# Lagrangians with reduced-order Euler-Lagrange equations 

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## Abstract

Any Lagrangian form of order $k$ obtained by horizontalization of a form of order $k-1$ gives rise to Euler-Lagrange equations of order strictly less than $2 k$.

But these are not the only possibilities. For example, with two independent variables, the horizontalization of a first-order 2-form gives a Lagrangian quadratic in the second-order variables; but there are also cubic second-order Lagrangians with third-order Euler-Lagrange equations.

## Abstract (continued)

In this talk I shall show first that any Lagrangian of order $k$ with Euler-Lagrange equations of order less than $2 k$ must be a polynomial in the $k$-th order variables of order not greater than the number of different symmetric multi-indices of length $k$.

I shall then describe a geometrical construction, based on Peter Olver's idea of differential hyperforms, which gives rise to Lagrangians with reduced-order Euler-Lagrange equations.

I believe (and might be able to prove, though this is not guaranteed!) that every such Lagrangian arises in this way.

## The Euler-Lagrange equations

Let $L$ be a Lagrangian in a single independent variable $x$, $n$ independent variables $u^{\alpha}$, and $n$ derivative variables $u_{x}^{\alpha}$.

The Euler-Lagrange equations are

$$
\frac{\partial L}{\partial u^{\beta}}-\frac{d}{d x} \frac{\partial L}{\partial u_{x}^{\beta}}=0
$$

and expanding the total derivative $d / d x$ gives

$$
\frac{\partial L}{\partial u^{\beta}}-\frac{\partial^{2} L}{\partial x \partial u_{x}^{\beta}}-u_{x}^{\alpha} \frac{\partial^{2} L}{\partial u^{\alpha} \partial u_{x}^{\beta}}-u_{x x}^{\alpha} \frac{\partial^{2} L}{\partial u_{x}^{\alpha} \partial u_{x}^{\beta}}
$$

In general these equations are second-order, but if $L$ is linear in the variables $u_{x}^{\alpha}$ then they are first-order.

## The Euler-Lagrange equations (2)

Now suppose there are $m$ independent variables $x^{i}, n$ independent variables $u^{\alpha}$, and $m n$ derivative variables $u_{i}^{\alpha}$.

The Euler-Lagrange equations are now

$$
\frac{\partial L}{\partial u^{\beta}}-\frac{d}{d x^{j}} \frac{\partial L}{\partial u_{j}^{\beta}}=0
$$

and expanding the total derivative $d / d x^{j}$ now gives

$$
\frac{\partial L}{\partial u^{\beta}}-\frac{\partial^{2} L}{\partial x^{j} \partial u_{j}^{\beta}}-u_{j}^{\alpha} \frac{\partial^{2} L}{\partial u^{\alpha} \partial u_{j}^{\beta}}-u_{i j}^{\alpha} \frac{\partial^{2} L}{\partial u_{i}^{\alpha} \partial u_{j}^{\beta}}
$$

In general these equations are second-order, but if $L$ is linear in the variables $u_{i}^{\alpha}$ then they are first-order. But ...

## The Euler-Lagrange equations (3)

$$
\frac{\partial L}{\partial u^{\beta}}-\frac{\partial^{2} L}{\partial x^{j} \partial u_{j}^{\beta}}-u_{j}^{\alpha} \frac{\partial^{2} L}{\partial u^{\alpha} \partial u_{j}^{\beta}}-u_{i j}^{\alpha} \frac{\partial^{2} L}{\partial u_{i}^{\alpha} \partial u_{j}^{\beta}}
$$

The equations can be first-order even when $L$ is not linear: for example $L=f(x, u)\left(u_{i}^{\alpha} u_{j}^{\beta}-u_{j}^{\alpha} u_{i}^{\beta}\right)$

These Lagrangians come from the geometric construction of horizontalization on jet bundles:
with a fibred manifold $\pi: E \rightarrow M$, any differential form $\omega$ on $E$ gives a horizontal differential form $\mathrm{h}(\omega)$ on $J^{1} \pi$

For instance, $\mathrm{h}\left(d u^{\alpha} \wedge d u^{\beta}\right)=\left(u_{i}^{\alpha} u_{j}^{\beta}-u_{j}^{\alpha} u_{i}^{\beta}\right) d x^{i} \wedge d x^{j}$

## The Euler-Lagrange equations (4)

The same applies for higher-order Lagrangians.
If the Lagrangian $L$ has order $k$, the Euler-Lagrange equations are generically of order $2 k$

$$
\sum_{|I|=0}^{k}(-1)^{|I|} \frac{d^{|I|}}{d x^{I}} \frac{\partial L}{\partial u_{I}^{\beta}}=0
$$

where $I \in \mathbb{N}^{k}$ is a symmetric multi-index:

$$
\text { if } u_{I}^{\beta}=u_{i_{1} i_{2} \cdots i_{k}}^{\beta} \text { then } I(i)=\left|\left\{i_{r}: i_{r}=i\right\}\right|
$$

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$$

The geometry of the multi-index space is important:

$$
\begin{aligned}
& |I|=\sum_{i=1}^{m} I(i) \text { is the length of } I \text {; } \\
& \|I\|^{2}=\sum_{i=1}^{m}(I(i))^{2} \text { is the square Euclidean norm of } I
\end{aligned}
$$

## Reduced-order Euler-Lagrange equations

$$
\sum_{|J|=0}^{k}(-1)^{|J|} \frac{d^{|J|}}{d x^{J}} \frac{\partial L}{\partial u_{J}^{\beta}}=0
$$

Each total derivative $d / d x^{j}$ increases the order of its argument by one, so that the terms of order $2 k$ come from

$$
\sum_{|J|=k}(-1)^{k} \frac{d^{|J|}}{d x^{J}} \frac{\partial L}{\partial u_{J}^{\beta}} \quad \text { and equal } \quad \sum_{|I|=|J|=k}(-1)^{k} u_{I+J}^{\alpha} \frac{\partial^{2} L}{\partial u_{I}^{\alpha} \partial u_{J}^{\beta}}
$$

The equations will have order less than $2 k$ if, and only if, for each multi-index $H$ of length $2 k$,

$$
\sum_{I+J=H} \frac{\partial^{2} L}{\partial u_{I}^{\alpha} \partial u_{J}^{\beta}}=0
$$

## The polynomial condition

Euler-Lagrange equations:

$$
\sum_{|J|=0}^{k}(-1)^{|J|} \frac{d^{|J|}}{d x^{J}} \frac{\partial L}{\partial u_{J}^{\beta}}=0
$$

Condition for lower order equations: whenever $|H|=2 k$ then

$$
\sum_{I+J=H} \frac{\partial^{2} L}{\partial u_{I}^{\alpha} \partial u_{J}^{\beta}}=0
$$

Theorem
A necessary condition for the Euler-Lagrange equations to have order less than $2 k$ is that $L$ is a polynomial in the highest-order derivatives $u_{I}^{\alpha},|I|=k$, of order at most $p_{k}$
where $p_{k}$ is the number of distinct multi-indices of length $k$

## Proof of the polynomial condition

A necessary condition for the Euler-Lagrange equations to have order less than $2 k$ is that $L$ is a polynomial in the highest-order derivatives $u_{I}^{\alpha},|I|=k$, of order at most $p_{k}$
Consider

$$
\frac{\partial^{p_{k}+1} L}{\partial u_{J_{1}}^{\alpha_{1}} \cdots \partial u_{J_{p_{k}}}^{\alpha_{p_{k}}} \partial u_{J_{p_{k}+1}}^{\alpha_{p_{k}+1}}}
$$

so at least two of the multi-indices must be the same - say $J_{1}=J_{2}$

Use the condition $\sum_{I+J=H} \frac{\partial^{2} L}{\partial u_{I}^{\alpha} \partial u_{J}^{\beta}}=0$ to put

$$
\frac{\partial^{p_{k}+1} L}{\partial u_{J_{1}}^{\alpha_{1}} \partial u_{J_{2}}^{\alpha_{2}} \partial u_{J_{3}}^{\alpha_{3}} \cdots}=\sum_{\substack{K_{1}+K_{2}=J_{1}+J_{2} \\\left(K_{1}, K_{2}\right) \neq\left(J_{1}, J_{2}\right)}}-\frac{\partial^{p_{k}+1} L}{\partial u_{K_{1}}^{\alpha_{1}} \partial u_{K_{2}}^{\alpha_{2}} \partial u_{J_{3}}^{\alpha_{3}} \cdots}
$$

## Proof of the polynomial condition (2)

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But each term on the RHS also has a repeated multi-index! So we can continue...

## Proof of the polynomial condition (2)

$$
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But each term on the RHS also has a repeated multi-index! So we can continue ...

But eventually, every term will have a repeated 'pure' multi-index $J$ (where $J(j)=k$ for some $j$, and $J(j)=0$ for $i \neq j$ )
and then $\sum_{J+J=H} \frac{\partial^{2} L}{\partial u_{J}^{\alpha} \partial u_{J}^{\beta}}=0$ implies that

$$
\frac{\partial^{2} L}{\partial u_{J}^{\alpha} \partial u_{J}^{\beta}}=0
$$

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The parallellogram rule for Euclidean norms!

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\frac{\partial^{p_{k}+1} L}{\partial u_{J}^{\alpha_{1}} \partial u_{J}^{\alpha_{2}} \partial u_{J_{3}}^{\alpha_{3}} \ldots}=\sum_{\substack{K_{1}+K_{2}=J+J \\\left(K_{1}, K_{2}\right) \neq(J, J)}}-\frac{\partial^{p_{k}+1} L}{\partial u_{K_{1}}^{\alpha_{1}} \partial u_{K_{2}}^{\alpha_{2}} \partial u_{J_{3}}^{\alpha_{3}} \ldots}
$$

we have $\|J\|^{2}+\|J\|^{2}=2\|J\|^{2}<\left\|K_{1}\right\|^{2}+\left\|K_{2}\right\|^{2}$
The sum of the square Euclidean norms in the terms keeps increasing, eventually giving $k+1$ pure multi-indices per term

## Proof of the polynomial condition (4)

Therefore

$$
\frac{\partial^{p_{k}+1} L}{\partial u_{J_{1}}^{\alpha_{1}} \cdots \partial u_{J_{p_{k}}}^{\alpha_{p_{k}}} \partial u_{J_{p_{k}+1}}^{\alpha_{p_{k}+1}}}=0
$$

so that $L$ is a polynomial in the $u_{J}^{\alpha},|J|=k$, of degree at most $p_{k}$.

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so that $L$ is a polynomial in the $u_{J}^{\alpha},|J|=k$, of degree at most $p_{k}$.

But this necessary condition is not sufficient: for instance, $L=\left(u_{x y}\right)^{2}$ has Euler-Lagrange equations $2 u_{x x y y}=0$

All the Lagrangians with lower-order equations appear to be determinants

Geometrically, determinants arise as the coefficients of wedge products $d x \wedge d y \wedge d z \wedge \ldots$

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Therefore

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All the Lagrangians with lower-order equations appear to be determinants

Geometrically, determinants arise as the coefficients of wedge products $d x \wedge d y \wedge d z \wedge \cdots$
... but also as coefficients of $d x^{2} \wedge d x d y \wedge d y^{2} \wedge \ldots$

## Differential hyperforms

Differential hyperforms were described in an unpublished paper by Peter Olver from 1982

They are covariant tensors with symmetry properties described by Young diagrams (ordinary differential forms are purely alternating, but hyperforms can have more complicated symmetries)

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Consider hyperforms on jet manifolds $J^{k} \pi$ that are

- horizontal over $M$, and
- wedge products of symmetric tensors (all of the same rank)

A $(p, q)$ hyperform is a section of $\bigwedge^{p} S^{q} T^{*} M$, pulled back to $J^{k} \pi$
These are generated over $C^{\infty}\left(J^{k} \pi\right)$ by $d x^{I_{1}} \wedge d x^{I_{2}} \wedge \cdots \wedge d x^{I_{p}}$ where $d x^{I}=d x^{i_{1}} d x^{i_{2}} \cdots d x^{i_{q}}$ with $I=\left(i_{1}, i_{2}, \cdots, i_{q}\right)$

## Affine $(1, q)$ hyperforms

A $(1, q)$ hyperform $(1 \leq q \leq k)$ is a horizontal symmetric tensor $\theta: J^{k} \pi \rightarrow S^{q} T^{*} M$

As $J^{k} \pi \rightarrow J^{k-1} \pi$ is an affine bundle, we say that $\theta$ is an affine $(1, q)$ hyperform if its restriction to each fibre of the bundle is an affine map: in coordinates

$$
\theta=\sum_{\substack{|I|=k \\|\mathcal{J}|=q}}\left(\theta_{\alpha \mathcal{J}}^{I} u_{I}^{\alpha}+\theta_{\mathcal{J}}\right) d x^{\mathcal{J}}
$$

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$$

These affine $(1, q)$ hyperforms are too general. We shall restrict attention to special affine $(1, q)$ hyperforms

## Special affine $(1, q)$ hyperforms

The affine bundle $J^{k} \pi \rightarrow J^{k-1} \pi$ has associated vector bundle $V \pi \otimes S^{k} T^{*} M \rightarrow J^{k-1} \pi$

The fibre-affine map $\theta$ has an associated fibre-linear 'difference map' $\bar{\theta}: V \pi \otimes S^{k} T^{*} M \rightarrow S^{q} T^{*} M$

We say that $\theta$ is a special affine $(1, q)$ hyperform if there is a tensor $\tilde{\theta} \in V \pi^{*} \otimes S^{k-q} T M$ such that the difference map $\bar{\theta}$ is given by contraction of elements of its domain $V \pi \otimes S^{k} T^{*} M$ with $\tilde{\theta}$.
In coordinates (where $\theta_{\alpha}^{I}$ are the coordinates of $\tilde{\theta}$ )

$$
\theta=\sum_{\substack{|I|=k-q \\|\mathcal{J}|=q}}\left(\theta_{\alpha}^{I} u_{I+\mathcal{J}}^{\alpha}+\theta_{\mathcal{J}}\right) d x^{\mathcal{J}}
$$

## Special affine $(1, q)$ hyperforms - example

$$
\theta=\sum_{\substack{|I|=k-q \\|\mathcal{J}|=q}}\left(\theta_{\alpha}^{I} u_{I+\mathcal{J}}^{\alpha}+\theta_{\mathcal{J}}\right) d x^{\mathcal{J}}
$$

In the special case where $q=1$ we have

$$
\theta=\sum_{|I|=k-1}\left(\theta_{\alpha}^{I} u_{I+1_{j}}^{\alpha}+\theta_{j}\right) d x^{j}
$$

the ordinary horizontalization of the 1-form
$\sum_{|I|=k-1} \theta_{\alpha}^{I} d u_{I}^{\alpha}+\theta_{j} d x^{j}$
There is no invariant operation of horizontalization for hyperforms when $q \geq 2$; but special affine $(1, q)$ hyperforms generalize the images of the horizontalization operator on ordinary 1-forms

## Hyperaffine $\left(p_{q}, q\right)$ hyperforms

A $\left(p_{q}, q\right)$ hyperform $\omega$ is a section of the line bundle $\bigwedge^{p_{q}} S^{q} T^{*} M$, pulled back to $J^{k} \pi$

It is hyperaffine if it is generated by wedge products of special affine hyperforms $\theta=\sum_{|I|=k-q,|\mathcal{J}|=q}\left(\theta_{\alpha}^{I} u_{I+\mathcal{J}}^{\alpha}+\theta_{\mathcal{J}}\right) d x^{\mathcal{J}}$

If $\omega=\omega_{q} d x^{\mathcal{J}_{1}} \wedge d x^{\mathcal{J}_{2}} \wedge \cdots \wedge d x^{\mathcal{J}_{p q}}$ then $\omega_{q}$ is a linear combination of determinants (or their minors)

$$
\left|\begin{array}{cccc}
u_{I_{1}+\mathcal{J}_{1}}^{\alpha_{1}} & u_{I_{1}+\mathcal{J}_{2}}^{\alpha_{1}} & \cdots & u_{I_{1}+\mathcal{J}_{p_{q}}}^{\alpha_{1}} \\
u_{I_{2}+\mathcal{J}_{1}}^{\alpha_{2}} & u_{I_{2}+\mathcal{J}_{2}}^{\alpha_{2}} & \cdots & u_{I_{2}+\mathcal{J}_{p_{q}}}^{\alpha_{2}} \\
\vdots & \vdots & \ddots & \vdots \\
u_{I_{p_{q}}+\mathcal{J}_{1}}^{\alpha_{p_{q}}} & u_{I_{p_{q}}+\mathcal{J}_{2}}^{\alpha_{p_{q}}} & \cdots & u_{I_{p_{q}+\mathcal{J}_{p_{q}}}}^{\alpha_{p_{1}}}
\end{array}\right|
$$

## What does this have to do with Lagrangians?

A Lagrangian $m$-form $\lambda$ defines local Lagrangian functions $L$ by

$$
\lambda=L d x^{1} \wedge d x^{2} \wedge \cdots \wedge d x^{m}
$$

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A Lagrangian $m$-form $\lambda$ defines local Lagrangian functions $L$ by

$$
\lambda=L d x^{1} \wedge d x^{2} \wedge \cdots \wedge d x^{m}
$$

Say that $\lambda$ is hyperaffine if, in any coordinate system,

$$
L=\omega_{1}+\omega_{2} \cdots+\omega_{k}
$$

where each $\omega_{q}$ is the coefficient of a hyperaffine hyperform

$$
\omega=\omega_{q} d x^{\mathcal{J}_{1}} \wedge d x^{\mathcal{J}_{2}} \wedge \cdots \wedge d x^{\mathcal{J}_{p_{q}}}
$$

This is independent of the coordinate system
In new coordinates $(\tilde{x}, \tilde{u})$, the volume $d x^{1} \wedge d x^{2} \wedge \cdots \wedge d x^{m}$ changes by the Jacobian determinant $J(\tilde{x}, x)$, whereas each hypervolume $d x^{\mathcal{J}_{1}} \wedge d x^{\mathcal{J}_{2}} \wedge \cdots \wedge d x^{\mathcal{J}_{p q}}$ changes by a power of $J(\tilde{x}, x)$

## Euler-Lagrange equations of hyperaffine Lagrangians

 TheoremIf $L$ is the Lagrangian function of a hyperaffine Lagrangian then the Euler-Lagrange equations have reduced order

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It is sufficient to show this for a determinant

$$
\Delta=\left|\begin{array}{cccc}
u_{I_{1}+\mathcal{J}_{1}}^{\alpha_{1}} & u_{I_{1}+\mathcal{J}_{2}}^{\alpha_{1}} & \cdots & u_{I_{1}+\mathcal{J}_{p_{q}}}^{\alpha_{1}} \\
u_{I_{2}+\mathcal{J}_{1}}^{\alpha_{2}} & u_{I_{2}+\mathcal{J}_{2}}^{\alpha_{2}} & \cdots & u_{I_{2}+\mathcal{J}_{p_{q}}}^{\alpha_{2}} \\
\vdots & \vdots & \ddots & \vdots \\
u_{I_{p_{q}}+\mathcal{J}_{1}}^{\alpha_{p_{q}}} & u_{I_{p_{q}+\mathcal{J}_{2}}^{\alpha_{p_{q}}}} & \cdots & u_{I_{p_{q}+\mathcal{J}_{p_{q}}}^{\alpha_{p_{1}}}}
\end{array}\right|
$$

so write $\Delta$ as

$$
\Delta=\sum_{\sigma \in \mathfrak{S}_{h}} \varepsilon_{\sigma} u_{I_{1}+\mathcal{J}_{\sigma(1)}}^{\alpha_{1}} u_{I_{2}+\mathcal{J}_{\sigma(2)}}^{\alpha_{2}} \cdots u_{I_{h}+\mathcal{J}_{\sigma(h)}}^{\alpha_{h}}
$$

Euler-Lagrange equations of hyperaffine Lagrangians (2)

$$
\Delta=\sum_{\sigma \in \mathfrak{S}_{h}} \varepsilon_{\sigma} u_{I_{1}+\mathcal{J}_{\sigma(1)}}^{\alpha_{1}} u_{I_{2}+\mathcal{J}_{\sigma(2)}}^{\alpha_{2}} \cdots u_{I_{h}+\mathcal{J}_{\sigma(h)}}^{\alpha_{h}}
$$

Substituting in the Euler-Lagrange equations gives

$$
\sum_{|\mathrm{K}|=k} \frac{d^{|\mathrm{K}|}}{d x^{\mathrm{K}}} \frac{\partial L}{\partial u_{\mathrm{K}}^{\beta}}=\sum_{\substack{1 \leq r, s \leq h \\ s \neq r}} \sum_{\sigma \in \mathfrak{S}_{h}} \delta_{\beta}^{\alpha_{r}} \varepsilon_{\sigma} \Phi_{r s \sigma} u_{I_{r}+I_{s}+\mathcal{J}_{\sigma(r)}+\mathcal{J}_{\sigma(s)}}^{\alpha_{s}}
$$

where the coefficients $\Phi_{r s \sigma}$ are

$$
\Phi_{r s \sigma}=u_{I_{1}+\mathcal{J}_{\sigma(1)}}^{\alpha_{1}} u_{I_{2}+\mathcal{J}_{\sigma(2)}}^{\alpha_{2}} \cdots \widehat{r} \cdots \hat{s} \cdots u_{I_{h}+\mathcal{J}_{\sigma(h)}}^{\alpha_{h}}
$$

Euler-Lagrange equations of hyperaffine Lagrangians (2)

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\Delta=\sum_{\sigma \in \mathfrak{S}_{h}} \varepsilon_{\sigma} u_{I_{1}+\mathcal{J}_{\sigma(1)}}^{\alpha_{1}} u_{I_{2}+\mathcal{J}_{\sigma(2)}}^{\alpha_{2}} \cdots u_{I_{h}+\mathcal{J}_{\sigma(h)}}^{\alpha_{h}}
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$$

where the coefficients $\Phi_{r s \sigma}$ are

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\Phi_{r s \sigma}=u_{I_{1}+\mathcal{J}_{\sigma(1)}}^{\alpha_{1}} u_{I_{2}+\mathcal{J}_{\sigma(2)}}^{\alpha_{2}} \cdots \widehat{r} \cdots \hat{s} \cdots u_{I_{h}+\mathcal{J}_{\sigma(h)}}^{\alpha_{h}}
$$

Fix $r \neq s$. Given $\sigma \in \mathfrak{S}_{h}$ put $\tilde{\sigma}=\sigma \circ(r, s) \neq \sigma$.
$\Phi_{r s \sigma}=\Phi_{r s \tilde{\sigma}}$ and $\varepsilon_{\sigma}=-\varepsilon_{\tilde{\sigma}}$ so all the terms cancel.

## Determinants

Established so far:

- If a Lagrangian function of order $k$ has reduced-order Euler-Lagrange equations then it is a polynomial of order at most $p_{k}$ in the variables $u_{H}^{\alpha}(|H|=k)$;
- Every hyperaffine Lagrangian has reduced-order Euler-Lagrange equations (and is a polynomial with a particular determinant structure)

I conjecture that every Lagrangian with reduced-order Euler-Lagrange equations has this particular determinant structure, and so is hyperaffine.

## Determinants (2)

A general polynomial Lagrangian function of order $k$ and degree $p_{k}$ is

$$
L=\sum_{r=0}^{p_{k}} A_{\alpha_{1} \alpha_{2} \cdots \alpha_{r}}^{H_{1} H_{2} \cdots H_{r}} u_{H_{1}}^{\alpha_{1}} u_{H_{2}}^{\alpha_{2}} \cdots u_{H_{r}}^{\alpha_{r}}
$$

with implicit sums over the indices and multi-indices, and with $|H|=k$

Can this be written as a linear combination of determinants

$$
\left|\begin{array}{cccc}
u_{I_{1}+\mathcal{J}_{1}}^{\alpha_{1}} & u_{I_{1}+\mathcal{J}_{2}}^{\alpha_{1}} & \cdots & u_{I_{1+}+\mathcal{J}_{r}}^{\alpha_{1}} \\
u_{I_{2}+\mathcal{J}_{1}}^{\alpha_{2}} & u_{I_{2}+\mathcal{J}_{2}}^{\alpha_{2}} & \cdots & u_{I_{2}+\mathcal{J}_{r}}^{\alpha_{2}} \\
\vdots & \vdots & \ddots & \vdots \\
u_{I_{r}+\mathcal{J}_{1}}^{\alpha_{r}} & u_{I_{r}+\mathcal{J}_{2}}^{\alpha_{r}} & \cdots & u_{I_{r}+\mathcal{J}_{r}}^{\alpha_{r}}
\end{array}\right| \quad|\quad| \mathcal{J}|\quad| \quad|I|=k-q, \quad 1 \leq q \leq k, \quad 0 \leq r \leq p_{q} ?
$$

## Introduction

## Determinants (3)

Consider homogeneous polynomials $A_{\alpha_{1} \alpha_{2} \cdots \alpha_{r}}^{H_{1} H_{2} \cdots H_{r}} u_{H_{1}}^{\alpha_{1}} u_{H_{2}}^{\alpha_{2}} \cdots u_{H_{r}}^{\alpha_{r}}$ In the case $r=2$ there is a constructive proof

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Partition the quadratic terms by $H_{1}+H_{2}=H$ and put

$$
\psi_{H}=\sum_{H_{1}+H_{2}=H} A_{\alpha_{1} \alpha_{2}}^{H_{1} H_{2}} u_{H_{1}}^{\alpha_{1}} u_{H_{2}}^{\alpha_{2}}
$$

Choose a term $A_{\alpha_{1} \alpha_{2}}^{K_{1} K_{2}} u_{K_{1}}^{\alpha_{1}} u_{K_{2}}^{\alpha_{2}}$ arbitrarily, so from E-L we have

$$
A_{\alpha_{1} \alpha_{2}}^{K_{1} K_{2}}=\sum_{H_{1}+H_{2}=H,\left(H_{1}, H_{2}\right) \neq\left(K_{1}, K_{2}\right)}-A_{\alpha_{1} \alpha_{2}}^{H_{1} H_{2}}
$$

and so

$$
\psi_{H}=\sum_{H_{1}+H_{2}=H} A_{\alpha_{1} \alpha_{2}}^{H_{1} H_{2}}\left(u_{H_{1}}^{\alpha_{1}} u_{H_{2}}^{\alpha_{2}}-u_{K_{1}}^{\alpha_{1}} u_{K_{2}}^{\alpha_{2}}\right)
$$

## Introduction

## Determinants (4)

For cubic and higher terms, there is no obvious algorithm to give an explicit construction
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A possible approach would use an abstract dimension argument:
The number of variables $u_{I}^{\alpha},|I|=k$, is known, and so the dimension of the space of homogeneous polynomials of degree $r$ is also known

The number of $\mathrm{E}-\mathrm{L}$ constraints for quadratic polynomials is known, so the number of constraints for degree $r$ polynomials can in principle be calculated

The theorem will be proved if there are enough independent $r \times r$ determinants of the correct type

