A geometric Iwatsuka type effect in quantum layers

Matěj Tušek*

joint work with Pavel Exner (ÚJF AV ČR) a Tomáš Kalvoda (ČVUT v Praze)

*Fakulta jaderná a fyzikálně inženýrská, ČVUT v Praze; Ústav jaderné fyziky, AV ČR

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Outline

Model

We consider a spinless non-relativistic quantum particle which is

1. confined to the **plane** and exposed to an external **translationally invariant magnetic field** that is perpendicular to the plane
2. confined to a layer Ω of fixed width constructed along a **translationally invariant surface** and exposed to an ambient **constant magnetic field**.

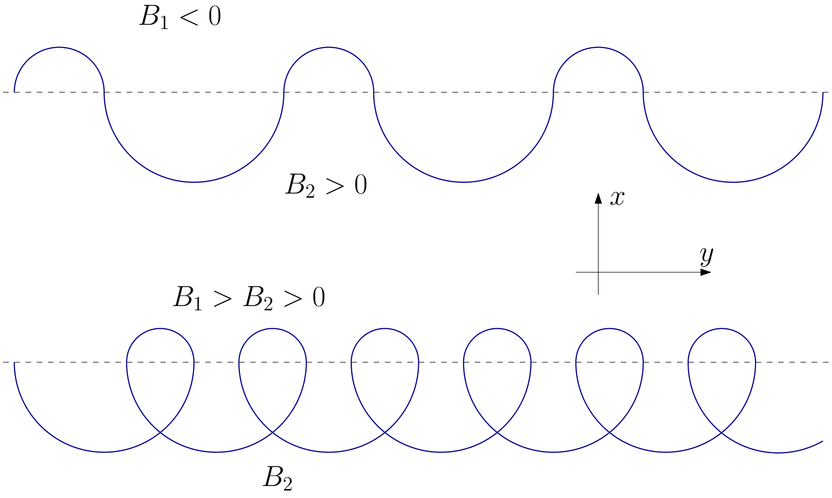
The corresponding Hamiltonian is

1. $H = -\partial_x^2 + (-i\partial_y + A_y(x))^2$ with $A_y(x) = \int_0^x B(t)\,\mathrm{d}t$
2. $H_\Omega = (-i\nabla + A)^2$ with $A = B_0(0, x, 0)$, $B_0 > 0$, acting on the $L^2(\Omega)$ and with the Dirichlet boundary condition on $\partial\Omega$.

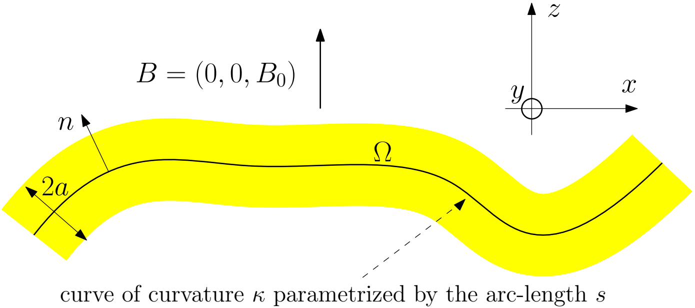
What is known or conjectured?

1. There are many classes of fields B, for which H is purely absolutely continuous. It's conjectured that as long as B is translationally invariant and non-constant, H is always purely absolutely continuous [B. Simon, 1987].
2. Recently, we discovered several classes of layers, for which the spectrum of H_Ω, in its entirety or a part of it, becomes absolutely continuous. We are convinced that our sufficient conditions for absolute continuity are far from being necessary.

Motivation-classical trajectories for "magnetic steps"



Motivation-thin layers: duality between curvature and magnetic field



As the width $2a$ of the layer Ω tends to zero, we have

$$\|(H_\Omega - (\pi/2a)^2 + k)^{-1} - (h_{\text{eff}} + k)^{-1} \oplus 0\| = \mathcal{O}(a),$$

where

$$h_{\text{eff}} = -\partial_s^2 + (-i\partial_y + B_0 x(s))^2 - \frac{1}{4}\kappa^2(s)$$

[D. Krejčiřík, N. Raymond, M.T.; 2015].
Remark that the effective magnetic field is $B_0\dot{x} = B \cdot n$.

L^2 space of $\mathscr{H}$-valued functions

$\mathscr{H}$ ... separable Hilbert space
(M, μ) ... σ-finite measure space

Definition

$L^2(M, \mathrm{d}\mu; \mathscr{H})$ stands for the space of all (equivalence classes of a.e. equal) measurable functions on M with values in $\mathscr{H}$ that are square integrable, i.e.,

$$\int_M \|f(m)\|^2_{\mathscr{H}} \, \mathrm{d}\mu(m) < +\infty.$$

With the following dot product

$$\langle f, g \rangle := \int_M \langle f(m), g(m) \rangle_{\mathscr{H}} \, \mathrm{d}\mu(m),$$

this space turns into a Hilbert space.

Remark: By definition, f is measurable iff $\langle \varphi, f(m) \rangle_{\mathscr{H}}$ is measurable for all $\varphi \in \mathscr{H}$.

L^2 space of $\mathscr{H}$-valued functions–cont.

Example:
$\mu \ldots$ finite sum of point measures at points $m_1, \ldots, m_k$
$\Rightarrow$ any $f \in L^2(M, \mathrm{d}\mu; \mathscr{H})$ is determined by the k–tuple $(f(m_1), \ldots, f(m_k))$
$\Rightarrow L^2(M, \mathrm{d}\mu; \mathscr{H})$ is isomorphic to $\bigoplus_{i=1}^{k} \mathscr{H}$

Definition

We will call $L^2(M, \mathrm{d}\mu; \mathscr{H})$ a constant fiber direct integral (with a fiber $\mathscr{H}$). We shall write

$$L^2(M, \mathrm{d}\mu; \mathscr{H}) \equiv \int_M^{\oplus} \mathscr{H},$$

whenever we want to put more emphasis on the fibers $\mathscr{H}$ rather than the points of M.

Decomposable operator

Definition

A function $A(.)$ from M to $\mathscr{B}(\mathscr{H})$ is called measurable iff $\langle\varphi, A(.)\psi\rangle$ is measurable for each $\varphi, \psi \in \mathscr{H}$. $L^\infty(M, \mathrm{d}\mu; \mathscr{B}(\mathscr{H}))$ denotes the space of (equivalence classes of a.e. equal) measurable functions from M to $\mathscr{B}(\mathscr{H})$ with

$$\|A\|_\infty := \operatorname*{ess\,sup}_{m\in M} \|A(m)\|_{\mathscr{B}(\mathscr{H})} < +\infty.$$

Definition

A bounded operator A on $\int_M^\oplus \mathscr{H}$ is said to be decomposed by the direct integral decomposition iff there is a function $A(.) \in L^\infty(M, \mathrm{d}\mu; \mathscr{B}(\mathscr{H}))$ so that for all $\psi \in \mathscr{H}$,

$$(A\psi)(m) = A(m)\psi(m).$$

We then call A **decomposable** and write

$$A = \int_M^\oplus A(m)\,\mathrm{d}\mu(m).$$

The operators $A(m)$ are called the **fibers** of A.

Direct integral of selfadjoint operators

Definition

A function $A(.)$ from M to (not necessarily bounded) selfadjoint operators on $\mathscr{H}$ is called measurable iff $(A(.)+i)^{-1}$ is measurable. Given such a function, we define an operator A on $\int_M^{\oplus} \mathscr{H}$ with domain

$$\mathrm{Dom}(A) = \left\{ \psi \in \int_M^{\oplus} \mathscr{H} \,|\, \psi(m) \in \mathrm{Dom}(A(m))\; a.e.;\; \int_M \|A(m)\psi(m)\|_{\mathscr{H}}^2 \, \mathrm{d}\mu(m) < +\infty \right\}$$

by

$$(A\psi)(m) = A(m)\psi(m).$$

We write

$$A = \int_M^{\oplus} A(m)\, \mathrm{d}\mu(m).$$

Properties of direct integral of selfadjoint operators

Theorem

Let $A = \int_M^{\oplus} A(m)\,\mathrm{d}\mu(m)$, *where* $A(.)$ *is measurable and* $A(m)$ *is selfadjoint for every* m. *Then*

- A *is selfadjoint,*
- *for any bounded Borel function* F *on* $\mathbb{R}$,
$$F(A) = \int_M^{\oplus} F(A(m))\,\mathrm{d}\mu,$$
- $\lambda \in \sigma(A)$ *iff for all* $\varepsilon > 0$,
$$\mu(\{m |\, \sigma(A(m)) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \neq \emptyset\}) > 0.$$
- $\lambda \in \sigma_p(A)$ *iff*
$$\mu(\{m |\, \lambda \in \sigma_p(A(m))\}) > 0.$$
- *if each* $A(m)$ *has purely absolutely continuous spectrum, then so does* A

Absolutely continuous spectrum

- For any selfadjoint operator A on a Hilbert space $\mathscr{H}$, $A = \int_{\mathbb{R}} \lambda \, \mathrm{d}P_A(\lambda)$, where $P_A(\lambda) = P_A((-\infty, \lambda])$ is its projection-valued measure.
- $\mathscr{H}_{ac} := \{\psi \in \mathscr{H} \,|\, \lambda \mapsto \|P_A(\lambda)\psi\|^2$ is absolutely continuous$\}$ ... the subspace of absolute continuity
- $\sigma_{ac}(A) := \sigma(A|_{\mathscr{H}_{ac}})$

Definition

A is **purely absolutely continuous** iff $\sigma(A) = \sigma_{ac}(A)$, i.e., $\mathscr{H} = \mathscr{H}_{ac}$.

Theorem (RAGE)

For every $R > 0$,

$$\psi \in \mathscr{H}_{ac} \oplus \mathscr{H}_{sc} \Leftrightarrow \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \|\chi_{\{|x|<R\}} e^{-itA} \psi\| \, \mathrm{d}t = 0.$$

It does not distinguish between the absolutely continuous and singularly continuous subspaces. However, in many physically interesting situations, $\mathscr{H}_{sc} = \emptyset$.

Theorem (N. Filonov, A.V. Sobolev, 2006)

Assume that $A = \int_M^{\oplus} A(m)\,\mathrm{d}\mu(m)$, *where each* $A(m)$ *is is selfadjoint and with compact resolvent, which in turn is a real-analytic function of* $m \in M$. *Then*

- $\sigma_{sc}(A) = \emptyset$
- $\sigma_{pp}(A)$ *is at most discrete (it consists only of isolated points without finite accumulation point) and each eigenvalue is of infinite multiplicity.*

Theorem (see, e.g., Reed & Simon: Vol. IV, 1978)

In addition to the assumptions above, let all eigenvalue branches of $A(m)$ *are non-constant. Then* A *is purely absolutely continuous.*

Proof (for μ being, e.g., the Lebesgue measure):

- eigenvalue branches are real-analytic functions of $m \in M$
- recall that $\lambda \in \sigma_p(A)$ iff $\mu(\{m|\, \lambda \in \sigma_p(A(m))\}) > 0$
- therefore, if $\lambda \in \sigma_p(A)$ then for some eigenvalue branch, say λ_p, $\mu(\{m|\, \lambda_p(m) = \lambda\}) > 0$
- thus, λ_p is constant, due to analyticity - a contradiction

Planar Iwatsuka model – direct integral decomposition

- $H = -\partial_x^2 + (-i\partial_y + A_y(x))^2 + W(x)$ with $A_y(x) = \int_0^x B(t)\,\mathrm{d}t$ and $B, W \in L^\infty$
- H is e.s.a. on $C_0^\infty(\mathbb{R}^2) \subset L^2(\mathbb{R}^2, \mathrm{d}x\,\mathrm{d}y)$
- after partial Fourier transform $(y \leftrightarrow \xi)$: $H \equiv -\partial_x^2 + (\xi + A_y(x))^2 + W(x)$ on $L^2(\mathbb{R}^2, \mathrm{d}x\,\mathrm{d}\xi)$
- $H \equiv \int_{\mathbb{R}}^{\oplus} H[\xi]$, where

$$H[\xi] := -\,\mathrm{d}_x^2 + (\xi + A_y(x))^2 + W(x)$$

  acts on $L^2(\mathbb{R}, \mathrm{d}x)$
- Example (**Landau Hamiltonian**): $W = 0, B = B_0 = const. \Rightarrow A_y(x) = B_0 x \Rightarrow$ all $H[\xi]$ are unitarily equivalent $\Rightarrow \sigma(H) = \sigma_p(H) = \{(2n-1)B_0|, n \in \mathbb{N}\}$

In a straightforward manner one can show

1. $\{H[\xi] | \, \xi \in \mathbb{R}\}$ forms an analytic family of type (B) $\Rightarrow$ analyticity of eigenvalues
2. if $|\lim_{x\to\pm\infty} A_y(x)| = +\infty$, then the resolvent of $H[\xi]$ is compact $\Rightarrow$ purely discrete spectrum of $H[\xi]$; moreover, one can prove that all **eigenvalues are simple**.

Non-constancy of eigenvalues

The trickiest part is to prove that

3. all the eigenvalues of $H[\xi]$ are non-constant as functions of ξ.

There are essentially two strategies:

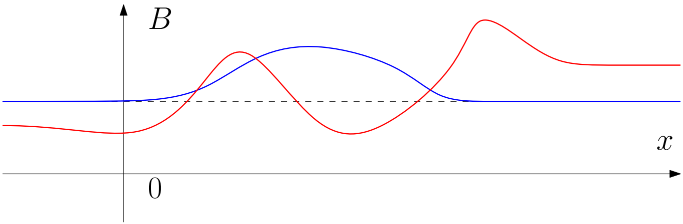
1. Use the Feynman-Helmann formula that gives the derivative of an eigenvalue w.r.t. a parameter. It is useful in situations when $B = B(x)$ is locally non-constant.
2. Use a comparison argument that gives asymptotic behaviour of eigenvalues at $\xi = \pm\infty$. It proves useful when the magnetic field behaves differently at $\pm\infty$. One can also compare asymptotic behaviour of eigenvalues at infinity with some estimates on the eigenvalues at a fixed point.

The second method strongly relies on the minimax principle together with the fact that the eigenvalues are simple, which will not necessarily be longer true in the case of layers!

Sufficient conditions for non-constancy of eigenvalues

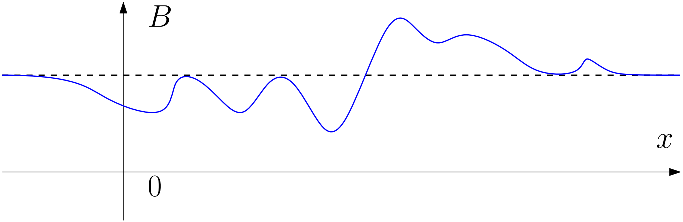
[A. Iwatsuka, 1985] $W \equiv 0$, $B \in C^\infty$ is bounded and above a positive constant and either of the following hold

- $\limsup_{x \to \pm\infty} B(x) < \liminf_{x \to \mp\infty} B(x)$
- $B = B(x)$ is constant away from a compact interval and there exists x_0 such that $B'(x_-)B'(x_+) \leq 0$ for all $x_- \leq x_0 \leq x_+$.



Sufficient conditions for non-constancy of eigenvalues – cont.

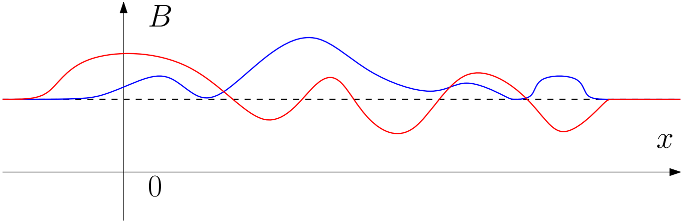
[M. Măntoiu, R. Purice, 1997] $W \equiv 0$, $B \in C^\infty$ is bounded, above a positive constant and non-constant and there exists a point x_0 such that for all x_1, x_2 with $x_1 \leq x_0 \leq x_2$ one has either $B(x_1) \leq B(x_0) \leq B(x_2)$ or $B(x_1) \geq B(x_0) \geq B(x_2)$



Sufficient conditions for non-constancy of eigenvalues – cont.

[P. Exner, H. Kovařík, 2000] $W \equiv 0$, $B(x) = B_0 + b(x)$, where $B_0 > 0$ and b is bounded, piecewise continuous and compactly supported and either of the following holds

- b is non-zero and **does not change sign**
- let $[a_l, a_r]$ be the smallest closed interval that contains $\operatorname{supp} b$; there are $c, \delta > 0$ and $m \in \mathbb{N}$ such that $|b(x)| \geq c(x - a_l)^m$ or $|b(x)| \geq c(a_r - x)^m$ for all $x \in [a_l, a_l + \delta)$ or $x \in (a_r - \delta, a_r]$, respectively.



Sufficient conditions for non-constancy of eigenvalues – cont.

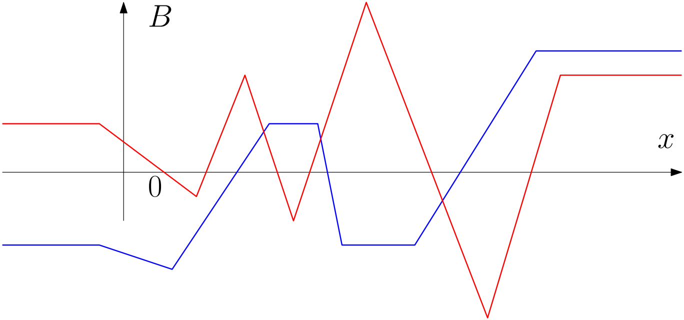
[M.T., 2016] $B, W \in L^\infty(\mathbb{R};\mathbb{R})$ are such that either of the following holds

- $\underline{B}_\pm > 0 \wedge \underline{B}_+ \geq \overline{B}_- \wedge (\overline{W}_- - \underline{W}_+ < \underline{B}_+ - \overline{B}_-)$
- $\underline{B}_+ > 0 \wedge \overline{B}_- < 0.$

Here, for any $f \in L^\infty(\mathbb{R},\mathbb{R})$,

$$\underline{f}_+ = \sup_{a\in\mathbb{R}} \operatorname*{ess\,inf}_{t\in(a,+\infty)} f(t) \qquad \overline{f}_+ = \inf_{a\in\mathbb{R}} \operatorname*{ess\,sup}_{t\in(a,+\infty)} f(t)$$

$$\underline{f}_- = \sup_{a\in\mathbb{R}} \operatorname*{ess\,inf}_{t\in(-\infty,a)} f(t) \qquad \overline{f}_- = \inf_{a\in\mathbb{R}} \operatorname*{ess\,sup}_{t\in(-\infty,a)} f(t).$$

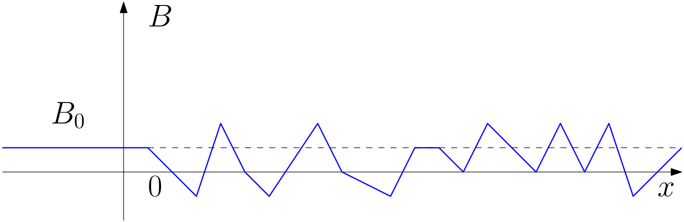


Sufficient conditions for non-constancy of eigenvalues – cont.

[P. Exner, T. Kalvoda, M.T., 2017] $B(t) = B_0(1 + b(t))$ with $B_0 > 0$ and

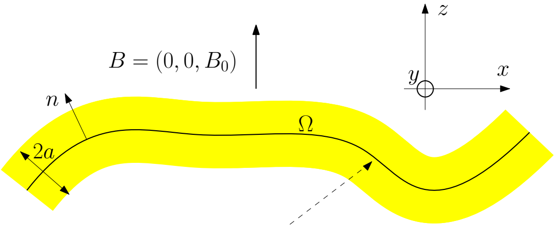
(i) $b \in L^2_{\mathrm{loc}}(\mathbb{R})$,
(ii) $b(t) = 0$ for all $t < 0$,
(iii) $\int_0^x b(t)\,\mathrm{d}t \leq 0$ holds for all $x \geq 0$,
(iv) there are $\alpha \in (-1, 0)$, $x_1 \geq 0$ such that $\int_0^x b(t)\,\mathrm{d}t > \alpha x$ holds for all $x \geq x_1$.

$W \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$ is such that $W(x) = \theta(x)W(x) \leq 0$, and either $b \not\equiv 0$ or $W \not\equiv 0$.



Nevertheless, the conjecture has not been proved yet under a general assumption $B \neq const.$! In particular, there is no general result in the case when $\lim_{x\to-\infty} B(x) = \lim_{x\to+\infty} B(x)$ but B is not compactly supported and non-constant.

Curved layers – direct integral decomposition



$H_\Omega = (-i\nabla + A)^2$ is unitarily equivalent to $\int_{\mathbb{R}}^{\oplus} H[\xi]\,\mathrm{d}\xi$, where

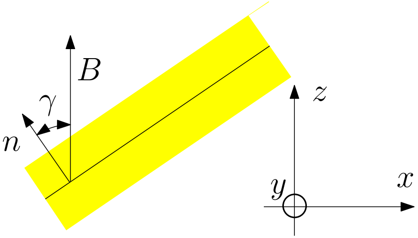
$$H[\xi] := -\partial_s f(s,u)^{-2}\partial_s + (\xi + \tilde{A}_2(s,u))^2 - a^{-2}\partial_u^2 + V(s,u)$$

acts on $L^2(\mathbb{R} \times (-1,1), \mathrm{d}s\,\mathrm{d}u)$. Here,

$$f(s,u) = 1 - au\kappa(s),$$
$$\tilde{A}_2(s,u) = B_0(x(s) - au\dot{\kappa}(s)),$$

and $V(s,u)$ depends on $\kappa, \dot{\kappa}, \ddot{\kappa}$ and $\lim_{a\to 0+} V(s,u) = -\frac{1}{4}\kappa(s)^2$.

Inclined layer



$$H[\xi] = -\partial_s^2 + (\xi + B_0(s\cos\gamma - au\sin\gamma))^2 - a^{-2}\partial_u^2$$

1. $\gamma \in (-\pi/2, \pi/2)$: $H[\xi]$ is unitarily equivalent to $H[0] \Rightarrow \sigma(H) = \sigma_p(H) = \sigma(H(0))$ (infinitely degenerate eigenvalues); especially, for $\gamma = 0$: 2D Landau Hamiltonian + Dirichlet Laplacian on $(-a, a)$
2. $\gamma = \pi/2$: each $H[\xi]$ is separable: $1D$ free Hamiltonian (purely absolutely continuous)+ an operator on a $(-a, a) \Rightarrow H[\xi]$ is purely a.c. $\Rightarrow \sigma(H) = \sigma_{ac}(H)$

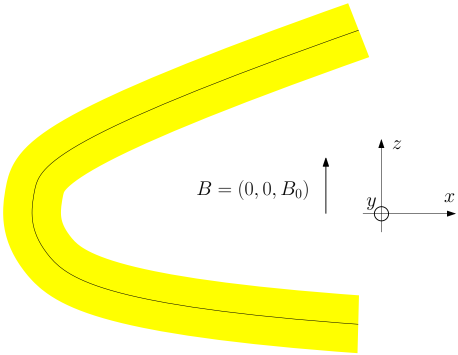
Easy part

If we assume that $\kappa, V_- \in L^\infty$, and $|\lim_{s\to\pm\infty} x(s)| = +\infty$, then

1. $H[\xi]$ forms an analytic family of type (B)
2. resolvent of $H[\xi]$ is compact

To show that the eigenvalue branches of $H[\xi]$ are non-constant is much more difficult that in the previous case when the fiber Hamiltonian was ordinary differential operator. In particular, it may occur that eigenvalue branches cross. However, it may not happen that a non-constant branch crosses a constant branch. In fact, if there is a constant branch then it has to be isolated from the rest of the spectrum.

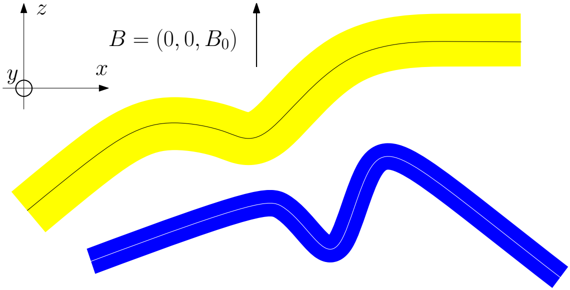
One-sided-fold layer



$$\lim_{s\to\pm\infty} x(s) = +\infty$$

Proof: $H[\xi] \geq \underbrace{-\partial_s f^{-2}\partial_s - a^{-2}\partial_u^2 + \tilde{A}_2^2}_{>0} + \underbrace{\xi^2 - 2\xi\|(\tilde{A}_2)_-\|_\infty - \|V_-\|_\infty}_{\xrightarrow{\xi\to+\infty} +\infty}$

Bent, asymptotically flat layer



$\exists s_\pm \in \mathbb{R}, \alpha_\pm \in (0,1] : \dot{x}(s) = \begin{cases} \alpha_+ & s > s_+ \\ \alpha_- & s < s_- \end{cases}$, $\alpha_+ \neq \alpha_-$, and $a < const.B_0^{-1/2}|\alpha_-^2 - \alpha_+^2|$

Fragments of the proof:

- for $\xi = \pm\infty$, $H[\xi]$ acts like operators on the inclined flat layer ($\dot{x}(s) = \alpha_\mp$)
- these limit operators may be compared to the sum of 1D harmonic oscillator and the Dirichlet Laplacian on $(-a, a)$
- if there is a constant eigenvalue branch of $H[\xi]$ then it is given by the same Rayleigh quotient both at $\pm\infty$

Thin layers

- Recall that, as $a \to 0+$, H_Ω is well approximated (after regularization) by

$$h_{\text{eff}} = -\partial_s^2 + (-i\partial_y + B_0 x(s))^2 - \frac{1}{4}\kappa^2(s),$$

  which is just a special case of the Iwatsuka Hamiltonian with an extra scalar potential,

$$H = -\partial_x^2 + (-i\partial_y + A_y(x))^2 + W(x).$$

- Both our results on non-constancy of eigenvalues of $H[\xi]$ apply to $h_{\text{eff}}[\xi]$.
- Since, for sufficiently small a, the lowest eigenvalues of $H_\Omega[\xi]$ are near $(\pi/2a)^2 + \sigma_p(h_{\text{eff}}[\xi])$, they are non-constant as well.
- This means that for any $E > 0$, there exists $a_E > 0$ s.t., for all $a < a_E$,

$$\sigma(H_\Omega - (\pi/2a)^2) \cap (0, E) = \sigma_{ac}(H_\Omega - (\pi/2a)^2) \cap (0, E).$$


THE END IS NEAR
THIS WILL NEVER END
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Thank you for your attention!

- M. T. On an extension of the Iwatsuka model. J. Phys. A, vol 49, no 36, 2016. arXiv:1601.02670
- P. Exner, T. Kalvoda, and M. T. A geometric Iwatsuka type effect in quantum layers. arXiv:1701.057140