Point-transformation structures on classes of differential equations

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Heaviside's aphorism "Mathematics is an experimental science. . ."

Mathematics is an experimental science, and definitions do not come first, but later on.

Oliver Heaviside

On operators in physical mathematics, part II,
Proceedings of the Royal Society of London **54** (1893), p. 121.

http://homepage.math.uiowa.edu/∼jorgen/heavisidequotesource.html

There is nothing by Heaviside in "Bartlett's Familiar Quotations," but the first phrase of Heaviside's aphorism, "Mathematics is an experimental science" is widely quoted. A web search can find it in dozens of places, but only a few of the ones found by Google, for example, continues the quotation to the end of the sentence!

> *In the preceding, I have purposely avoided giving any definition of 'equivalence.' Believing in example rather than precept, I have preferred to let the formulae, and the method of obtaining them, speak for themselves. Besides that, I could not give a satisfactory definition which I could feel sure would not require subsequent revision.* **Mathematics is an experimental science, and definitions do not come first, but later on.** *They make themselves, when the nature of the subject has developed itself. It would be absurd to lay down the law beforehand. Perhaps, therefore, the best thing I can do is to describe briefly several successive stages of knowledge related to equivalent and divergent series, being approximately representative of personal experience.*

A system of differential equations $\mathcal{L}$: $L(x, u_{(p)}) = 0$

$L = (L^1, \dots, L^k)$
$x = (x^1, \dots, x^n)$ are the independent variables
$u = (u^1, \dots, u^m)$ are the dependent variables
$u_{(p)}$ is the set of all derivatives of u of order $\leqslant p$ w.r.t. x

Within the local approach the system $\mathcal{L}$ is treated as a system of algebraic equations in the jet space $J^{(p)}$ of the order p.

$$u^a_\alpha = \frac{\partial^{|\alpha|} u^a}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, \qquad \begin{aligned} &\alpha = (\alpha_1, \dots, \alpha_n), \\ &|\alpha| = \alpha_1 + \dots + \alpha_n. \end{aligned}$$

$$\mathcal{L} = \{(x, u_{(p)}) \in J^{(p)} \mid L(x, u_{(p)}) = 0\}$$

Example. $\mathcal{L}$: $u_t = u_{xx}$

$$J^{(2)} \sim \{(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx})\}$$

Point transformations

A *point transformation* in a space is an invertible smooth mapping of an open domain in this space into the same domain.

A point transformation in the space of (x, u):

$$g\colon \begin{array}{l}\tilde{x} = X(x,u),\\ \tilde{u} = U(x,u),\end{array} \quad \left|\frac{\partial(X,U)}{\partial(x,u)}\right| \neq 0 \quad \longrightarrow \quad \mathrm{pr}_{(r)}g\colon \begin{array}{l}\tilde{x} = X(x,u),\ \tilde{u} = U(x,u),\\ \tilde{u}_\alpha = U^\alpha(x, u_{(r')}),\ r' = |\alpha| \leq r.\end{array}$$

Example. $\mathcal{L}\colon\ u_t = u_{xx}$

$$\tilde{t} = t + \delta_0,\ \tilde{x} = x + \delta_1,\ \tilde{u} = u,\ \delta_0, \delta_1 = \mathrm{const} \quad \Rightarrow \quad \tilde{u}_{\tilde{t}} = u_t,\ \tilde{u}_{\tilde{x}\tilde{x}} = u_{xx},$$
$$\Rightarrow \quad u_t = u_{xx} \quad \to \quad \tilde{u}_{\tilde{t}} = \tilde{u}_{\tilde{x}\tilde{x}}$$

$$\tilde{t} = t,\ \tilde{x} = x + \delta_1 t,\ \tilde{u} = u,\ \delta_1 = \mathrm{const} \quad \Rightarrow \quad \tilde{u}_{\tilde{t}} = u_t - \delta_1 u_x,\ \tilde{u}_{\tilde{x}\tilde{x}} = u_{xx},$$
$$\Rightarrow \quad u_t = u_{xx} \quad \to \quad \tilde{u}_{\tilde{t}} = \tilde{u}_{\tilde{x}\tilde{x}} - \delta_1 \tilde{u}_{\tilde{x}}$$

The most general solution obtainable from a given solution $u = f(t, x)$ by group transformations is of the form

$$\tilde{u} = \frac{\varepsilon_3}{\sqrt{1 + 4\varepsilon_6 t}} e^{-\frac{\varepsilon_5 x + \varepsilon_6 x^2 - \varepsilon_5^2 t}{1 + 4\varepsilon_6 t}} f\left(\frac{\varepsilon_4^2 t}{1 + 4\varepsilon_6 t} - \varepsilon_2, \frac{\varepsilon_4(x - 2\varepsilon_5 t)}{1 + 4\varepsilon_6 t} - \varepsilon_1\right) + v(t, x), \quad (1)$$

where $\varepsilon_1, \dots, \varepsilon_6$ are arbitrary constants, $\varepsilon_6 \neq 0$ and $v(t, x)$ is an arbitrary solution to the linear heat equation.

Classes of differential equations

The central notion underlying the theory of group classification is an appropriate definition of a class of (systems of) differential equations.

Ingredients:

- a system of differential equations $\mathcal{L}_\theta$: $L(x, u_{(p)}, \theta_{(q)}(x, u_{(p)})) = 0$
  parameterized arbitrary elements $\theta(x, u_{(p)}) = (\theta^1(x, u_{(p)}), \dots, \theta^k(x, u_{(p)}))$
  $x = (x^1, \dots, x^n)$ are the independent variables
  $u = (u^1, \dots, u^m)$ are the dependent variables
  $u_{(p)}$ is the set of all derivatives of u of order $\leqslant p$ w.r.t. x
  $\theta_{(q)}$ is the set of derivatives of θ of order $\leqslant q$ w.r.t. x and $u_{(p)}$
- $\mathcal{S} = \{\theta \mid S(x, u_{(p)}, \theta_{(q')}(x, u_{(p)})) = 0,\ \Sigma(x, u_{(p)}, \theta_{(q')}(x, u_{(p)})) \neq 0\ (> 0, < 0, \dots)\}$
  both x and $u_{(p)}$ play the role of independent variables

The inequalities might be essential to guarantee that each element of the class has some common properties with all other elements of the same class.

Definition

The set $\{\mathcal{L}_\theta \mid \theta \in \mathcal{S}\}$ denoted by $\mathcal{L}|_\mathcal{S}$ is called a *class of differential equations* defined by the parameterized form of systems $\mathcal{L}_\theta$ and the set $\mathcal{S}$ of the arbitrary elements θ.

Example. $\mathcal{L}_\theta$: $u_t = (D(u)u_x)_x, \quad \theta = D \qquad$ [Ovsiannikov, 1959]

$$\mathcal{S}: \quad D_t = D_x = D_{u_t} = D_{u_x} = D_{u_{tt}} = D_{u_{tx}} = D_{u_{xx}} = 0, \quad D \neq 0$$

The example: precise definition of the class

Parameterized representation of equations:

$$\mathcal{W}: \quad u_{tt} = f(x, u_x)u_{xx} + g(x, u_x)$$

with

two independent variables t and x
one dependent variable u
the related jet space is of second order, $(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx})$
two arbitrary elements $\theta = (f, g)$

The auxiliary system for θ:

- $f = f(x, u_x),\ g = g(x, u_x) \quad \sim \quad \begin{array}{l} f_t = f_u = f_{u_t} = f_{u_{tt}} = f_{u_{tx}} = f_{u_{xx}} = 0, \\ g_t = g_u = g_{u_t} = g_{u_{tt}} = g_{u_{tx}} = g_{u_{xx}} = 0 \end{array}$
- "wave equations" $\quad \sim \quad f \neq 0$

  The equations with $f = 0$ completely differs from the equations with $f \neq 0$!
- "nonlinear equations" $\quad \sim \quad (f_{u_x}, g_{u_x u_x}) \neq (0, 0)$

  Nonlinear and linear equations of this form are not mixed by point transformations and have quite different Lie symmetry properties. Moreover, linear wave equations of this form were already well investigated within the framework of classical symmetry analysis.

pth order ODEs in the canonical form
$n = m = 1, \quad u^{(p)} = F(x, u, u', \dots, u^{(p-1)})$
[Lie, the 1870s–1890s, especially $p = 2$; ...]

Linear second-order partial differential equations in two independent variables
$n = 2,\ m = 1,\ p = 2,$
$A^{11}(x,y)u_{xx} + A^{12}(x,y)u_{xy} + A^{22}(x,y)u_{yy} + B^1(t,x)u_x + B^2(t,x)u_y + C(t,x)u = 0,$
$(A^{11}, A^{12}, A^{22}) \neq (0,0,0)$
[Lie, the 1881, over $\mathbb{C}$; Ovsiannikov, the 1970s, over $\mathbb{R}$; ...]

Linear second-order evolution equations
$n = 2,\ m = 1,\ p = 2 \qquad u_t = A(t,x)u_{xx} + B(t,x)u_x + C(t,x)u,\ A \neq 0$
[Lie, the 1881, over $\mathbb{C}$; Ovsiannikov, the 1970s, over $\mathbb{R}$; ...]

(In general) nonlinear wave equations
$n = 2,\ m = 1,\ p = 2 \qquad u_{tx} = F(u)$
[Lie, 1881, contact symmetry transformations; ...]

Gauge equivalence

Additional problem: the correspondence $\theta \to \mathcal{L}_\theta$ may not be injective

The values θ and $\tilde\theta$ of arbitrary elements are called *gauge-equivalent* if $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde\theta}$ are the same system of differential equations.

Example. $(1+1)$-dimensional nonlinear Schrödinger equations with potentials and modular nonlinearities

$$i\psi_t + \psi_{xx} + f(|\psi|)\psi + V(t,x)\psi = 0, \quad f' \neq 0$$
$$\tilde f = f + \beta,\ \tilde V = V - \beta \quad \to \quad i\psi_t + \psi_{xx} + \tilde f(|\psi|)\psi + \tilde V\psi = 0$$
$$S = S(t,x,|\psi|) = f(|\psi|) + V(t,x){:}$$
$$\psi S_\psi - \psi^* S_{\psi^*} = 0, \quad \psi S_{\psi t} + \psi^* S_{\psi^* t} = \psi S_{\psi x} + \psi^* S_{\psi^* x} = 0, \quad \psi S_\psi + \psi^* S_{\psi^*} \neq 0$$

Example. Variable coefficient nonlinear diffusion–convection equations

$$f(x)u_t = (g(x)A(u)u_x)_x + h(x)B(u)u_x, \quad fghA \neq 0, \quad (A_u, B_u) \neq (0,0)$$

(f,g,h,A,B) and $(\tilde f, \tilde g, \tilde h, \tilde A, \tilde B)$ correspond to the same equation iff

$$\tilde f = \varepsilon_1 \varphi\, f, \quad \tilde g = \varepsilon_1 \varepsilon_2^{-1} \varphi\, g, \quad \tilde h = \varepsilon_1 \varepsilon_3^{-1} \varphi\, h, \quad \tilde A = \varepsilon_2 A, \quad \tilde B = \varepsilon_3 (B + \varepsilon_4 A),$$

where $\varphi = e^{-\varepsilon_4 \int \frac{h(x)}{g(x)} dx}$, ε_i $(i = 1, \dots, 4)$ are arbitrary constants, $\varepsilon_1 \varepsilon_2 \varepsilon_3 \neq 0$

Subclasses of differential equations

In the course of studying point transformations in a complicated class of differential equations, it is often helpful to consider subclasses of this class.

A *subclass* is singled out from the class $\mathcal{L}|_{\mathcal{S}}$ by attaching additional equations or inequalities to the auxiliary system $\mathcal{S}$.

Properties:

The intersection of a finite number of subclasses of $\mathcal{L}|_{\mathcal{S}}$ is also a subclass in $\mathcal{L}|_{\mathcal{S}}$, which is defined by the union of the additional auxiliary systems associated with the intersecting sets.

The complement of a subclass in the class $\mathcal{L}|_{\mathcal{S}}$ is also a subclass of $\mathcal{L}|_{\mathcal{S}}$ only in special cases, e.g., if either the additional system of equations or the additional system of inequalities is empty.

The union $\mathcal{L}|_{\mathcal{S}'} \cup \mathcal{L}|_{\mathcal{S}''} = \mathcal{L}|_{\mathcal{S}' \cup \mathcal{S}''}$ and the difference $\mathcal{L}|_{\mathcal{S}'} \setminus \mathcal{L}|_{\mathcal{S}''} = \mathcal{L}|_{\mathcal{S}' \setminus \mathcal{S}''}$ of the subclasses $\mathcal{L}|_{\mathcal{S}'}$ and $\mathcal{L}|_{\mathcal{S}''}$ in the class $\mathcal{L}|_{\mathcal{S}}$ also are subclasses of $\mathcal{L}|_{\mathcal{S}}$ if the additional systems of equations or the additional system of inequalities of these subclasses coincide.

Examples of subclasses: generalized nonlinear Schrödinger equations

$$i\psi_t + G(t,x,\psi,\psi^*,\psi_x,\psi_x^*)\psi_{xx} + F(t,x,\psi,\psi^*,\psi_x,\psi_x^*) = 0$$

$$i\psi_t + G(t,x,\psi,\psi^*)\psi_{xx} + F(t,x,\psi,\psi^*,\psi_x,\psi_x^*) = 0$$

$$i\psi_t + G(t,x)\psi_{xx} + F(t,x,\psi,\psi^*,\psi_x,\psi_x^*) = 0$$

$$i\psi_t + G(t,x)\psi_{xx} + F(t,x,\psi,\psi^*) = 0$$

$$i\psi_t + \psi_{xx} + F(t,x,\psi,\psi^*,\psi_x,\psi_x^*) = 0$$

$$i\psi_t + \psi_{xx} + F(t,x,\psi,\psi^*) = 0$$

$$i\psi_t + \psi_{xx} + S(t,x,|\psi|)\psi = 0, \quad (\psi\partial_\psi - \psi^*\partial_{\psi^*})(F/\psi) = 0$$

$$i\psi_t + \psi_{xx} + f(\rho)\psi + V(t,x)\psi = 0, \quad S_{\rho t} = S_{\rho x} = 0,\ S_\rho \neq 0, \quad \rho = |\psi|$$

1. General case $\rho f''/f' \neq \mathrm{const} \in \mathbb{R}$
2. $f = \sigma \ln \rho$, $\sigma \in \mathbb{C} \setminus \{0\}$
3. $f = \sigma\rho^\gamma$, $\sigma \in \mathbb{C} \setminus \{0\}$, $\gamma \in \mathbb{R} \setminus \{0\}$

Point transformations in classes of differential equations

Point transformations related to $\mathcal{L}|_\mathcal{S}$ form different structures.

$\mathrm{T}(\theta,\tilde\theta) =$ the set of point transformations in the space of (x,u) mapping $\mathcal{L}_\theta$ to $\mathcal{L}_{\tilde\theta}$, $\theta,\tilde\theta\in\mathcal{S}$

Symmetry groups

$T(\theta,\theta) =$ the *maximal point symmetry (pseudo)group G_θ* of $\mathcal{L}_\theta$

$G^\cap = G^\cap(\mathcal{L}|_\mathcal{S}) = \bigcap_{\theta\in\mathcal{S}} G_\theta$ is the *kernel of the maximal point symmetry groups* of systems from $\mathcal{L}|_\mathcal{S}$ (resp. *the kernel group* of $\mathcal{L}|_\mathcal{S}$).

$\mathrm{T}(\theta,\tilde\theta)\neq\varnothing \sim$ $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde\theta}$ are similar w.r.t. point transformations
Then $\mathrm{T}(\theta,\tilde\theta) = \varphi^0\circ G_\theta = G_{\tilde\theta}\circ\varphi^0$, where φ^0 is a fixed transformation from $\mathrm{T}(\theta,\tilde\theta)$.

Definition

The **equivalence groupoid** of the class $\mathcal{L}|_\mathcal{S}$ is

$$\mathcal{G}^\sim = \mathcal{G}^\sim(\mathcal{L}|_\mathcal{S}) = \{(\theta,\tilde\theta,\varphi)|\ \theta,\tilde\theta\in\mathcal{S}, \varphi\in\mathrm{T}(\theta,\tilde\theta)\}.$$

Elements of $\mathcal{G}^\sim$ are called *admissible transformations* of $\mathcal{L}|_\mathcal{S}$. This formalizes the notions of form-preserving [Kingsto&Sophocleous, 1991,1998] or allowed [Winternitz&Gazeau, 1992] transformations.

Equivalence group

Definition

The *(usual) equivalence group* $G^\sim = G^\sim(\mathcal L|_\mathcal S)$ of the class $\mathcal L|_\mathcal S$ is the (pseudo)group of point transformations in the space of $(x, u_{(p)}, \theta)$ that

- are projectable to the space of $(x, u_{(p')})$ for any $0 \le p' \le p$,
- are consistent with the contact structure on the space of $(x, u_{(p)})$ and
- map every system from the class $\mathcal L|_\mathcal S$ to another system from the same class.

The equivalence group $G^\sim$ gives rise to a subgroupoid of the equivalence groupoid $\mathcal G^\sim$,

$$G^\sim \ni \mathcal T \to \{(\theta, \mathcal T\theta, \pi_*\mathcal T) \mid \theta \in \mathcal S\} \subset \mathcal G^\sim.$$

Roughly speaking, $G^\sim$ is the set of equivalence transformations
$=$ the set admissible transformations that can be applied to any $\theta \in \mathcal S$.

Folklore proposition

The kernel group $G^\cap$ of $\mathcal L|_\mathcal S$ is naturally embedded into the (usual) equivalence group $G^\sim$ of $\mathcal L|_\mathcal S$ via trivial (identical) prolongation of the kernel transformations to the arbitrary elements. The associated subgroup $\hat G^\cap$ of $G^\sim$ is normal.

Normalized classes of differential equations

It is convenient to characterize and estimate transformational properties of classes of differential equations in terms of normalization.

Definition

A class $\mathcal{L}|_{\mathcal{S}}$ is *normalized* (in the usual sense) if its (usual) equivalence group $G^{\sim}$ induces its equivalence groupoid $\mathcal{G}^{\sim}$.

$\sim\quad \forall(\theta,\tilde\theta,\varphi)\in\mathcal{G}^{\sim}\ \exists\mathcal{T}\in G^{\sim}\colon\ \tilde\theta=\mathcal{T}\theta$ and $\varphi=\mathcal{T}|_{(x,u)}$

Definition

A class $\mathcal{L}|_{\mathcal{S}}$ is *semi-normalized* (in the usual sense) if its equivalence groupoid $\mathcal{G}^{\sim}$ is induced by transformations from $G^{\sim}$ and from the maximal point symmetry groups of its equations.

$\sim\quad \forall(\theta,\tilde\theta,\varphi)\in\mathcal{G}^{\sim}\ \exists\mathcal{T}\in G^{\sim}$ and $\exists\tilde\varphi\in G_\theta\colon\ \tilde\theta=\mathcal{T}\theta$ and $\varphi=\mathcal{T}|_{(x,u)}\circ\tilde\varphi$

$\sim\quad$ arbitrary similar systems from $\mathcal{L}|_{\mathcal{S}}$ are related via transformations from $G^{\sim}$

If $\mathcal{L}|_{\mathcal{S}}$ is normalized in the usual sense, then

- it is semi-normalized in the usual sense,
- its kernel group $G^{\cap}$ is a normal subgroup of G_θ for each $\theta\in\mathcal{S}$,
- $G_\theta\leqslant G^{\sim}|_{(x,u)}$ for each $\theta\in\mathcal{S}$.

Computation of equivalence groupoid

To establish the normalization properties of the class $\mathcal{L}|_\mathcal{S}$ one should compute its equivalence groupoid $\mathcal{G}^\sim$, which is realized using the direct method.

- Here one fixes two arbitrary systems from the class,

$$\mathcal{L}_\theta\colon\ L(x,u_{(p)},\theta(x,u_{(p)}))=0 \quad\text{and}\quad \mathcal{L}_{\tilde\theta}\colon\ L(\tilde x,\tilde u_{(p)},\tilde\theta(\tilde x,\tilde u_{(p)}))=0,$$

- and aims to find the (nondegenerate) point transformations,

$$\varphi\colon\quad \tilde x_i=X^i(x,u),\ \ \tilde u^a=U^a(x,u),\ \ i=1,\dots,n,\ \ a=1,\dots,m,$$

  connecting them.
- For this, one changes the variables in the system $\mathcal{L}_{\tilde\theta}$ by expressing the derivatives $\tilde u_{(p)}$ in terms of $u_{(p)}$ and derivatives of the functions X^i and U^a as well as by substituting X^i and U^a for $\tilde x_i$ and $\tilde u^a$, respectively.
- The requirement that the resulting transformed system has to be satisfied identically for solutions of $\mathcal{L}_\theta$ leads to the (nonlinear) system of determining equations for the transformation components of φ.
- One solves the system of determining equations.

$$\mathcal{E}\colon\quad u_t = H(t,x,u_0,u_1,\dots,u_p),\quad p\geqslant 2,\quad H_{u_p}\neq 0,$$

where $u_j\equiv\partial^j u/\partial x^j$, $u_0\equiv u$, and $u_x=u_1$.

The contact transformations mapping a (fixed) equation $\mathcal{E}$: $u_t=H$ into another equation $\tilde{\mathcal{E}}$: $\tilde u_{\tilde t}=\tilde H$ are well known [Magadeev,1993] to have the form

$$\tilde t = T(t),\quad \tilde x = X(t,x,u,u_x),\quad \tilde u = U(t,x,u,u_x).$$

The nondegeneracy assumptions: $\quad T_t\neq 0,\quad \mathrm{rank}\begin{pmatrix} X_x & X_u & X_{u_x}\\ U_x & U_u & U_{u_x}\end{pmatrix}=2$

The contact condition: $\quad (U_x+U_uu_x)X_{u_x}=(X_x+X_uu_x)U_{u_x}\quad\Longrightarrow\quad \tilde u_{\tilde x}=V(t,x,u,u_x),$

where $\quad V=\dfrac{U_x+U_uu_x}{X_x+X_uu_x}\quad$ if $\quad X_x+X_uu_x\neq 0\quad$ or $\quad V=\dfrac{U_{u_x}}{X_{u_x}}\quad$ if $\quad X_{u_x}\neq 0$

$$\Longrightarrow\quad \tilde u_{\tilde t}=\frac{U_u-X_uV}{T_t}u_t+\frac{U_t-X_tV}{T_t},\quad \tilde u_k\equiv\frac{\partial^k\tilde u}{\partial\tilde x^k}=\left(\frac{1}{D_xX}D_x\right)^{k-1}V$$

$$\Longrightarrow\quad \tilde H=\frac{U_u-X_uV}{T_t}H+\frac{U_t-X_tV}{T_t}$$

The equiv. group $G^\sim_{\rm c}$ generates the whole set of admissible contact transformations in the class, i.e., the class is normalized [ROP & Kunzinger & Eshraghi,2010] w.r.t. contact transformations.

Proposition

The class of evolution equations is contact-normalized.

The class of evolution equations is also point-normalized.

The point equivalence group $G^\sim_{\rm p}$:

$$\tilde t=T(t),\quad \tilde x=X(t,x,u),\quad \tilde u=U(t,x,u),\quad \tilde H=\frac{\Delta}{T_tD_xX}H+\frac{U_tD_xX-X_tD_xU}{T_tD_xX},$$

where $T_t\neq0$ and $\Delta=X_xU_u-X_uU_x\neq0$.

The point equivalence group of the subclass of quasilinear evolution equations (i.e., $H_{u_pu_p}\neq0$) is the same, and this subclass is normalized.

pth order ODEs in the canonical form [Lie, the 1880s, especially $p=2$; ...]
$n=m=1,\quad u^{(p)}=F(x,u,u',\dots,u^{(p-1)})$
Arbitrary local diffeomorphisms in the space of (x,u)
Arbitrary contact transformations

Systems of pth order ODEs in the canonical form
$n=1$, arbitrary m, $\quad u^{(p)}=F(x,u,u',\dots,u^{(p-1)})$, $u=(u^1,\dots,u^m)$, $F=(F^1,\dots,F^m)$
Arbitrary local diffeomorphisms in the space of (x,u)

Linear (in general, inhomogeneous) second-order PDEs in two independent variables,
$n=2,\ m=1,\ p=2,$
$A^{11}(x,y)u_{xx}+A^{12}(x,y)u_{xy}+A^{22}(x,y)u_{yy}+B^1(t,x)u_x+B^2(t,x)u_y+C(t,x)u=D(t,x),$
$(A^{11},A^{12},A^{22})\neq(0,0,0)$
[Lie, the 1881, over $\mathbb{C}$; Ovsiannikov, the 1970s, over $\mathbb{R}$; ...]

Linear (in general, inhomogeneous) second-order evolution equations
$n=2,\ m=1,\ p=2\qquad u_t=A(t,x)u_{xx}+B(t,x)u_x+C(t,x)u+D(t,x),\ A\neq0$
[Lie, the 1880s, over $\mathbb{C}$; Ovsiannikov, the 1970s, over $\mathbb{R}$; ...]
$\tilde t=T(t),\ \tilde x=X(t,x),\ \tilde u=U^1(t,x)u+U^0(t,x),\ T_tX_xU^1\neq0$
$\longrightarrow$ Linear (in general, inhomogeneous) pth order evolution equations

$$u_t + uu_x + f(t,x)u_{xx} = 0, \quad f \neq 0$$ **This class is normalized!**

[Kingston&Sophocleous, 1991]

$$G^{\sim}: \quad \tilde t = \frac{\alpha t+\beta}{\gamma t+\delta}, \ \ \tilde x = \frac{\kappa x+\mu_1 t+\mu_0}{\gamma t+\delta}, \ \ \tilde u = \frac{\kappa(\gamma t+\delta)u-\kappa\gamma x+\mu_1\delta-\mu_0\gamma}{\alpha\delta-\beta\gamma},$$
$$\tilde f = \frac{\kappa^2}{\alpha\delta-\beta\gamma}f,$$

where α, β, γ, δ, μ_0, μ_1 and κ are constants; α, β, γ, δ are defined up to a nonzero multiplier, $\alpha\delta-\beta\gamma\neq 0$ and $\kappa\neq 0$.

$$\mathfrak g^{\sim} = \langle \widetilde P^t, \widetilde P^x, \widetilde D^t, \widetilde D^x, \widetilde G, \widetilde \Pi\rangle,$$

where

$$\widetilde P^t = \partial_t, \quad \widetilde P^x = \partial_x, \quad \widetilde D^t = t\partial_t - u\partial_u - f\partial_f, \quad \widetilde D^x = x\partial_x + u\partial_u + 2f\partial_f,$$
$$\widetilde G = t\partial_x + \partial_u, \quad \widetilde \Pi = t^2\partial_t + tx\partial_x + (x-tu)\partial_u,$$

$$u_t + uu_x + f(t)u_{xx} = 0, \quad f \neq 0$$ **This class is also normalized!**

Generalized nonlinear Schrödinger equations

Normalized classes:

$$i\psi_t + G(t,x,\psi,\psi^*,\psi_x,\psi_x^*)\psi_{xx} + F(t,x,\psi,\psi^*,\psi_x,\psi_x^*) = 0$$

$$i\psi_t + G(t,x,\psi,\psi^*)\psi_{xx} + F(t,x,\psi,\psi^*,\psi_x,\psi_x^*) = 0$$

$$i\psi_t + G(t,x)\psi_{xx} + F(t,x,\psi,\psi^*,\psi_x,\psi_x^*) = 0$$

$$i\psi_t + G(t,x)\psi_{xx} + F(t,x,\psi,\psi^*,\psi_x,\psi_x^*) = 0$$

$$i\psi_t + \psi_{xx} + F(t,x,\psi,\psi^*,\psi_x,\psi_x^*) = 0$$

$$i\psi_t + \psi_{xx} + F(t,x,\psi,\psi^*) = 0$$

$$i\psi_t + \psi_{xx} + S(t,x,\rho)\psi = 0, \quad S_\rho \neq 0, \quad \rho = |\psi|$$

✗ $i\psi_t + \psi_{xx} + f(|\psi|)\psi + V\psi = 0$ **This class is not normalized!**

What we can do no. 1: partition into normalized subclasses.

1. General case $\rho f_{\rho\rho}/f_\rho \neq \mathrm{const} \in \mathbb{R}$
2. $f = \sigma \ln|\psi|$, $\sigma \in \mathbb{C} \setminus \{0\}$
3. $f = \sigma|\psi|^\gamma$, $\sigma \in \mathbb{C} \setminus \{0\}$, $\gamma \in \mathbb{R} \setminus \{0\}$

Generalization of equivalence groups

A class is not normalized.
What we can do no. 2: generalize of the notion of equivalence group.

Recall

The *(usual) equivalence group* $G^{\sim} = G^{\sim}(\mathcal{L}|_{\mathcal{S}})$ of the class $\mathcal{L}|_{\mathcal{S}}$ is the (pseudo)group of point transformations in the space of $(x, u_{(p)}, \theta)$ that

- are projectable to the space of $(x, u_{(p')})$ for any $0 \leq p' \leq p$,
- are consistent with the contact structure on the space of $(x, u_{(p)})$ and
- map every system from the class $\mathcal{L}|_{\mathcal{S}}$ to another system from the same class.

There exist several generalizations of the notion of equivalence group, in which restrictions for equivalence transformations (projectability or locality with respect to arbitrary elements) are weakened within the point-transformation framework.

generalized equivalence group $G^{\sim}_{\rm gen} = G^{\sim}_{\rm gen}(\mathcal{L}|_{\mathcal{S}})$ [Meleshko, 1994]:
transformation components for x and u may depends on arbitrary elements

$\longrightarrow$ normalization in the generalized sense

$\longrightarrow$ semi-normalization in the generalized sense

$n=2,\ m=1,\ p\geqslant 2,\ u_t=\partial u/\partial t,\ u_k=\partial^k u/\partial x^k,\ \theta=(A^0,\dots,A^p,B,C),$

$$u_t+C(t,x)uu_x=\sum_{k=0}^{p}A^k(t,x)u_k+B(t,x). \qquad (2)$$

$$\mathcal{S}:\quad A^k_{u_\alpha}=0,\ k=0,\dots,p,\quad B_{u_\alpha}=0,\quad C_{u_\alpha}=0,\quad |\alpha|\leqslant p,\quad A^pC\neq 0,$$

where $\alpha=(\alpha_1,\alpha_2)$, $\alpha_1,\alpha_2\in\mathbb{N}\cup\{0\}$, $|\alpha|=\alpha_1+\alpha_2$, and $u_\alpha=\partial^{|\alpha|}u/\partial t^{\alpha_1}\partial x^{\alpha_2}$.

Proposition

The class (2) is normalized. Its equivalence group $G^\sim_{(2)}$ consists of the transformations in the joint space of (t,x,u,θ) whose (t,x,u)-components are of the form

$$\tilde t=T(t),\quad \tilde x=X(t,x),\quad \tilde u=U^1(t)u+U^0(t,x),$$

where $T=T(t)$, $X=X(t,x)$, $U^1=U^1(t)$ and $U^0=U^0(t,x)$ are arbitrary smooth functions of their arguments such that $T_tX_xU^1\neq 0$.

$\tilde C=\dfrac{X_x}{T_tU^1}C$. Gauge: $C=1$. $X=X^1(t)x+X^0(t)$, $U^1=\dfrac{X^1}{T_t}$.

Then $\tilde A^1=\dfrac{X^1}{T_t}A^1+U^0-\dfrac{X^1_tx+X^0_t}{T_t}$. Further gauge: $A^1=0$. $U^0=\dfrac{X^1_tx+X^0_t}{T_t}$.

Both the subclasses with $C=1$ and $(C,A^1)=(1,0)$ are normalized in the usual sense.

General Burgers–Korteweg–de Vries equations

$$u_t + C(t,x)uu_x = \sum_{k=0}^{p} A^k(t,x)u_k + B(t,x), \quad CA^p \neq 0.$$

$$\tilde{t} = T(t), \quad \tilde{x} = X(t,x), \quad \tilde{u} = U^1(t)u + U^0(t,x)$$

Alternative gauge: $A^p = 1$. $X = X^1(t)x + X^0(t)$, where $(X^1)^p = T_t$.
The subclass is normalized in the usual sense.

Further gauge: $A^1 = 0$. $U^0 = \dfrac{X^1_t x + X^0_t}{X^1 C} U^1$.
The subclass with $(A^p, A^1) = (1, 0)$ is not normalized in the usual sense.

It becomes normalized in the generalized sense if formally extend the arbitrary-element tuple θ' with the derivatives of C as new arbitrary elements, $Z^0 := C_t$ and $Z^k := C_k$, $k = 1, \dots, p$, and prolong equivalence transformations to them.

The subclass with $A^k_x=0$, $k=0,\dots,p$, $C_x=0$ and $B_x=0$ needs the extension of θ with $Y^1_t=A^0$, $Y^2_t=Ce^{Y^1}$.
Its equivalence groupoid:

$$\tilde t=T(t),\quad \tilde x=X^1(t)x+X^0(t),\quad \tilde u=U^1(t)u+\frac{X^1_tU^1}{X^1C}x+U^{00}(t),\quad\text{with}$$

$$\left(\frac{X^1_t}{C(X^1)^2}\right)_t=A^0\frac{X^1_t}{C(X^1)^2}\quad\sim\quad X^1=\frac{1}{\varepsilon_1Y^2+\varepsilon_0},$$

where ε_1 and ε_0 are arbitrary constants with $(\varepsilon_1,\varepsilon_0)\neq(0,0)$.
Its generalized equivalence group:

$$\tilde t=\bar T(t,Y^1,Y^2),\quad \tilde x=\bar X^1x+\bar X^0(t,Y^1,Y^2),\quad \bar X^1:=\frac{1}{\varepsilon_1Y^2+\varepsilon_0},$$

$$\tilde u=\bar U^1(t,Y^1,Y^2)\big(u-\varepsilon_1\bar X^1e^{Y^1}x\big)+\bar U^{00}(t,Y^1,Y^2),\dots$$

Its effective generalized equivalence group:

$$\tilde t=T(t),\quad \tilde x=X^1\big(x+X^{01}(t)Y^2+X^{00}(t)\big),\quad X^1:=\frac{1}{\varepsilon_1Y^2+\varepsilon_0},$$

$$\tilde u=V(t)\left(\frac{u}{X^1}-e^{Y^1}(\varepsilon_1x-\varepsilon_0X^{01}+\varepsilon_1X^{00})\right),\dots$$

Example. Variable coefficient nonlinear diffusion–convection equations

$$f(x)u_t = (g(x)A(u)u_x)_x + h(x)B(u)u_x, \quad fghA \neq 0, \quad (A_u, B_u) \neq (0,0)$$

The usual equivalence group $G^{\sim}$ consists of the transformations

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = X(x), \quad \tilde{u} = \delta_3 u + \delta_4,$$
$$\tilde{f} = \frac{\varepsilon_1 \delta_1}{X_x} f, \quad \tilde{g} = \varepsilon_1 \varepsilon_2^{-1} X_x\, g, \quad \tilde{h} = \varepsilon_1 \varepsilon_3^{-1} h, \quad \tilde{A} = \varepsilon_2 A, \quad \tilde{B} = \varepsilon_3 B,$$

where δ_j $(j = 1, \ldots, 4)$ and ε_i $(i = 1, \ldots, 3)$ are arbitrary constants, $\delta_1\delta_3\varepsilon_1\varepsilon_2\varepsilon_3 \neq 0$, X is an arbitrary smooth function of x, $X_x \neq 0$.

The generalized extended equivalence group $\hat{G}^{\sim}$ is formed by the transformations

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = X(x), \quad \tilde{u} = \delta_3 u + \delta_4,$$
$$\tilde{f} = \frac{\varepsilon_1 \delta_1 \varphi}{X_x} f, \quad \tilde{g} = \varepsilon_1 \varepsilon_2^{-1} X_x \varphi\, g, \quad \tilde{h} = \varepsilon_1 \varepsilon_3^{-1} \varphi\, h, \quad \tilde{A} = \varepsilon_2 A, \quad \tilde{B} = \varepsilon_3 (B + \varepsilon_4 A),$$

where δ_j $(j = 1, \ldots, 4)$ and ε_i $(i = 1, \ldots, 4)$ are arbitrary constants, $\delta_1\delta_3\varepsilon_1\varepsilon_2\varepsilon_3 \neq 0$, X is an arbitrary smooth function of x, $X_x \neq 0$, $\varphi = e^{-\varepsilon_4 \int \frac{h(x)}{g(x)} dx}$.

The equivalence group $G^{\sim}$ of the class $\mathcal{L}|_{\mathcal{S}}$ may contain transformations which act only on arbitrary elements and do not really change systems, i.e., which generate gauge admissible transformations.

In general, transformations of this type can be considered as trivial (gauge) equivalence transformations and form the *gauge* subgroup

$$G^{\mathrm{g}\sim} = \{\Phi \in G^{\sim} \mid \Phi x = x,\ \Phi u = u,\ \Phi\theta \overset{g}{\sim} \theta\}$$

of the equivalence group $G^{\sim}$.

Moreover, $G^{\mathrm{g}\sim}$ is a normal subgroup of $G^{\sim}$.

Example. Variable coefficient nonlinear diffusion–convection equations

$$f(x)u_t = (g(x)A(u)u_x)_x + h(x)B(u)u_x, \quad fghA \neq 0, \quad (A_u, B_u) \neq (0,0)$$

The gauge equivalence group $G^{\mathrm{g}\sim}$ consists of the transformations

$$\tilde f = \varepsilon_1\varphi\, f, \quad \tilde g = \varepsilon_1\varepsilon_2^{-1}\varphi\, g, \quad \tilde h = \varepsilon_1\varepsilon_3^{-1}\varphi\, h, \quad \tilde A = \varepsilon_2 A, \quad \tilde B = \varepsilon_3(B + \varepsilon_4 A),$$

where $\varphi = e^{-\varepsilon_4 \int \frac{h(x)}{g(x)}dx}$, ε_i $(i = 1, \ldots, 4)$ are arbitrary constants, $\varepsilon_1\varepsilon_2\varepsilon_3 \neq 0$

Definition

Given a class of differential equations $\mathcal{L}|_{\mathcal{S}}$ with equivalence groupoid $\mathcal{G}^{\sim}$ and (usual) equivalence group $G^{\sim}$, suppose that there exists a normal subgroup H of $G^{\sim}$, and for each $\theta \in \mathcal{S}$ the point symmetry group G_θ of the system $\mathcal{L}_\theta \in \mathcal{L}|_{\mathcal{S}}$ contains a subgroup N_θ such that the family $\mathcal{N}_{\mathcal{S}} = \{N_\theta \mid \theta \in \mathcal{S}\}$ of all these subgroups satisfies the following properties:

1. $H|_{(x,u)} \cap N_\theta = \{\mathrm{id}\}$ for any $\theta \in \mathcal{S}$.
2. $N_{\mathcal{T}\theta} = \mathcal{T}|_{(x,u)} N_\theta (\mathcal{T}|_{(x,u)})^{-1}$ for any $\theta \in \mathcal{S}$ and any $\mathcal{T} \in H$.
3. For any $(\theta^1, \theta^2, \varphi) \in \mathcal{G}^{\sim}$ there exist $\varphi^1 \in N_{\theta^1}$, $\varphi^2 \in N_{\theta^2}$ and $\mathcal{T} \in H$ such that $\theta^2 = \mathcal{T}\theta^1$ and $\varphi = \varphi^2(\mathcal{T}|_{(x,u)})\varphi^1$.

We then say that the class of differential equations $\mathcal{L}|_{\mathcal{S}}$ is *uniformly semi-normalized* with respect to the subgroup H of $G^{\sim}$ and the symmetry-subgroup family $\mathcal{N}_{\mathcal{S}}$.

Here $\mathcal{T}|_{(x,u)}$ and $H|_{(x,u)}$ respectively denote the restrictions of $\mathcal{T}$ and H to the space with local coordinates (x,u), $H|_{(x,u)} = \{\mathcal{T}|_{(x,u)} \mid \mathcal{T} \in H\}$, and id is the identity transformation in this space.

Splitting symmetry groups in uniformly semi-normalized classes

Each normalized class of differential equations is uniformly semi-normalized with respect to the improper subgroup $H = G^{\sim}$ and the trivial family $\mathcal{N}_{\mathcal{S}}$, where for each θ the group N_θ consists of just the identity transformation, $\mathcal{N}_{\mathcal{S}} = \{\{\mathrm{id}\} \mid \theta \in \mathcal{S}\}$.

Each uniformly semi-normalized class is semi-normalized.

At the same time, there are semi-normalized classes that are not uniformly semi-normalized,

Theorem

Suppose that the class of differential equations $\mathcal{L}|_{\mathcal{S}}$ is uniformly semi-normalized with respect to a subgroup H of $G^{\sim}$ and a symmetry-subgroup family $\mathcal{N}_{\mathcal{S}} = \{N_\theta \mid \theta \in \mathcal{S}\}$. Then for each $\theta \in \mathcal{S}$ the point symmetry group G_θ of the system $\mathcal{L}_\theta \in \mathcal{L}|_{\mathcal{S}}$ splits over N_θ. More specifically,

- N_θ is a normal subgroup of G_θ,
- $G^{\mathrm{ess}}_\theta = H|_{(x,u)} \cap G_\theta$ is a subgroup of G_θ, and
- the group G_θ is a semidirect product of G^{ess}_θ acting on N_θ, $G_\theta = G^{\mathrm{ess}}_\theta \ltimes N_\theta$.

Infinitesimal transformations

As the study of point transformations of differential equations usually involves cumbersome and sophisticated calculations, instead of finite point transformations one may consider their infinitesimal counterparts.

This leads to a certain linearization of the related problem which essentially simplifies the whole consideration.

In the framework of the infinitesimal approach, a (pseudo)group G of point transformations is replaced by the Lie algebra $\mathfrak{g}$ of vector fields on the same space, which are generators of one-parametric local subgroups of G.

the point symmetry (pseudo)group $G_\theta \quad \longrightarrow \quad$ the *maximal Lie invariance algebra* $\mathfrak{g}_\theta$ of $\mathcal{L}_\theta$
$=$ the set of vector fields in the space of (x, u) generating one-parametric subgroups of G_θ

the kernel group $G^\cap \quad \longrightarrow \quad \mathfrak{g}^\cap = \mathfrak{g}^\cap(\mathcal{L}|_\mathcal{S}) = \bigcap_{\theta\in\mathcal{S}} \mathfrak{g}_\theta$, the *kernel algebra* of $\mathcal{L}|_\mathcal{S}$

the equivalence group $G^\sim \quad \longrightarrow \quad$ the *equivalence algebra* $\mathfrak{g}^\sim$ of $\mathcal{L}_\theta =$ the set of generators of one-parametric groups of equivalence transformations for the class $\mathcal{L}|_\mathcal{S}$.

These generators are vector fields in the space of $(x, u_{(p)}, \theta)$ which are projectable to the space of $(x, u_{(p')})$ for any $0 \leq p' \leq p$ and whose projections to the space of $(x, u_{(p)})$ are the p-th order prolongations of the corresponding projections to the space of (x, u).

Corollary

The trivial prolongation $\hat{\mathfrak{g}}^\cap$ of $\mathfrak{g}^\cap$ to the arbitrary elements is an ideal in $\mathfrak{g}^\sim$.

Problem of group classification

The solution of the group classification problem by Lie–Ovsiannikov for a class $\mathcal{L}|_\mathcal{S}$ includes the construction of the following elements:

- the equivalence group $G^\sim$ of $\mathcal{L}|_\mathcal{S}$,
- the kernel algebra $\mathfrak{g}^\cap = \bigcap_{\theta\in\mathcal{S}} \mathfrak{g}_\theta$ of $\mathcal{L}|_\mathcal{S}$,
- an exhaustive list of $G^\sim$-equivalent extensions of $\mathfrak{g}^\cap$ in $\mathcal{L}|_\mathcal{S}$, i.e., an exhaustive list of $G^\sim$-equivalent values of θ with the corresponding maximal Lie invariance algebras $\mathfrak{g}_\theta$ for which $\mathfrak{g}_\theta \neq \mathfrak{g}^\cap$.

More precisely, the *classification list* consists of pairs $(\mathcal{S}_\gamma, \{\mathfrak{g}_\theta, \theta\in\mathcal{S}_\gamma\})$, $\gamma\in\Gamma$:

- For each $\gamma\in\Gamma$ $\mathcal{L}|_{\mathcal{S}_\gamma}$ is a subclass of $\mathcal{L}|_\mathcal{S}$,
- $\mathfrak{g}_\theta \neq \mathfrak{g}^\cap$ for any $\theta\in\mathcal{S}_\gamma$ and the structures of the algebras $\mathfrak{g}_\theta$ are similar for all $\theta\in\mathcal{S}_\gamma$; in particular, the algebras $\mathfrak{g}_\theta$, $\theta\in\mathcal{S}_\gamma$, have the same dimension or display the same arbitrariness of algebra parameters in the infinite-dimensional case.
- for any $\theta\in\mathcal{S}$ with $\mathfrak{g}_\theta \neq \mathfrak{g}^\cap$ there exists $\gamma\in\Gamma$ such that $\theta\in\mathcal{S}_\gamma \bmod G^\sim$,
- all elements from $\bigcup_{\gamma\in\Gamma}\mathcal{S}_\gamma$ are $G^\sim$-inequivalent.

In all examples of group classification presented in the literature the set Γ was finite.

If the class $\mathcal{L}|_\mathcal{S}$ is not semi-normalized, the classification list may include equations similar with respect to point transformations which do not belong to $G^\sim$.

The construction of such *additional* equivalences can be considered as one further step of the algorithm of group classification.

Algebraic method of group classification

Underlying facts:

- For each fixed value of the arbitrary elements the solution space of the determining equations is associated with a Lie algebra of vector fields.
- If systems of differential equations are similar with respect to a point transformation then its push-forward relates the corresponding maximal Lie invariance algebras.

This is why any version of the algebraic method involves, in some way, the classification of algebras of vector fields up to certain equivalence induced by point transformations.

The key questions:

- What set of vector fields should be classified?
- What kind of equivalence should be used?

$\mathfrak{g}^{\cup}=\bigcup_{\theta\in\mathcal{S}}\mathfrak{g}_\theta=$ the set of vector fields for which the system of determining equations is consistent with respect to the arbitrary elements with the auxiliary system $\mathcal{S}$ of the class $\mathcal{L}|_\mathcal{S}$.

$\Longrightarrow$ The set $\mathfrak{g}^{\cup}$ can be obtained at the onset of group classification, independently from deriving the maximal Lie invariance algebras of equations from the class $\mathcal{L}|_\mathcal{S}$.

We can extend $\mathfrak{g}^{\cup}$ to its linear span $\mathfrak{g}^{\langle\rangle}=\langle\mathfrak{g}_\theta|\theta\in\mathcal{S}\rangle$. Often $\mathfrak{g}^{\cup}=\mathfrak{g}^{\langle\rangle}$.

Via push-forwarding of vector fields, equivalence (resp. admissible) point transformations for $\mathcal{L}|_\mathcal{S}$ induce an equivalence relation on algebras contained in $\mathfrak{g}^{\cup}$.

We should classify, up to the above equivalence relation, only appropriate algebras. An algebra $\mathfrak{s}$ contained in $\mathfrak{g}^{\cup}$ is called *appropriate* if it is the maximal Lie invariance algebra of an equation from the class $\mathcal{L}|_\mathcal{S}$. The simplest restriction for $\mathfrak{s}$ is $\mathfrak{s}\supset\mathfrak{g}^{\cap}$. The condition that $\mathfrak{s}$ is really maximal Lie invariance algebras for an equation from $\mathcal{L}|_\mathcal{S}$ is more nontrivial to verify.

Substituting the basis elements of each algebra obtained into the determining equations gives a compatible system for values of the arbitrary elements associated with Lie symmetry extensions.

This whole construction is based on the following assertion:

Proposition

Let $\mathcal{S}_i$ be the subset of $\mathcal{S}$ that consists of all arbitrary elements for which the corresponding equations from $\mathcal{L}|_\mathcal{S}$ are invariant with respect to the same algebra of vector fields, $i=1,2$. Then the algebras $\mathfrak{g}^\cap(\mathcal{L}|_{\mathcal{S}_1})$ and $\mathfrak{g}^\cap(\mathcal{L}|_{\mathcal{S}_2})$ are similar with respect to push-forwards of vector fields by transformations from $G^\sim$ (resp. point transformations) if and only if the subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ are mapped to each other by transformations from $G^\sim$ (resp. point transformations).

Proposition

Let $\mathfrak{a}$ be a subalgebra of the equivalence algebra $\mathfrak{g}^{\sim}$ of the class $\mathcal{L}|_{\mathcal{S}}$, $\mathfrak{a} \subset \mathfrak{g}^{\sim}$, and let $\theta^0(x, u_{(r)}) \in \mathcal{S}$ be a value of the tuple of arbitrary elements θ for which the algebraic equation $\theta = \theta^0(x, u_{(r)})$ is invariant with respect to $\mathfrak{a}$. Then the differential equation $\mathcal{L}_{\theta^0}$ is invariant with respect to the projection of $\mathfrak{a}$ to the space of variables (x, u).

Proposition

Let $\mathcal{S}_i$ be the subset of $\mathcal{S}$ that consists of tuples of arbitrary elements for which the corresponding algebraic equations are invariant with respect to the same subalgebra of $\mathfrak{g}^{\sim}$ and let $\mathfrak{a}_i$ be the maximal subalgebra of $\mathfrak{g}^{\sim}$ for which $\mathcal{S}_i$ satisfies this property, $i = 1, 2$. Then the subalgebras $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are equivalent with respect to the adjoint action of $G^{\sim}$ if and only if the subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ are mapped to each other by transformations from $G^{\sim}$.

$\Longrightarrow$ To construct particular cases of symmetry extensions, we can classify subalgebras of $\mathfrak{g}^{\sim}$ instead of algebras of vector fields contained in $\mathfrak{g}^{\cup}$.

This procedure is called *preliminary group classification* [Akhatov, Gazizov, Ibragimov, 1989].

$u_{tt} = f(x, u_x)u_{xx} + g(x, u_x)$

- Ibragimov N.H., Torrisi M. and Valenti A., Preliminary group classification of equations $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$, *J. Math. Phys.* **32** (1991), 2988–2995.
- Harin A.O., On a countabe-dimensional subalgebra of the equivalence algebra for equations $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$, *J. Math. Phys.* **34** (1993), 3676–3682.
- Ibragimov N.H. and Khabirov S.V., Contact transformation group classification of nonlinear wave equations, *Nonlin. Dyn.* **22** (2000), 61–71.
- Morozov O., Contact Equivalence Problem for Nonlinear Wave Equations. arxiv:math-ph/0306007.
- Ibragimov N.H., Torrisi M. and Valenti A., Differential invariants of nonlinear equations $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$, *Commun. Nonlinear Sci. Numer. Simul.* **9** (2004), 69–80.
- Khabirov S.V., A property of the determining equations for an algebra in the group classification problem for wave equations, *Sibirsk. Mat. Zh.* **50** (2009), 647–668, in Russian; translation in *Sib. Math. J.*, 50:515–532, 2009.

This talk:

- Bihlo A., Dos Santos Cardoso-Bihlo E. and Popovych R.O., Complete group classification of a class of nonlinear wave equations, J. Math. Phys. 53 (2012), 123515, 32 pp., arXiv:1106.4801.

$$\mathcal{W}\colon\quad u_{tt}=f(x,u_x)u_{xx}+g(x,u_x),\quad f\neq0,\quad (f_{u_x},g_{u_xu_x})\neq(0,0)$$

The equivalence algebra $\mathfrak{g}^{\sim}$:

$$\mathcal{D}^u=u\partial_u+u_x\partial_{u_x}+g\partial_g,\quad \mathcal{D}^t=t\partial_t-2f\partial_f-2g\partial_g,\quad \mathcal{P}^t=\partial_t,$$
$$\mathcal{D}(\varphi)=\varphi\partial_x-\varphi_xu_x\partial_{u_x}+2\varphi_xf\partial_f+\varphi_{xx}u_xf\partial_g,$$
$$\mathcal{G}(\psi)=\psi\partial_u+\psi_x\partial_{u_x}-\psi_{xx}f\partial_g,\quad \mathcal{F}^1=t\partial_u,\quad \mathcal{F}^2=t^2\partial_u+2\partial_g,$$

where $\varphi=\varphi(x)$ and $\psi=\psi(x)$ run through the set of smooth functions of x

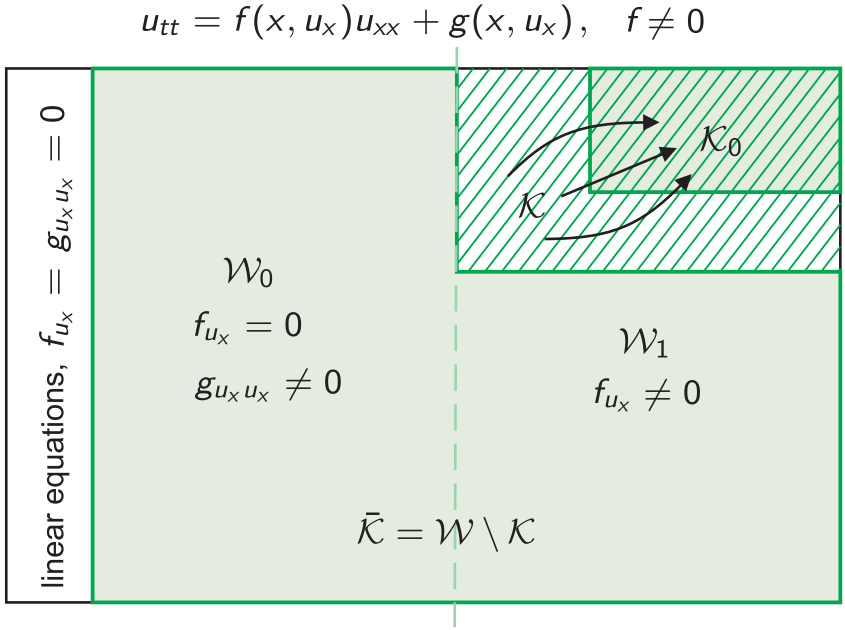
Theorem

The equivalence group $G^{\sim}=G^{\sim}(\mathcal{W})$ of the class $\mathcal{W}$ consists of the transformations

$$\tilde t=c_1t+c_0,\quad \tilde x=\varphi(x),\quad \tilde u=c_2u+c_4t^2+c_3t+\psi(x),$$
$$\tilde u_{\tilde x}=\frac{c_2u_x+\psi_x}{\varphi_x},\quad \tilde f=\frac{\varphi_x^2}{c_1^2}f,\quad \tilde g=\frac{1}{c_1^2}\left(c_2g+\frac{c_2u_x+\psi_x}{\varphi_x}\varphi_{xx}f-\psi_{xx}f+2c_4\right),$$

where c_0, …, c_4 are arbitrary constants satisfying the condition $c_1c_2\neq0$ and φ and ψ run through the set of smooth functions of x, $\varphi_x\neq0$.

The class $\mathcal{W}$ admits three independent discrete equivalence transformations:
$(t,x,u,f,g)\quad\mapsto(-t,x,u,f,g),\quad\mapsto(t,-x,u,f,g)$ and $\mapsto(t,x,-u,f,-g)$.



$u_{tt} = f(x, u_x)u_{xx} + g(x, u_x)\,, \quad f \neq 0$
linear equations, $f_{u_x} = g_{u_x u_x} = 0$
$\mathcal{K}_0$
$\mathcal{K}$
$\mathcal{W}_0$
$f_{u_x} = 0$
$g_{u_x u_x} \neq 0$
$\mathcal{W}_1$
$f_{u_x} \neq 0$
$\bar{\mathcal{K}} = \mathcal{W} \setminus \mathcal{K}$

Nonlinear wave equations

Two subclasses of $\mathcal{W}$: $\mathcal{W}_0 \sim f_{u_x} = 0,\ g_{u_x u_x} \neq 0$ and $\mathcal{W}_1 \sim f_{u_x} \neq 0$

Theorem

The subclass $\mathcal{W}_0$ is normalized and saved by any transformation from $\mathrm{T}(\mathcal{W})$. $G^{\sim}(\mathcal{W}_0) = G^{\sim}$.

$$\mathcal{K}_0 = \{u_{tt} = \pm u_x^{-4} u_{xx} + \mu(x) u_x^{-3}\} \quad \subset \quad \mathcal{K} \sim (f, g) = (\pm u_x^{-4}, \mu(x) u_x^{-3}) \bmod G^{\sim} \quad \subset \quad \mathcal{W}_1$$

Theorem

The subclass $\mathcal{K}_0$ is normalized. The subclass $\mathcal{K}$ is semi-normalized with respect to $G^{\sim}$. Any admissible transformation in $\mathcal{K}$ is generated by $G^{\sim}$ or is represented as a composition of $(\theta_1, \theta_2, \mathcal{T}_1)$, $(\theta_2, \theta_2, \mathcal{T}_2)$ and $(\theta_2, \theta_3, \mathcal{T}_3)$, where $\theta_1 = (f, g)$, $\theta_2 = (\pm u_x^{-4}, \mu u_x^{-3})$, $\theta_3 = (\tilde f, \tilde g)$ and $\mathcal{T}_1$, $\mathcal{T}_3$ are equivalence transformations and $\mathcal{T}_2$: $\tilde t = 1/t$, $\tilde x = x$, $\tilde u = u/t$ is a symmetry transformation of $\mathcal{L}_{\theta_2}$.

Theorem

The complement $\bar{\mathcal{K}}_1 = \mathcal{W}_1 \setminus \mathcal{K}$ and, therefore, $\bar{\mathcal{K}} = \mathcal{W} \setminus \mathcal{K}$ are normalized with respect to $G^{\sim}$. $G^{\sim}(\bar{\mathcal{K}})_1 = G^{\sim}(\bar{\mathcal{K}}) = G^{\sim}$.

Corollary

The entire class $\mathcal{W}$ is semi-normalized. Hence

group classification of $\mathcal{W}$ up to $G^{\sim}$-equivalence	$\sim$	group classification of $\mathcal{W}$ up to general point equivalence

Conclusion

group classification of $\mathcal{W}$ $=$
group classification of $\mathcal{K}_0$ $+$ group classification of $\bar{\mathcal{K}}$

The subclasses $\mathcal{K}_0$ and $\bar{\mathcal{K}}$ are normalized. $\Longrightarrow$

We can carry out complete group classification by the algebraic method!

Thank you for your attention!