}$: $\ln f(z)$ is an entire function,
- $\int(1+|t|^3)|\mathcal{F}[\varphi](t)|dt < \infty.$

Then $\sqrt{n}\varphi_{jj}^{\circ}(M_n)$ converges in distribution to a random variable ξ such that $\forall x \in \mathbb{R}$

$$\mathbf{E}\{e^{ix\xi}\} = \exp\{-x^2 V_{m.el.}^W[\varphi]/2 + x^{*2}\} \cdot f(x^*),$$

where $x^* = x\int_{-2}^2 \varphi(\mu)\mu\rho_{scl}(\mu)d\mu$, and

$$V_{m.el.}^W[\varphi] = V_{m.el.}^{GOE}[\varphi] + \kappa_4\Big|\int_{-2}^2 \varphi(\mu)(1-\mu^2)\rho_{scl}(\mu)d\mu\Big|^2.$$

Pizzo, A., Renfrew, D., Soshnikov, A. (2011). Fluctuations of matrix entries of regular functions of Wigner matrices. *Journal of Statistical Physics*, 146(3), 550-591.

Sample Covariance Matrices

All results concerning the CLT for linear eigenvalue statistics and limiting probability law for the fluctuations of $\sqrt{n}\varphi_{jj}(M_n)$ remain valid (with corresponding modifications) for the Sample Covariance Matrix

$$M = n^{-1}XX^T, \quad X = \begin{bmatrix} X_1 & X_2 & \dots & X_m \end{bmatrix},$$

where $m/n \to c \in (0, \infty), \quad n \to \infty$.

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O. Guedon, A. Lytova, A. Pajor, and L. Pastur, *The Central Limit Theorem for linear eigenvalue statistics of the sum of rank one projections on independent vectors*. Spectral Theory and Differential Equations. V. A. Marchenko 90th Anniversary Collection

Ideas of the proof. CLT for GOE.

Theorem.
Let $\widehat{M}_n = n^{-1/2}\widehat{W}_n$ be the GOE,

$$\widehat{W}_{jk} \sim N(0, 1+\delta_{jk}), \quad j \le k, \qquad \text{are independent.}$$

Let $\mathcal{N}_n[\varphi]$ be the linear eigenvalue statistic corresponding to a bounded test function φ with bounded derivative. Then $\mathcal{N}_n^\circ[\varphi]$ converges in distribution to the Gaussian random variable with zero mean and variance

$$V_{GOE}[\varphi] = \frac{1}{2\pi^2}\int_{-2}^{2}\int_{-2}^{2}\left(\frac{\varphi(\lambda_1) - \varphi(\lambda_2}{\lambda_1 - \lambda_2}\right)^2 \frac{(4-\lambda_1\lambda_2)d\lambda_1 d\lambda_2}{\sqrt{4-\lambda_1^2}\sqrt{4-\lambda_2^2}}.$$

Ideas of the proof.

We show that if $Z_n(x) = \mathbf{E}\big\{ \exp\{ix\mathcal{N}_n^\circ[\varphi]\}\big\}$, then for any $x \in \mathbb{R}$

$$\lim_{n\to\infty} Z_n(x) = Z(x), \quad \lim_{n\to\infty} Z_n'(x) = -xV_{GOE}Z(x).$$

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Proposition

Let $\xi = \{\xi_\ell\}_{\ell=1}^p$ be independent Gaussian random variables of zero mean, and $\Phi : \mathbb{R}^p \to \mathbb{C}$ be a differentiable function with polynomially bounded partial derivatives Φ_ℓ', $\ell = 1, ..., p$. Then we have

$$\mathbf{E}\{\xi_\ell \Phi(\xi)\} = \mathbf{E}\{\xi_\ell^2\}\mathbf{E}\{\Phi_\ell'(\xi)\}, \ \ell = 1, ..., p,$$

and

$$\mathbf{Var}\{\Phi(\xi)\} \le \sum_{\ell=1}^p \mathbf{E}\{\xi_\ell^2\}\mathbf{E}\left\{|\Phi_\ell'(\xi)|^2\right\}.$$

The first formula is a version of the integration by parts. The second is a version of the Poincaré inequality.

Ideas of the proof. CLT for Wigner Matrices, $\mu_4 = 3$.

Theorem (AL, Pastur'09). Let $M_n = n^{-1/2} W_n$ be a Wigner matrix:

- $W_{jk} = W_{kj} \in \mathbb{R}$,
- W_{jk}, $j \le k$, are independent,
- $\mathbf{E}\{W_{jk}\} = 0, \quad \mathbf{E}\{W_{jk}^2\} = (1 + \delta_{jk})$,
- the fifth absolute moments of matrix entries are uniformly bounded, the third and the fourth moments, $\mu_{3,4} = \mathbf{E}\{W_{jk}^{3,4}\}$, do not depend on j, k, n when $j \neq k$.
- $\mathbf{E}\{W_{jk}^4\} = 3$, $j \neq k$.

Suppose also that

$$\varphi : \mathbb{R} \to \mathbb{R} : \quad \int (1 + |t|^5)|F[\varphi](t)|dt < \infty.$$

Then $\mathcal{N}_n^\circ[\varphi]$ converges in distribution to the Gaussian random variable with zero mean and variance $V_{GOE}[\varphi]$.

Ideas of the proof. An interpolation trick.

Proposition.(Khoruzhenko, Khorunzhy, Pastur, 1995)

If $\mathbf{E}\{|\xi|^{p+2}\} < \infty$ and $\Phi \in C^{p+1}$ with bounded partial derivatives, then

$$\mathbf{E}\{\xi\Phi(\xi)\} = \sum_{\ell=0}^{p} \frac{\kappa_{\ell+1}}{\ell!}\mathbf{E}\{\Phi^{(\ell)}(\xi)\} + \varepsilon_p,$$

$$|\varepsilon_p| \le C_p \mathbf{E}\{|\xi|^{p+2}\} \sup_{t\in\mathbb{R}} |\Phi^{(p+1)}(t)|.$$

An interpolation matrix: $M(s) = s^{1/2}M + (1-s)^{1/2}\widehat{M}, \quad 0 \le s \le 1.$

Here M and $\widehat{M}$ are independent Wigner and GOE matrices with equal moments up to the fourth order.

$$\mathbf{E}\Big\{e^{ix\,\mathrm{Tr}\;\varphi(M)^\circ}\Big\} - \mathbf{E}\Big\{e^{ix\,\mathrm{Tr}\;\varphi(\widehat{M})^\circ}\Big\} = \int_0^1 \frac{\partial}{\partial s}\mathbf{E}\left\{e^{ix\,\mathrm{Tr}\;\varphi(M(s))^\circ}\right\} ds$$

$$= \frac{ix}{2}\int_0^1 \Big\{\frac{1}{\sqrt{ns}}\sum_{j,k=1}^{n}\mathbf{E}\{\Phi(M(s))W_{jk}\} - \frac{1}{\sqrt{n(1-s)}}\sum_{j,k=1}^{n}\mathbf{E}\{\Phi(M(s))\widehat{W}_{jk}\}\Big\}ds$$

$$= O(n^{-1/2}).$$

Opole, Poland

Thank you!