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Lagrangians with reduced-order Euler-Lagrange equations

D.J. Saunders Ostrava, June 2017



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Abstract

Any Lagrangian form of order k obtained by horizontalization of a form of order k-1 gives rise to Euler-Lagrange equations of order strictly less than 2k.

But these are not the only possibilities. For example, with two independent variables, the horizontalization of a first-order 2-form gives a Lagrangian quadratic in the second-order variables; but there are also cubic second-order Lagrangians with third-order Euler-Lagrange equations.

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Abstract (continued)

Ab

In this talk I shall show first that any Lagrangian of order k with Euler-Lagrange equations of order less than 2k must be a polynomial in the k-th order variables of order not greater than the number of different symmetric multi-indices of length k.

I shall then describe a geometrical construction, based on Peter Olver's idea of differential hyperforms, which gives rise to Lagrangians with reduced-order Euler-Lagrange equations.

I believe (and might be able to prove, though this is not guaranteed!) that every such Lagrangian arises in this way.



The Euler-Lagrange equations

Let L be a Lagrangian in a single independent variable x, n independent variables u^{α} , and n derivative variables u_x^{α} .

The Euler-Lagrange equations are

$$\frac{\partial L}{\partial u^{\beta}} - \frac{d}{dx} \frac{\partial L}{\partial u_x^{\beta}} = 0$$

and expanding the total derivative d/dx gives

$$\frac{\partial L}{\partial u^{\beta}} - \frac{\partial^2 L}{\partial x \partial u_x^{\beta}} - u_x^{\alpha} \frac{\partial^2 L}{\partial u^{\alpha} \partial u_x^{\beta}} - u_{xx}^{\alpha} \frac{\partial^2 L}{\partial u_x^{\alpha} \partial u_x^{\beta}}$$

In general these equations are second-order, but if L is linear in the variables u^α_x then they are first-order.

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The Euler-Lagrange equations (2)

Now suppose there are m independent variables x^i , n independent variables u^{α} , and mn derivative variables u^{α}_i .

The Euler-Lagrange equations are now

$$\frac{\partial L}{\partial u^{\beta}} - \frac{d}{dx^{j}} \frac{\partial L}{\partial u_{j}^{\beta}} = 0$$

and expanding the total derivative d/dx^j now gives

$$\frac{\partial L}{\partial u^{\beta}} - \frac{\partial^2 L}{\partial x^j \partial u_j^{\beta}} - u_j^{\alpha} \frac{\partial^2 L}{\partial u^{\alpha} \partial u_j^{\beta}} - u_{ij}^{\alpha} \frac{\partial^2 L}{\partial u_i^{\alpha} \partial u_j^{\beta}}$$

In general these equations are second-order, but if L is linear in the variables u_i^α then they are first-order. But \ldots

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The Euler–Lagrange equations (3)

$$\frac{\partial L}{\partial u^{\beta}} - \frac{\partial^2 L}{\partial x^j \partial u^{\beta}_j} - u^{\alpha}_j \frac{\partial^2 L}{\partial u^{\alpha} \partial u^{\beta}_j} - u^{\alpha}_{ij} \frac{\partial^2 L}{\partial u^{\alpha}_i \partial u^{\beta}_j}$$

The equations can be first-order even when L is not linear: for example $L=f(x,u)\big(u_i^\alpha u_j^\beta-u_j^\alpha u_i^\beta\big)$

These Lagrangians come from the geometric construction of *horizontalization* on jet bundles:

with a fibred manifold $\pi: E \to M$, any differential form ω on Egives a horizontal differential form $h(\omega)$ on $J^1\pi$

For instance, $h(du^lpha \wedge du^eta) = (u^lpha_i u^eta_j - u^lpha_j u^eta_i) dx^i \wedge dx^j$

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The Euler-Lagrange equations (4)

The same applies for higher-order Lagrangians.

If the Lagrangian L has order k, the Euler–Lagrange equations are generically of order 2k

$$\sum_{|I|=0}^k (-1)^{|I|} \frac{d^{|I|}}{dx^I} \frac{\partial L}{\partial u_I^\beta} = 0$$

where $I \in \mathbb{N}^k$ is a symmetric multi-index:

if
$$u_{I}^{\beta}=u_{i_{1}i_{2}\cdots i_{k}}^{\beta}$$
 then $I(i)=|\{i_{r}:i_{r}=i\}|$

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The Euler-Lagrange equations (4)

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if
$$u_{I}^{\beta}=u_{i_{1}i_{2}\cdots i_{k}}^{\beta}$$
 then $I(i)=|\{i_{r}:i_{r}=i\}|$

The geometry of the multi-index space is important:

$$|I| = \sum_{i=1}^{m} I(i)$$
 is the *length* of *I*;
 $||I||^2 = \sum_{i=1}^{m} (I(i))^2$ is the square Euclidean norm of *I*

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Reduced-order Euler-Lagrange equations

$$\sum_{|J|=0}^{k} (-1)^{|J|} \frac{d^{|J|}}{dx^J} \frac{\partial L}{\partial u_J^\beta} = 0$$

Each total derivative d/dx^j increases the order of its argument by one, so that the terms of order 2k come from

$$\sum_{|J|=k} (-1)^k \frac{d^{|J|}}{dx^J} \frac{\partial L}{\partial u_J^\beta} \quad \text{and equal} \quad \sum_{|I|=|J|=k} (-1)^k u_{I+J}^\alpha \frac{\partial^2 L}{\partial u_I^\alpha \partial u_J^\beta}$$

The equations will have order less than 2k if, and only if, for each multi-index H of length 2k,

$$\sum_{I+J=H} \frac{\partial^2 L}{\partial u_I^\alpha \partial u_J^\beta} = 0$$

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The polynomial condition

Euler-Lagrange equations:

$$\sum_{J|=0}^{k} (-1)^{|J|} \frac{d^{|J|}}{dx^J} \frac{\partial L}{\partial u_J^\beta} = 0$$

Condition for lower order equations: whenever |H| = 2k then

$$\sum_{I+J=H} \frac{\partial^2 L}{\partial u_I^\alpha \partial u_J^\beta} = 0$$

Theorem

A necessary condition for the Euler-Lagrange equations to have order less than 2k is that L is a polynomial in the highest-order derivatives u_I^{α} , |I| = k, of order at most p_k

where p_k is the number of distinct multi-indices of length k

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Proof of the polynomial condition

A necessary condition for the Euler-Lagrange equations to have order less than 2k is that L is a polynomial in the highest-order derivatives u_{I}^{α} , |I| = k, of order at most p_k

Consider

$$\frac{\partial^{p_k+1}L}{\partial u_{J_1}^{\alpha_1}\cdots \partial u_{J_{p_k}}^{\alpha_{p_k}}\partial u_{J_{p_k+1}}^{\alpha_{p_k+1}}}$$

so at least two of the multi-indices must be the same — say $J_1=J_2\,$

Use the condition
$$\sum_{I+J=H} \frac{\partial^2 L}{\partial u_I^{\alpha} \partial u_J^{\beta}} = 0 \text{ to put}$$
$$\frac{\partial^{p_k+1}L}{\partial u_{J_1}^{\alpha_1} \partial u_{J_2}^{\alpha_2} \partial u_{J_3}^{\alpha_3} \cdots} = \sum_{\substack{K_1+K_2=J_1+J_2\\(K_1,K_2) \neq (J_1,J_2)}} -\frac{\partial^{p_k+1}L}{\partial u_{K_1}^{\alpha_1} \partial u_{K_2}^{\alpha_2} \partial u_{J_3}^{\alpha_3} \cdots}$$

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Proof of the polynomial condition (2)

$$\frac{\partial^{p_{k}+1}L}{\partial u_{J_{1}}^{\alpha_{1}}\partial u_{J_{2}}^{\alpha_{2}}\partial u_{J_{3}}^{\alpha_{3}}\cdots} = \sum_{\substack{K_{1}+K_{2}=J_{1}+J_{2}\\(K_{1},K_{2})\neq(J_{1},J_{2})}} -\frac{\partial^{p_{k}+1}L}{\partial u_{K_{1}}^{\alpha_{1}}\partial u_{K_{2}}^{\alpha_{2}}\partial u_{J_{3}}^{\alpha_{3}}\cdots}$$

But each term on the RHS also has a repeated multi-index! So we can continue ...

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Proof of the polynomial condition (2)

$$\frac{\partial^{p_{k}+1}L}{\partial u_{J_{1}}^{\alpha_{1}}\partial u_{J_{2}}^{\alpha_{2}}\partial u_{J_{3}}^{\alpha_{3}}\cdots} = \sum_{\substack{K_{1}+K_{2}=J_{1}+J_{2}\\(K_{1},K_{2})\neq(J_{1},J_{2})}} -\frac{\partial^{p_{k}+1}L}{\partial u_{K_{1}}^{\alpha_{1}}\partial u_{K_{2}}^{\alpha_{2}}\partial u_{J_{3}}^{\alpha_{3}}\cdots}$$

But each term on the RHS also has a repeated multi-index! So we can continue ...

But eventually, every term will have a repeated 'pure' multi-index J (where J(j) = k for some j, and J(j) = 0 for $i \neq j$)

and then $\sum_{J+J=H} \frac{\partial^2 L}{\partial u^\alpha_J \partial u^\beta_J} = 0$ implies that

$$\frac{\partial^2 L}{\partial u_J^\alpha \partial u_J^\beta} = 0$$

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Proof of the polynomial condition (3)

Why is this process guaranteed to terminate?



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Proof of the polynomial condition (3)

Why is this process guaranteed to terminate?

The parallellogram rule for Euclidean norms!

$$2\|x\|^{2} \leq 2\|x\|^{2} + 2\|y\|^{2} = \|x + y\|^{2} + \|x - y\|^{2}$$

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with equality when y = 0,

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Proof of the polynomial condition (3)

Why is this process guaranteed to terminate?

The parallellogram rule for Euclidean norms!

$$2\|x\|^{2} \leq 2\|x\|^{2} + 2\|y\|^{2} = \|x + y\|^{2} + \|x - y\|^{2}$$

with equality when y = 0, so that in

$$\frac{\partial^{p_k+1}L}{\partial u_J^{\alpha_1}\partial u_J^{\alpha_2}\partial u_{J_3}^{\alpha_3}\cdots} = \sum_{\substack{K_1+K_2=J+J\\(K_1,K_2)\neq (J,J)}} -\frac{\partial^{p_k+1}L}{\partial u_{K_1}^{\alpha_1}\partial u_{K_2}^{\alpha_2}\partial u_{J_3}^{\alpha_3}\cdots}$$

we have $||J||^2 + ||J||^2 = 2||J||^2 < ||K_1||^2 + ||K_2||^2$

The sum of the square Euclidean norms in the terms keeps increasing, eventually giving k+1 pure multi-indices per term

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Proof of the polynomial condition (4)

Therefore

$$\frac{\partial^{p_k+1}L}{\partial u_{J_1}^{\alpha_1}\cdots \partial u_{J_{p_k}}^{\alpha_{p_k}}\partial u_{J_{p_k+1}}^{\alpha_{p_k+1}}} = 0$$

so that L is a polynomial in the u_{J}^{α} , |J| = k, of degree at most p_k .

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Proof of the polynomial condition (4)

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so that L is a polynomial in the u^{lpha}_J , |J|=k, of degree at most p_k .

But this necessary condition is not sufficient: for instance, $L = (u_{xy})^2$ has Euler–Lagrange equations $2u_{xxyy} = 0$

All the Lagrangians with lower-order equations appear to be determinants

Geometrically, determinants arise as the coefficients of wedge products $dx \wedge dy \wedge dz \wedge \cdots$

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Proof of the polynomial condition (4)

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All the Lagrangians with lower-order equations appear to be determinants

Geometrically, determinants arise as the coefficients of wedge products $dx \wedge dy \wedge dz \wedge \cdots$

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... but also as coefficients of $dx^2 \wedge dx dy \wedge dy^2 \wedge \cdots$

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Differential hyperforms

Differential hyperforms were described in an unpublished paper by Peter Olver from 1982

They are covariant tensors with symmetry properties described by Young diagrams (ordinary differential forms are purely alternating, but hyperforms can have more complicated symmetries)

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Differential hyperforms

Differential hyperforms were described in an unpublished paper by Peter Olver from 1982

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Consider hyperforms on jet manifolds $J^k\pi$ that are

- horizontal over M, and
- wedge products of symmetric tensors (all of the same rank)

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Consider hyperforms on jet manifolds $J^k\pi$ that are

- horizontal over M, and
- wedge products of symmetric tensors (all of the same rank)

A (p,q) hyperform is a section of $\bigwedge^p S^q T^*M,$ pulled back to $J^k\pi$

These are generated over $C^{\infty}(J^k\pi)$ by $dx^{I_1} \wedge dx^{I_2} \wedge \cdots \wedge dx^{I_p}$ where $dx^I = dx^{i_1}dx^{i_2}\cdots dx^{i_q}$ with $I = (i_1, i_2, \cdots, i_q)$

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Affine (1,q) hyperforms

A (1,q) hyperform $(1\leq q\leq k)$ is a horizontal symmetric tensor $\theta:J^k\pi\to S^qT^*M$

As $J^k \pi \to J^{k-1} \pi$ is an affine bundle, we say that θ is an *affine* (1,q) hyperform if its restriction to each fibre of the bundle is an affine map: in coordinates

$$\theta = \sum_{\substack{|I|=k\\|\mathcal{J}|=q}} \left(\theta^{I}_{\alpha\mathcal{J}} u^{\alpha}_{I} + \theta_{\mathcal{J}} \right) dx^{\mathcal{J}}$$

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Affine (1,q) hyperforms

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$$\theta = \sum_{\substack{|I|=k\\|\mathcal{J}|=q}} \left(\theta^{I}_{\alpha\mathcal{J}} u^{\alpha}_{I} + \theta_{\mathcal{J}} \right) dx^{\mathcal{J}}$$

These affine (1, q) hyperforms are too general. We shall restrict attention to special affine (1, q) hyperforms

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Special affine (1,q) hyperforms

The affine bundle $J^k\pi\to J^{k-1}\pi$ has associated vector bundle $V\pi\otimes S^kT^*M\to J^{k-1}\pi$

The fibre-affine map θ has an associated fibre-linear 'difference map' $\bar{\theta}: V\pi\otimes S^kT^*M\to S^qT^*M$

We say that θ is a special affine (1,q) hyperform if there is a tensor $\tilde{\theta} \in V\pi^* \otimes S^{k-q}TM$ such that the difference map $\bar{\theta}$ is given by contraction of elements of its domain $V\pi \otimes S^kT^*M$ with $\tilde{\theta}$.

In coordinates (where $heta^I_lpha$ are the coordinates of $ilde{ heta}$)

$$\theta = \sum_{\substack{|I|=k-q\\|\mathcal{J}|=q}} \left(\theta^{I}_{\alpha} u^{\alpha}_{I+\mathcal{J}} + \theta_{\mathcal{J}} \right) dx^{\mathcal{J}}$$

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Special affine (1, q) hyperforms — example

$$\theta = \sum_{\substack{|I|=k-q\\|\mathcal{J}|=q}} \left(\theta^{I}_{\alpha} u^{\alpha}_{I+\mathcal{J}} + \theta_{\mathcal{J}} \right) dx^{\mathcal{J}}$$

In the special case where q = 1 we have

$$\theta = \sum_{|I|=k-1} \left(\theta^I_\alpha u^\alpha_{I+1_j} + \theta_j \right) dx^j$$

the ordinary horizontalization of the 1-form $\sum_{|I|=k-1}\theta^I_\alpha du^\alpha_I+\theta_j dx^j$

There is no invariant operation of horizontalization for hyperforms when $q \ge 2$; but special affine (1, q) hyperforms generalize the images of the horizontalization operator on ordinary 1-forms

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Hyperaffine (p_q, q) hyperforms

A (p_q,q) hyperform ω is a section of the line bundle $\bigwedge^{p_q} S^q T^*M$, pulled back to $J^k\pi$

It is *hyperaffine* if it is generated by wedge products of special affine hyperforms $\theta = \sum_{|I|=k-q, |\mathcal{J}|=q} (\theta^I_{\alpha} u^{\alpha}_{I+\mathcal{J}} + \theta_{\mathcal{J}}) dx^{\mathcal{J}}$

If $\omega = \omega_q dx^{\mathcal{J}_1} \wedge dx^{\mathcal{J}_2} \wedge \cdots \wedge dx^{\mathcal{J}_{pq}}$ then ω_q is a linear combination of determinants (or their minors)

$$\begin{vmatrix} u_{I_1+\mathcal{J}_1}^{\alpha_1} & u_{I_1+\mathcal{J}_2}^{\alpha_1} & \cdots & u_{I_1+\mathcal{J}_{p_q}}^{\alpha_1} \\ u_{I_2+\mathcal{J}_1}^{\alpha_2} & u_{I_2+\mathcal{J}_2}^{\alpha_2} & \cdots & u_{I_2+\mathcal{J}_{p_q}}^{\alpha_2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{I_{p_q}+\mathcal{J}_1}^{\alpha_{p_q}} & u_{I_{p_q}+\mathcal{J}_2}^{\alpha_{p_q}} & \cdots & u_{I_{p_q}+\mathcal{J}_{p_q}}^{\alpha_{p_1}} \end{vmatrix}$$

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What does this have to do with Lagrangians? A Lagrangian *m*-form λ defines local Lagrangian functions *L* by

 $\lambda = L \, dx^1 \wedge dx^2 \wedge \dots \wedge dx^m$

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What does this have to do with Lagrangians? A Lagrangian *m*-form λ defines local Lagrangian functions *L* by

$$\lambda = L \, dx^1 \wedge dx^2 \wedge \dots \wedge dx^m$$

Say that λ is *hyperaffine* if, in any coordinate system,

$$L = \omega_1 + \omega_2 \dots + \omega_k$$

where each ω_a is the coefficient of a hyperaffine hyperform

$$\omega = \omega_q dx^{\mathcal{J}_1} \wedge dx^{\mathcal{J}_2} \wedge \dots \wedge dx^{\mathcal{J}_{p_q}}$$

This is independent of the coordinate system In new coordinates (\tilde{x}, \tilde{u}) , the volume $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m$ changes by the Jacobian determinant $J(\tilde{x}, x)$, whereas each hypervolume $dx^{\mathcal{J}_1} \wedge dx^{\mathcal{J}_2} \wedge \cdots \wedge dx^{\mathcal{J}_{p_q}}$ changes by a power of $J(\tilde{x}, x)$ うして ふゆう ふほう ふほう うらつ

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Euler-Lagrange equations of hyperaffine Lagrangians **Theorem**

If L is the Lagrangian function of a hyperaffine Lagrangian then the Euler-Lagrange equations have reduced order

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Euler-Lagrange equations of hyperaffine Lagrangians **Theorem**

If L is the Lagrangian function of a hyperaffine Lagrangian then the Euler-Lagrange equations have reduced order

It is sufficient to show this for a determinant

$$\Delta = \begin{vmatrix} u_{I_1+\mathcal{J}_1}^{\alpha_1} & u_{I_1+\mathcal{J}_2}^{\alpha_1} & \cdots & u_{I_1+\mathcal{J}_{pq}}^{\alpha_1} \\ u_{I_2+\mathcal{J}_1}^{\alpha_2} & u_{I_2+\mathcal{J}_2}^{\alpha_2} & \cdots & u_{I_2+\mathcal{J}_{pq}}^{\alpha_2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{I_{pq}+\mathcal{J}_1}^{\alpha_{pq}} & u_{I_{pq}+\mathcal{J}_2}^{\alpha_{pq}} & \cdots & u_{I_{pq}+\mathcal{J}_{pq}}^{\alpha_{p1}} \end{vmatrix}$$

so write Δ as

$$\Delta = \sum_{\sigma \in \mathfrak{S}_h} \varepsilon_{\sigma} u_{I_1 + \mathcal{J}_{\sigma(1)}}^{\alpha_1} u_{I_2 + \mathcal{J}_{\sigma(2)}}^{\alpha_2} \cdots u_{I_h + \mathcal{J}_{\sigma(h)}}^{\alpha_h}$$

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Euler-Lagrange equations of hyperaffine Lagrangians (2)

$$\Delta = \sum_{\sigma \in \mathfrak{S}_h} \varepsilon_{\sigma} u_{I_1 + \mathcal{J}_{\sigma(1)}}^{\alpha_1} u_{I_2 + \mathcal{J}_{\sigma(2)}}^{\alpha_2} \cdots u_{I_h + \mathcal{J}_{\sigma(h)}}^{\alpha_h}$$

Substituting in the Euler-Lagrange equations gives

$$\sum_{|\mathbf{K}|=k} \frac{d^{|\mathbf{K}|}}{dx^{\mathbf{K}}} \frac{\partial L}{\partial u_{\mathbf{K}}^{\beta}} = \sum_{\substack{1 \le r,s \le h \\ s \ne r}} \sum_{\sigma \in \mathfrak{S}_{h}} \delta_{\beta}^{\alpha_{r}} \varepsilon_{\sigma} \Phi_{rs\sigma} u_{I_{r}+I_{s}+\mathcal{J}_{\sigma(r)}+\mathcal{J}_{\sigma(s)}}^{\alpha_{s}}$$

where the coefficients $\Phi_{rs\sigma}$ are

$$\Phi_{rs\sigma} = u_{I_1 + \mathcal{J}_{\sigma(1)}}^{\alpha_1} u_{I_2 + \mathcal{J}_{\sigma(2)}}^{\alpha_2} \cdots \widehat{r} \cdots \widehat{s} \cdots u_{I_h + \mathcal{J}_{\sigma(h)}}^{\alpha_h}$$

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Euler-Lagrange equations of hyperaffine Lagrangians (2)

$$\Delta = \sum_{\sigma \in \mathfrak{S}_h} \varepsilon_{\sigma} u_{I_1 + \mathcal{J}_{\sigma(1)}}^{\alpha_1} u_{I_2 + \mathcal{J}_{\sigma(2)}}^{\alpha_2} \cdots u_{I_h + \mathcal{J}_{\sigma(h)}}^{\alpha_h}$$

Substituting in the Euler-Lagrange equations gives

$$\sum_{|\mathbf{K}|=k} \frac{d^{|\mathbf{K}|}}{dx^{\mathbf{K}}} \frac{\partial L}{\partial u_{\mathbf{K}}^{\beta}} = \sum_{\substack{1 \le r, s \le h \\ s \ne r}} \sum_{\sigma \in \mathfrak{S}_{h}} \delta_{\beta}^{\alpha_{r}} \varepsilon_{\sigma} \Phi_{rs\sigma} u_{I_{r}+I_{s}+\mathcal{J}_{\sigma(r)}+\mathcal{J}_{\sigma(s)}}^{\alpha_{s}}$$

where the coefficients $\Phi_{rs\sigma}$ are

$$\Phi_{rs\sigma} = u_{I_1 + \mathcal{J}_{\sigma(1)}}^{\alpha_1} u_{I_2 + \mathcal{J}_{\sigma(2)}}^{\alpha_2} \cdots \widehat{r} \cdots \widehat{s} \cdots u_{I_h + \mathcal{J}_{\sigma(h)}}^{\alpha_h}$$

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Fix $r \neq s$. Given $\sigma \in \mathfrak{S}_h$ put $\tilde{\sigma} = \sigma \circ (r, s) \neq \sigma$. $\Phi_{rs\sigma} = \Phi_{rs\tilde{\sigma}}$ and $\varepsilon_{\sigma} = -\varepsilon_{\tilde{\sigma}}$ so all the terms cancel.

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Determinants

Established so far:

- If a Lagrangian function of order k has reduced-order Euler-Lagrange equations then it is a polynomial of order at most p_k in the variables u_H^{α} (|H| = k);
- Every hyperaffine Lagrangian has reduced-order Euler-Lagrange equations (and is a polynomial with a particular determinant structure)

I conjecture that every Lagrangian with reduced-order Euler-Lagrange equations has this particular determinant structure, and so is hyperaffine.

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Determinants (2)

A general polynomial Lagrangian function of order k and degree p_k is

$$L = \sum_{r=0}^{p_k} A^{H_1 H_2 \cdots H_r}_{\alpha_1 \alpha_2 \cdots \alpha_r} u^{\alpha_1}_{H_1} u^{\alpha_2}_{H_2} \cdots u^{\alpha_r}_{H_r}$$

with implicit sums over the indices and multi-indices, and with $\left| H \right| = k$

Can this be written as a linear combination of determinants

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Determinants (3)

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 $\text{Consider homogeneous polynomials } A^{H_1H_2\cdots H_r}_{\alpha_1\alpha_2\cdots\alpha_r}u^{\alpha_1}_{H_1}u^{\alpha_2}_{H_2}\cdots u^{\alpha_r}_{H_r}$

In the case r = 2 there is a constructive proof

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Determinants (3)

Consider homogeneous polynomials $A^{H_1H_2\cdots H_r}_{\alpha_1\alpha_2\cdots\alpha_r}u^{\alpha_1}_{H_1}u^{\alpha_2}_{H_2}\cdots u^{\alpha_r}_{H_r}$ In the case r=2 there is a constructive proof

Partition the quadratic terms by $H_1 + H_2 = H$ and put

$$\psi_H = \sum_{H_1 + H_2 = H} A_{\alpha_1 \alpha_2}^{H_1 H_2} u_{H_1}^{\alpha_1} u_{H_2}^{\alpha_2}$$

Choose a term $A^{K_1K_2}_{lpha_1lpha_2} u^{lpha_1}_{K_1} u^{lpha_2}_{K_2}$ arbitrarily, so from E–L we have

$$A_{\alpha_1\alpha_2}^{K_1K_2} = \sum_{H_1 + H_2 = H, (H_1, H_2) \neq (K_1, K_2)} -A_{\alpha_1\alpha_2}^{H_1H_2}$$

and so

$$\psi_H = \sum_{H_1 + H_2 = H} A_{\alpha_1 \alpha_2}^{H_1 H_2} (u_{H_1}^{\alpha_1} u_{H_2}^{\alpha_2} - u_{K_1}^{\alpha_1} u_{K_2}^{\alpha_2})$$

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Determinants (4)

For cubic and higher terms, there is no obvious algorithm to give an explicit construction

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(although ad-hoc methods work for all examples investigated)

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Determinants (4)

For cubic and higher terms, there is no obvious algorithm to give an explicit construction

(although ad-hoc methods work for all examples investigated)

A possible approach would use an abstract dimension argument:

The number of variables u_I^{α} , |I| = k, is known, and so the dimension of the space of homogeneous polynomials of degree r is also known

The number of E–L constraints for quadratic polynomials is known, so the number of constraints for degree r polynomials can in principle be calculated

The theorem will be proved if there are enough independent $r \times r$ determinants of the correct type