# Deformations of infinite-dimensional Lie algebras, exotic cohomology, and integrable nonlinear partial differential equations

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# Integrable differential equations

Integrable nonlinear partial differential equations:
Lax representations (zero-curvature representations, Wahlquist–Estabrook prolongation structures, integrable extensions, IST, differential coverings...):

- soliton solutions
- Bäcklund transformations
- nonlocal symmetries and nonlocal conservation laws
- recursion operators
- Darboux transformations
- ...

UNSOLVED PROBLEM: to find conditions of existence of a Lax representation for a given PDE.

# Lax representations

A Lax representation for a differential equation $\mathcal{E}$

$$F(x^i, u, u_{x^i}, u_{x^i x^j}, \ldots) = 0, \quad i, j, k \in \{1, \ldots, n\},$$

is defined by the over-determined system

$$w_{a,x^k} = T_{ak}(x^i, u, u_{x^i}, u_{x^i x^j}, \ldots, w_b), \qquad a, b \in \mathbb{N},$$

whose compatibility conditions coincide with $\mathcal{E}$.

Geometrically: a flat connection on $\mathcal{E} \times \mathcal{W} \to \mathcal{E}$,

$$D_{x^k} \mapsto \widetilde{D}_{x^k} = D_{x^k} + \sum_a T_{ak}(x^i, u, u_{x^i}, u_{x^i x^j}, \ldots, w_b) \frac{\partial}{\partial w_a}$$

such that $[\widetilde{D}_{x^i}, \widetilde{D}_{x^j}] = 0 \quad \Leftrightarrow \quad F(x^i, u, u_{x^i}, u_{x^i x^j}, \ldots) = 0,$

or by the ideal of the Wahlquist–Estabrook forms

$$\tau_a = dw_a - \sum_k T_{ak}(x^i, u, u_{x^i}, u_{x^i x^j}, \ldots, w_b)\, dx^k$$

such that

$$d\tau_a \equiv \sum_b \eta_{ab} \wedge \tau_b \mod \langle \text{contact forms on } \mathcal{E} \rangle.$$

# Lax representations

Definitions, details, examples:

- H.D. Wahlquist, F.B. Estabrook, J. Math. Phys., 1975, Vol. 16, 1–7.
- ...
- I.S. Krasil′shchik, A.M. Vinogradov, Acta Appl. Math., 1984, Vol. 2, 79–86, Acta Appl. Math., 1989, Vol. 15, 161–209.
- Krasil′shchik I.S., Lychagin V.V., Vinogradov A.M. Geometry of jet spaces and nonlinear partial differential equations. N.Y.: Gordon and Breach, 1986
- ...

# Exotic cohomology

Exotic (deformed, twisted, covariant, ...) cohomology: geometry of Poisson manifolds, Morse theory of multi-valued functionals, symplectic geometry, algebraic topology, theory of Lie algebras.

- A. Lichnerowicz, 1977,
- E. Witten 1982,
- M. Atiyah, R. Bott, 1984
- S.P. Novikov, 1986, 2002, 2005
- ...

# Exotic cohomology of Lie algebras

$\mathfrak{g}$ – Lie algebra over $\mathbb{R}$, $\rho\colon \mathfrak{g} \to \mathrm{End}(V)$ – its representation,

$C^k(\mathfrak{g}, V) = \mathrm{Hom}(\Lambda^k(\mathfrak{g}), V), \quad k \geq 1,$

$$d\theta(X_1, ..., X_{k+1}) = \sum_{q=1}^{k+1} (-1)^{q+1} \rho(X_q) \left(\theta(X_1, ..., \hat{X}_q, ..., X_{k+1})\right)$$
$$+ \sum_{1 \leq p < q \leq k+1} (-1)^{p+q} \theta([X_p, X_q], X_1, ..., \hat{X}_p, ..., \hat{X}_q, ..., X_{k+1}),$$

the Chevalley-Eilenberg differential complex

$$\dots \xrightarrow{d} C^k(\mathfrak{g}, V) \xrightarrow{d} C^{k+1}(\mathfrak{g}, V) \xrightarrow{d} \dots$$

Its cohomology

$$H^k(\mathfrak{g}, V) = \frac{\ker d\colon C^k(\mathfrak{g}, V) \longrightarrow C^{k+1}(\mathfrak{g}, V)}{\mathrm{im}\, d\colon C^{k-1}(\mathfrak{g}, V) \longrightarrow C^k(\mathfrak{g}, V)},$$

$\rho_0\colon X \mapsto 0 \Rightarrow$ cohomology with trivial coefficients $H^*(\mathfrak{g})$.

# Exotic cohomology of Lie algebras

Suppose $H^1(\mathfrak{g}) \neq \{0\}$, take $\omega$ such that $[\omega] \in H^1(\mathfrak{g})$. For $\lambda \in \mathbb{R}$ define

$$d_{\lambda\omega}\theta = d\theta + \lambda\,\omega \wedge \theta$$

From $d\omega = 0$ it follows that $d_{\lambda\omega}^2 = 0$, therefore the differential complex

$$\dots \xrightarrow{d_{\lambda\omega}} C^k(\mathfrak{g}, \mathbb{R}) \xrightarrow{d_{\lambda\omega}} C^{k+1}(\mathfrak{g}, \mathbb{R}) \xrightarrow{d_{\lambda\omega}} \dots$$

is defined. Its cohomology groups $H^*_{\lambda\omega}(\mathfrak{g})$ are referred to as exotic cohomology groups of $\mathfrak{g}$.

REMARK. $H^*_{\lambda\omega}(\mathfrak{g})$ coincide with the cohomology of $\mathfrak{g}$ with coefficients in the one-dimensional representation $\rho_{\lambda\omega} \colon \mathfrak{g} \to \mathbb{R}$, $\rho_{\lambda\omega} \colon X \mapsto \lambda\,\omega(X)$.

# Exotic cohomology of Lie algebras

EXAMPLE. Consider a Lie algebra $\mathfrak{h}$ with generators $X_0$, ... , $X_3$ and non-zero commutators

$$\begin{aligned}
[X_0, X_1] &= \quad -X_1, \\
[X_0, X_2] &= -2\, X_2, \\
[X_0, X_3] &= -3\, X_3, \\
[X_1, X_2] &= \quad -X_3.
\end{aligned}$$

For the dual 1-forms (Maurer–Cartan forms) $\omega_i$ such that $\omega_i(X_j) = \delta_{ij}$ the structure equations hold:

$$\begin{aligned}
d\omega_0 &= 0, \\
d\omega_1 &= \quad \omega_0 \wedge \omega_1, \\
d\omega_2 &= 2\, \omega_0 \wedge \omega_2, \\
d\omega_3 &= 3\, \omega_0 \wedge \omega_3 + \omega_1 \wedge \omega_2.
\end{aligned}$$

$H^1(\mathfrak{h}) = \mathbb{R}\,[\omega_0] = \mathbb{R}\,\omega_0.$

$H^2(\mathfrak{h}) = \{0\} \;\Rightarrow \mathfrak{h}$ has no central extensions.

# Exotic cohomology of Lie algebras

We have

$$H^2_{\lambda\,\omega_0}(\mathfrak{h}) = \begin{cases} \mathbb{R}\,[\omega_1 \wedge \omega_3], & \lambda = -4, \\ \mathbb{R}\,[\omega_2 \wedge \omega_3], & \lambda = -5, \\ \{0\}, & \lambda \notin \{-5, -4\} \end{cases}$$

From

$$d_{-4\,\omega_0}\,(\omega_1 \wedge \omega_3) = 0$$

it follows that equation

$$d_{-4\,\omega_0}\,\omega_4 = \omega_1 \wedge \omega_3$$

with unknown 1-form $\omega_4$ is compatible with the structure equations of $\mathfrak{h}$ $\Rightarrow$ two additional structure equations

$$d\omega_4 = 4\,\omega_0 \wedge \omega_4 + \omega_1 \wedge \omega_3,$$

$$d\omega_5 = 5\,\omega_0 \wedge \omega_5 + \omega_2 \wedge \omega_3,$$

$\Rightarrow$ two-dimensional extension $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \langle X_4, X_5 \rangle$,

$$[X_0, X_4] = -4\,X_4, \quad [X_1, X_3] = -X_4,$$

$$[X_0, X_5] = -5\,X_5, \quad [X_2, X_3] = -X_5.$$

# Cartan's method of equivalence & symmetries of DEs

The main idea: to apply the above trick to the structure equations of the contact symmetry pseudo-group of the PDE under the study.

For a given PDE $\mathcal{E}$ the Maurer–Cartan forms and the structure equations of the symmetry pseudo-gropu $Sym(\mathcal{E})$ can be found algorithmically (using derivation and operations of linear algebra) by means of É. Cartan's method of equivalence.


- É. Cartan. Œuvres Complètes. Paris: Gauthier - Villars, 1953
- P.J. Olver. Equivalence, invariants, and symmetry. Cambridge: CUP, 1995
- M. Fels, P.J. Olver. Acta Appl. Math., 1998, Vol. 51, 161–213
- O.M. J. Phys. A: Math. Gen., 2002, Vol. 35, 2965–2977
- O.M. J. Math. Sci., 2006, Vol. 135, 2680–2694

# Lax representations & exotic cohomology

EXAMPLE. The potential Khokhlov–Zabolotskaya equation

$$u_{yy} = u_{tx} + u_x\, u_{xx} \qquad (\mathcal{E})$$

The structure equations of $Sym(\mathcal{E})$:

$$\begin{aligned}
d\alpha &= 0,\\
d\Theta_0 &= \nabla\Theta_0 \wedge \Theta_0,\\
d\Theta_1 &= \alpha \wedge \Theta_1 + \nabla\Theta_1 \wedge \Theta_0 + \tfrac{2}{3}\nabla\Theta_0 \wedge \Theta_1,\\
d\Theta_2 &= 2\,\alpha \wedge \Theta_2 + \nabla\Theta_2 \wedge \Theta_0 + \tfrac{2}{3}\nabla\Theta_1 \wedge \Theta_1 + \tfrac{1}{3}\nabla\Theta_0 \wedge \Theta_2,\\
d\Theta_3 &= 3\,\alpha \wedge \Theta_3 + \nabla\Theta_3 \wedge \Theta_0 + \tfrac{2}{3}\nabla\Theta_2 \wedge \Theta_1 + \tfrac{1}{3}\nabla\Theta_1 \wedge \Theta_2,\\
d\Theta_4 &= 4\,\alpha \wedge \Theta_4 + \nabla\Theta_4 \wedge \Theta_0 + \tfrac{2}{3}\nabla\Theta_3 \wedge \Theta_1 + \tfrac{1}{3}\nabla\Theta_2 \wedge \Theta_2 - \tfrac{1}{3}\nabla\Theta_0 \wedge \Theta_4,
\end{aligned}$$

where $\Theta_m = \sum\limits_{n=0}^{\infty} \dfrac{h^n}{n!}\,\theta_{m,n}, \quad 0 \le m \le 4,$

$$\nabla\Theta_m = \frac{\partial}{\partial h}\Theta_m = \sum_{n=0}^{\infty} \frac{h^n}{n!}\,\theta_{m,n+1}, \quad dh = 0, \quad \theta_{3,0} = 0.$$

# Lax representations & exotic cohomology

The Maurer–Cartan forms:

$$\alpha = p^{-1}\,dp,$$

$$\theta_{0,0} = q\,dt,$$

$$\theta_{1,0} = p\,q^{2/3}\,(dy + a_1\,dt),$$

$$\theta_{2,0} = p^2\,q^{1/3}\,\big(dx + \tfrac{2}{3}\,a_1\,dy + a_2\,dt\big),$$

$$\theta_{4,0} = p^4\,q^{-1/3}\,(du - u_t\,dt - u_x\,dx - u_y\,dy).$$

REMARK: All the other Maurer–Cartan forms can be found inductively by integration:

$$d\theta_{0,0} = \theta_{0,1} \wedge \theta_{0,0} \quad \Rightarrow \quad dq \wedge dt = \theta_{0,1} \wedge q\,dt \quad \Rightarrow \quad \theta_{0,1} = \frac{dq}{q} + b_1\,dt,$$

$$d\theta_{0,1} = \theta_{0,2} \wedge \theta_{0,0} \quad \Rightarrow \quad \theta_{0,2} = db_1 + b_2\,dt,$$

$$d\theta_{2,0} = \theta_{0,3} \wedge \theta_{0,0} - \theta_{0,1} \wedge \theta_{0,2} \quad \Rightarrow \quad \theta_{0,3} = \ldots,$$

$$\ldots \ \ldots$$

# Lax representations & exotic cohomology

THEOREM.

$$H^1(Sym(\mathcal{E})) = \mathbb{R}\,\alpha,$$

$$H^2_{\lambda\,\alpha}(Sym(\mathcal{E})) = \begin{cases} \mathbb{R}\,[\Omega], & \lambda = -3, \\ \{0\}, & \lambda \neq -3, \end{cases}$$

where

$$\Omega = \theta_{3,1} \wedge \theta_{0,0} + \tfrac{2}{3}\,\theta_{2,1} \wedge \theta_{1,0} + \tfrac{1}{3}\,\theta_{1,1} \wedge \theta_{2,0}.$$

COROLLARY. Equation

$$d\omega = 3\,\alpha \wedge \omega + \Omega$$

is compatible with the structure equations of $Sym(\mathcal{E})$.

REMARK. Denote $\omega = \theta_{3,0} \quad \Rightarrow \quad \Theta_3 = \sum_{j=0}^{\infty} \frac{h^j}{j!}\,\theta_{3,j}.$

# Lax representations & exotic cohomology

Integrate

$$d\theta_{3,0} = 3\,\alpha \wedge \theta_{3,0} + \theta_{3,1} \wedge \theta_{0,0} + \tfrac{2}{3}\,\theta_{2,1} \wedge \theta_{1,0} + \tfrac{1}{3}\,\theta_{1,1} \wedge \theta_{2,0}$$

$\Rightarrow$

$$\theta_{3,0} = p^3\,\left(dv - v_x\,dx - \left(\tfrac{1}{3}\,v_x^3 - u_x\,v_x - u_y\right)\,dt - \left(\tfrac{1}{2}\,v_x^2 - u_x\right)\,dy\right)$$

the Wahlquist–Estabrook form of the Lax representation

$$\begin{cases} v_t &= \tfrac{1}{3}\,v_x^3 - u_x\,v_x - u_y, \\ v_y &= \tfrac{1}{2}\,v_x^2 - u_x \end{cases}$$

for the potential KhZ equation (G.M. Kuz′mina, 1967; J. Gibbons, 1988; I.M. Krichever, 1988).

# Lax representations & exotic cohomology

REMARK. The structure equations for $Sym(\mathcal{E})$:

$$
\begin{aligned}
d\alpha &= 0,\\
d\Theta_0 &= \nabla\Theta_0 \wedge \Theta_0,\\
d\Theta_1 &= \alpha \wedge \Theta_1 + \nabla\Theta_1 \wedge \Theta_0 + \tfrac{2}{3}\nabla\Theta_0 \wedge \Theta_1,\\
d\Theta_2 &= 2\,\alpha \wedge \Theta_2 + \nabla\Theta_2 \wedge \Theta_0 + \tfrac{2}{3}\nabla\Theta_1 \wedge \Theta_1 + \tfrac{1}{3}\nabla\Theta_0 \wedge \Theta_2,\\
d\Theta_3 &= 3\,\alpha \wedge \Theta_3 + \nabla\Theta_3 \wedge \Theta_0 + \tfrac{2}{3}\nabla\Theta_2 \wedge \Theta_1 + \tfrac{1}{3}\nabla\Theta_1 \wedge \Theta_2,\\
d\Theta_4 &= 4\,\alpha \wedge \Theta_4 + \nabla\Theta_4 \wedge \Theta_0 + \tfrac{2}{3}\nabla\Theta_3 \wedge \Theta_1 + \tfrac{1}{3}\nabla\Theta_2 \wedge \Theta_2 - \tfrac{1}{3}\nabla\Theta_0 \wedge \Theta_4,
\end{aligned}
$$

can be written in the form

$$
\begin{aligned}
d\alpha &= 0,\\
d\Theta_k &= k\,\alpha \wedge \Theta_k + \sum_{m=0}^{k} \left(1 - \tfrac{1}{3}\,m\right)\, \nabla\Theta_{k-m} \wedge \Theta_m, \qquad 0 \le k \le 4.
\end{aligned}
$$

# Lax representations & exotic cohomology

DEFINITION. For $n \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$ denote by $\mathfrak{G}(n, \varepsilon)$ the Lie algebra with the structure equations

$$d\alpha = 0,$$

$$d\Theta_k = k\,\alpha \wedge \Theta_k + \sum_{m=0}^{k} (1 + \varepsilon\, m)\ \nabla\Theta_{k-m} \wedge \Theta_m, \qquad 0 \le k \le n,$$

REMARK. $\mathfrak{G}(n, \varepsilon) = (\mathbb{R}_{n+1}[s] \otimes C^\omega(\mathbb{R})) \rtimes \mathbb{R}\,Y$,

where

- $\mathbb{R}_{n+1}[s] = \mathbb{R}[s]/\langle s^{n+1} = 0\rangle$;
- $C^\omega(\mathbb{R})$ is the space of real -analytic functions of $t$;
- the Lie bracket on $\mathbb{R}_{n+1}[s] \otimes C^\omega(\mathbb{R})$ is
$$[f, g]_\varepsilon = f\,g_t - g\,f_t + \varepsilon\,s\,(f_s\,g_t - g_s\,f_t);$$
- $Y = s\,\partial_s$.

REMARK. $\mathfrak{G}(n, \varepsilon)$ is a deformation of $\mathfrak{G}(n, 0)$.

# Lax representations & exotic cohomology

OBSERVATION. Suppose $\varepsilon = -\frac{1}{r}$ with $r \in \mathbb{N}$ and $n > r$. Then the 1-form $\theta_{r,0}$ has only two entries in the structure equations

$$d\alpha = 0,$$

$$d\Theta_k = k\,\alpha \wedge \Theta_k + \sum_{m=0}^{k} \left(1 - \tfrac{m}{r}\right) \nabla\Theta_{k-m} \wedge \Theta_m \qquad (*)$$

of $\mathfrak{G}(n, -\frac{1}{r})$, namely, the first equation from the series of equations for $\Theta_r$ is of the form

$$d\theta_{r,0} = r\,\alpha \wedge \theta_{r,0} + \Psi_r, \qquad (**)$$

where neither $\Psi_r$ nor the other equations in $(*)$ contain $\theta_{r,0}$.

CONCLUSION. Remove $(**)$ from $(*)$. Denote by $\tilde{\mathfrak{G}}(n, -\frac{1}{r})$ the Lie algebra with the obtained structure equations. Then

$$H^2_{-r\,\alpha}(\tilde{\mathfrak{G}}(n, -\tfrac{1}{r})) = \mathbb{R}\,[\Psi_r],$$

and $\mathfrak{G}(n, -\frac{1}{r})$ is an extension of $\tilde{\mathfrak{G}}(n, -\frac{1}{r})$ associated to $(**)$.

# Lax representations & exotic cohomology

EXAMPLE. $\mathfrak{G}(5,-\frac{1}{4})$ — integrate the structure equations:

$$\theta_{0,0} = \tfrac{1}{4}\, q^4\, dt, \qquad \theta_{1,0} = \tfrac{1}{3}\, p\, q^3\, (dy + a\, dt),$$

$$\theta_{2,0} = \tfrac{1}{2}\, p^2\, q^2\, (dx + 2\, a\, dy + (3\, a^2 - 2\, u_z)\, dt),$$

$$\theta_{3,0} = p^3\, q\, (dz + a\, dx + (a^2 - u_z)\, dy + (a^3 - 2\, a\, u_z - u_x)\, dt),$$

$$\theta_{5,0} = p^5\, q^{-1}\, (du - u_t\, dt - u_x\, dx - u_y\, dy - u_z\, dz).$$

Then integrate the equation for $\theta_{4,0}$ and rename $a = v_z$:

$$\theta_{4,0} = -p^4\, \Big(dv - \big(\tfrac{1}{4}\, v_z^4 - u_z\, v_z^2 - u_x\, v_z - u_y + \tfrac{1}{2}\, u_z^2\big)\, dt$$
$$- \big(\tfrac{1}{2}\, v_z^2 - u_z\big)\, dx - \big(\tfrac{1}{3}\, v_z^3 - u_z\, v_z - u_x\big)\, dy - v_z\, dz\Big).$$

# Lax representations & exotic cohomology

$\theta_{4,0} = 0 \quad \Rightarrow$

$$\begin{cases} v_t &= \frac{1}{4}\, v_z^4 - u_z\, v_z^2 - u_x\, v_z - u_y + \frac{1}{2}\, u_z^2, \\ v_y &= \frac{1}{3}\, v_z^3 - u_z\, v_z - u_x, \\ v_x &= \frac{1}{2}\, v_z^2 - u_z. \end{cases}$$

This is the Lax representation for the second system from the dKP hierarchy

$$\begin{cases} u_{xx} &= u_{yz} + u_z\, u_{zz}, \\ u_{xy} &= u_{tz} + u_z\, u_{xz} + u_x\, u_{zz}, \\ u_{yy} &= u_{tx} + u_x\, u_{xz} + u_z^2\, u_{zz}. \end{cases}$$

- V.E. Zakharov, 1980
- P.D. Lax, C.D. Levermore, 1983
- Y. Kodama, 1988
- B.A. Kupershmidt, 1990
- ...

# Lax representations & exotic cohomology

EXAMPLE. $\mathfrak{G}(6, -\frac{1}{5})$

$$\begin{cases} v_t &= \frac{1}{5}\, v_s^5 - u_s\, v_s^3 - u_z\, v_s^2 + (u_s^2 - u_x)\, v_s + u_z\, u_s - u_y, \\ v_y &= \frac{1}{4}\, v_s^4 - u_s\, v_s^2 - u_z\, v_s - u_x + \frac{1}{2}\, u_s^2, \\ v_x &= \frac{1}{3}\, v_s^3 - u_s\, v_s - u_z, \\ v_z &= \frac{1}{2}\, v_s^2 - u_s \end{cases}$$

$\Rightarrow$

$$\begin{cases} u_{zz} &= u_{xs} + u_s\, u_{ss}, \\ u_{xz} &= u_{ys} + u_s\, u_{zs} + u_z\, u_{ss}, \\ u_{yz} &= u_{ts} + u_s\, u_{xs} + u_z\, u_{zs} + u_x\, u_{ss}, \\ u_{xx} &= u_{yz} + u_z\, u_{zs} + u_s^2\, u_{ss}, \\ u_{xy} &= u_{tz} + u_z\, u_{xs} + (u_x + u_s^2)\, u_{zs} + 2\, u_z\, u_s\, u_{ss}, \\ u_{yy} &= u_{tx} + u_x\, u_{xs} + 2\, u_z\, u_s\, u_{zs} + (u_z^2 + u_s^3)\, u_{ss}. \end{cases}$$

# Lax representations & exotic cohomology

$\mathfrak{G}(3, -\frac{1}{2})$??

For $n > r$ consider the Lie algebra $\mathfrak{H}(n, r)$ with the structure equations

$$d\alpha = 0,$$

$$d\beta = r\,\alpha \wedge \beta,$$

$$d\Theta_{k'} = k'\,\alpha \wedge \Theta_{k'} + \sum_{m=0}^{k'} \left(1 - \tfrac{m}{r}\right) \nabla\Theta_{k'-m} \wedge \Theta_m,$$

$$d\Theta_{k''} = k''\,\alpha \wedge \Theta_{k''} + \sum_{m=0}^{k''} \left(1 - \tfrac{m}{r}\right) \nabla\Theta_{k''-m} \wedge \Theta_m + \beta \wedge \nabla\Theta_{k''-r},$$

$$k' \in \{0, \ldots, r-1\}, \;\; k'' \in \{r, \ldots, n\}.$$

REMARK: $\mathfrak{H}(n, r)$ is the right extension of $\mathfrak{G}(n, -1/r)$ by the outer derivative $Z$ such that $Z\colon s^k t^j \mapsto j s^{k+r} t^{j-1}$ for $k + r \leq n$, $Z\colon s^k t^j \mapsto 0$ for $k + r > n$, and $Z \circ Y - Y \circ Z = -r\,Y$.

# Lax representations & exotic cohomology

EXAMPLE. $\mathfrak{H}(3,2)$:

$$H^2_{\lambda\,\alpha}(\mathfrak{H}(3,2)) = \begin{cases} \mathbb{R}\,[\alpha\wedge\beta], & \lambda=-2,\\ \{0\}, & \lambda\neq -2. \end{cases}$$

$$d\tilde{\theta}_{2,0} = 2\,\alpha\wedge\tilde{\theta}_{2,0} + \theta_{2,1}\wedge\theta_{0,0} + \tfrac{1}{2}\,\theta_{1,1}\wedge\theta_{1,0} + \beta\wedge\theta_{0,1} + 2\,\alpha\wedge\beta$$

Integrate $\Rightarrow$ the Lax representation

$$\begin{cases} v_t &= \frac{1}{2}\,(\mathrm{e}^{v_x} - u_x)^2 - u_y,\\ v_y &= \mathrm{e}^{v_x} - u_x. \end{cases}$$

for the equation

$$u_{yy} = u_{tx} + u_x\,u_{xy}.$$

- V.S. Gerdjikov, 1988,
- M. Błaszak, 2002,
- M.V. Pavlov, 2003,
- E.V. Ferapontov, A. Moro, V.V. Sokolov, 2008.

# Lax representations & exotic cohomology

EXAMPLE. $\mathfrak{H}(4,3)$:

$$H^2_{\lambda\,\alpha}(\mathfrak{H}(4,3)) = \begin{cases} \mathbb{R}\,[\alpha\wedge\beta], & \lambda = -3, \\ \{0\}, & \lambda \neq -3. \end{cases}$$

$$\begin{aligned} d\tilde{\theta}_{3,0} = {} & 3\,\alpha\wedge\tilde{\theta}_{3,0} + \theta_{3,1}\wedge\theta_{0,0} + \tfrac{2}{3}\,\theta_{2,1}\wedge\theta_{1,0} + \tfrac{1}{3}\,\theta_{1,1}\wedge\theta_{2,0} \\ & + \beta\wedge\theta_{0,1} + 3\,\alpha\wedge\beta \end{aligned}$$

Integrate $\Rightarrow$ the Lax representation

$$\begin{cases} v_t &= \frac{1}{3}\,\left(\mathrm{e}^{v_x} - u_x\right)^3 - u_z\,\left(\mathrm{e}^{v_x} - u_x\right) - u_y, \\ v_y &= \frac{1}{2}\,\left(\mathrm{e}^{v_x} - u_x\right)^2 - u_z, \\ v_z &= \mathrm{e}^{v_x} - u_x, \end{cases}$$

for the system

$$\begin{cases} u_{yy} &= u_{tz} + u_z\,u_{zz}, \\ u_{zz} &= u_{xy} + u_x\,u_{xz}, \\ u_{yz} &= u_{tx} + u_x\,u_{xy} + u_z\,u_{xz}. \end{cases}$$

# Lax representations & exotic cohomology

THEOREM: Let $n \geq 2$, then $H^2_{\lambda\,\alpha}(\mathfrak{G}(n,\varepsilon)) \neq \{0\} \quad \Longleftrightarrow$ $\varepsilon = -\frac{2}{r}$ and $\lambda = -r$, where $r \in \{2,\ldots,n\}$. In this case

$$H^2_{-r\,\alpha}(\mathfrak{G}(n,-\tfrac{2}{r})) = \mathbb{R}\,[\Phi_r],$$

where

$$\Phi_r = \sum_{m=0}^{[r/2]} (r-2\,m)\;\theta_{r-m,0} \wedge \theta_{m,0}.$$

COROLLARY. Equation

$$d\omega = r\,\alpha \wedge \omega + \Phi_r.$$

is compatible with the structure equations of $\mathfrak{G}(n,-\frac{2}{r})$.

# Lax representations & exotic cohomology

EXAMPLE. $\mathfrak{G}(3, -\frac{2}{3})$:

the additional equation

$$d\omega = 3\,\alpha \wedge \omega + 3\,\theta_{0,0} \wedge \theta_{3,0} + \theta_{1,0} \wedge \theta_{2,0},$$

integrate $\Rightarrow$ the Lax representation

$$\begin{cases} v_t &= u - (u_x^2 + u_y)\, v_x, \\ v_y &= x - u_x\, v_x \end{cases}$$

for

$$u_{yy} = u_{tx} + (u_y - u_x^2)\, u_{xx} - 3\, u_x\, u_{xy}.$$

- O.M., J. Geom. Phys. 59 (2009), 1461–1475

# Lax representations & exotic cohomology

EXAMPLE. $\mathfrak{G}(4,-\frac{1}{2})$ ?? $\quad \mathfrak{H}(4,2)$:

the additional equation

$$d\omega = 4\,\alpha\wedge\omega + \theta_{1,0}\wedge\theta_{3,0} + 2\,\theta_{0,0}\wedge\theta_{4,0} + \beta\wedge\theta_{2,0}$$

$\Rightarrow$

$$\left\{\begin{array}{lcl} v_t & = & \left(u_y - u_x\,u_z - \frac{1}{2}\,u_z^3\right)\,v_z - u, \\ v_y & = & \left(u_x + \frac{1}{2}\,u_z\right)\,v_z + z. \end{array}\right.$$

$\Rightarrow$

$$\begin{aligned} u_{yy} = {} & u_{tx} + u_z\,u_{xy} - u_y\,u_{xz} + u_z\,u_{tz} + 2\,(u_x + u_z^2)\,u_{yz} \\ & - \left(u_x^2 + u_x\,u_z^2 + u_y\,u_z + \frac{1}{4}\,u_z^4\right)\,u_{zz}. \end{aligned}$$

# Lax representations & exotic cohomology

EXAMPLE. $\mathfrak{G}(5, -\frac{2}{5})$:

Integrate $\Rightarrow$ the Lax representation

$$\begin{cases} v_t &= 5\,\Big(2\,u - \big(u_x - \frac{4}{3}\,u_z\,u_s - \frac{2}{9}\,u_s^3\big)\;v_z \\ & \quad - \big(\frac{1}{3}\,u_y - \frac{11}{9}\,u_z\,u_s^2 + u_x\,u_s - u_z^2 - \frac{13}{81}\,u_s^4\big)\;v_s\Big)\,, \\ v_x &= 2\,z + \frac{1}{3}\,u_s\,v_z + \big(\frac{2}{9}\,u_s^2 + u_z\big)\;v_s, \\ v_y &= 6\,s + \big(3\,u_z + \frac{2}{3}\,u_s^2\big)\;v_z + \big(3\,(u_z\,u_s - u_x) + \frac{13}{27}\,u_s^3\big)\;v_s \end{cases}$$

for ...

# Lax representations & exotic cohomology

... the system

$$
\begin{cases}
u_{xz} = u_{ys} - u_s\,u_{xs} + u_s\,u_{zz} + \frac{1}{16}\left(12\,u_x + 11\,u_s^3 + 12\,u_z\,u_s\right)u_{ss} \\
\qquad +2\left(u_s^2 + u_z\right)u_{zs}, \\
u_{xx} = \frac{3}{4}\left(u_s^2 + 8\,u_z\right)u_{xs} + \frac{1}{2}\left(11\,u_s^3 + 18\,u_z\,u_s - 6\,u_x\right)u_{zs} \\
\qquad -4\,u_{yz} + \left(4\,u_z + 3\,u_s^2\right)u_{zz} + \frac{9}{8}\left(2\,u_z + u_s^2\right)\left(u_s^2 - 4\,u_z\right)u_{ss}, \\
u_{tz} = 10\,u_z\,u_{ys} - 2\,u_{xy} + \frac{1}{8}\left(u_s^3 - 12\,u_x + 4\,u_z\,u_s\right)u_{xs} \\
\qquad +\frac{1}{8}\left(63\,u_s^4 + 16\,u_z^2 - 48\,u_y + 172\,u_z\,u_s^2 - 36\,u_x\,u_s\right)u_{zs} \\
\qquad +\frac{1}{8}\left(18\,u_s^5 - 84\,u_z^2\,u_s + 60\,u_x\,u_z + 19\,u_z\,u_s^3\right)u_{ss} \\
\qquad +2\left(2\,u_s^3 - u_x + 5\,u_z\,u_s\right)u_{zz}, \\
u_{tx} = 8\,u_{yy} + \frac{1}{2}\left(21\,u_z\,u_s^2 + 24\,u_z^2 - 5\,u_x\,u_s + 4\,u_s^4 - 12\,u_y\right)u_{xs} \\
\qquad +\frac{1}{8}\left(7\,u_s^5 - 144\,u_z^2\,u_s - 32\,u_x\,u_z - 32\,u_z\,u_s^3 - 32\,u_x\,u_s^2\right)u_{zs} \\
\qquad -\frac{1}{32}\left(372\,u_z\,u_s^4 + 48\,u_x\,u_s\,u_z + 44\,u_x\,u_s^3 + 576\,u_z^3 + 1008\,u_z^2\,u_s^2\right. \\
\qquad \left.-96\,u_x^2 + 31\,u_s^6\right)u_{ss} + \left(u_s^4 - 2\,u_x\,u_s - 8\,u_z^2\right)u_{zz} + 10\,u_x\,u_{ys}, \\
u_{ts} = 8\,u_{yz} + \frac{1}{2}\left(u_s^2 + 4\,u_z\right)u_{xs} + 10\,u_s u_{ys} - \left(22 u_z u_s + 9 u_s^3 - 4 u_x\right)u_{zs} \\
\qquad -4\left(2\,u_z + u_s^2\right)u_{zz} - \frac{1}{2}\left(21\,u_z\,u_s^2 - 6\,u_x\,u_s + 8\,u_s^4 + 12\,u_y\right)u_{ss}.
\end{cases}
$$

# Conclusion & outlook

- The approach based on the exotic cohomology of symmetry pseudo-groups is successful in both describing known Lax representations and deriving new ones.
- It gives the solution to the problem of existence of a Lax representation in internal terms of the PDE and allows one to eliminate apriori assumptions about the possible form of the Lax representation.
- The approach is universal and can be used to analyze a lot of equations or Lie algebras with nontrivial second exotic cohomology.

# Conclusion & outlook

Generalizations:

- to describe right extensions of $\mathfrak{G}(n,\varepsilon)$ (to compute $H^1(\mathfrak{G}(n,\varepsilon),\mathfrak{G}(n,\varepsilon))$);
- to replace vector fields on $\mathbb{R}$ by vector fields on $\mathbb{R}^n$ in the constructions above. For example, Hamiltonian vector fields on $\mathbb{R}^2$ $\Rightarrow$ the heavenly equations and related equations (B. Kruglikov, O.M., 2012, 2015);
- $\alpha^1$, ... , $\alpha^m$ instead of $\alpha$,

  $d\alpha^i = \frac{1}{2}\, c^i_{jk}\, \alpha^j \wedge \alpha^k$ instead of $d\alpha = 0$.
- ...