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Recent advances on minimal systems and minimal spaces

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OP Výzkum
a vývoj pro inovace

Outline

Minimal maps and spaces

G. D. Birkhoff, *Quelques théorémes sur le mouvement des systémes dynamiques*, Bulletin de la Socité mathématiques de France, 40 (1912), 305-323:

Given a compact metric space X, a map $f : X \to X$ is called **minimal** if for any closed set $A \subset X$ such $f(A) \subseteq A$ we must have $A = X$ or $A = \emptyset$.

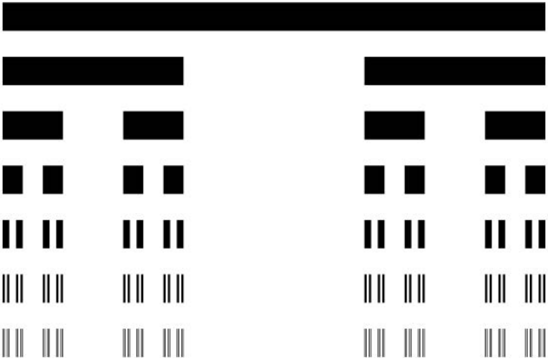
Equivalently, f is said to be **minimal** if the forward orbit $\{f^n(x) : n = 1, 2, \dots\}$ is dense in X.

In such a case X is called a **minimal space**.

What are minimal spaces?

- ***Any dynamical system*** on a compact space ***contains*** minimal subsystems
- Minimal sets/systems are ***building blocks*** for more complicated ones
- ***Periodic orbits*** are among the simplest examples
- Whether a given space admits a minimal map is still ***unknown for large classes of spaces***
- For ***n-manifolds*** the question is fully answered only for $n < 3$
- In higher dimensions mainly **isolated examples** are known
- If X is minimal, and D is its **decomposition into connected components** then the quotient space X/D must be the **Cantor set**, or a **finite set**.

The Cantor set



Note: Every compact metric space is a continuous image of the Cantor set.

Toy Models

Example

(Irrational rotations) Let $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ be the unit circle and $\alpha \notin \mathbb{Q}$. Then the irrational rotation

$$f(x) = x + \alpha$$

is minimal.

Example

(Denjoy homeomorphisms) The irrational rotation f can be modified by a "‘blow-up"’ of an orbit to form a minimal Cantor set homeomorphism.

Denjoy homeomorphisms

The process of inserting the intervals I_n into $S^1(1)$ to obtain a new circle $S^1(1+a)$ is expressed formally by a continuous map $g: S^1(1+a) \to S^1(1)$ which collapses each interval $I_n \subset S^1(1+a)$ to the corresponding point $x_n \in S^1(1)$ and is one-to-one outside I_*. We choose $a_n = \text{length}(I_n) > 0$ to satisfy

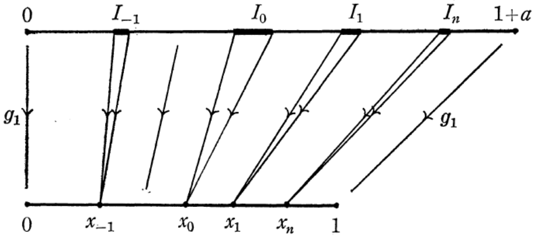
(1) $$a = \sum_{n \in \mathbf{Z}} a_n < \infty$$

so that the disjoint intervals I_n will all fit into $S^1(1+a)$ and

(2) $$\lim_{n \to \pm\infty} a_{n+1}/a_n = 1$$

so that f can be C^1. For example, $a_n = (1+n^2)^{-1}$ suffices. Then the map g is induced by the continuous map $g_1: [0, 1+a] \to [0, 1]$ defined (see Figure 4)

(3) $$g_1(y) = \limsup\{x_n \mid x_n + \sum_{x_k < x_n} a_k < y\} .$$



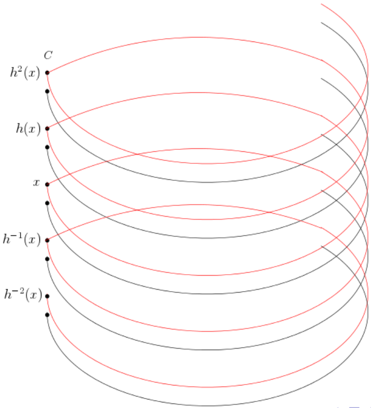
2-adic odometer

Example

Define a Cantor set homeomorphism $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ by “add one and carry”:
$\sigma(s) = (0, \dots, 0, 1, s_{k+1}, s_{k+2}, \dots)$, where $s_k = 0$ and $s_i = 1$ for all $i < k$ and
$\sigma(1, 1, 1, \dots) = (0, 0, 0, \dots)$.
The homeomorphism σ is minimal.

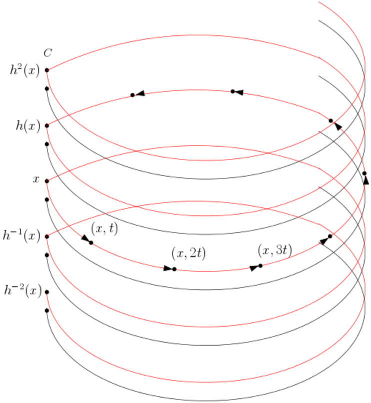
Suspensions

Let $h: C \to C$ be a minimal homeomorphism of a compact metric space C. The **suspension** of (C, h) is the space $X = C \times \mathbb{R}/_\sim$, where $\sim$ is the equivalence relation given by: $(x, y) \sim (p, q)$ if $y - q \in \mathbb{Z}$ and $p = h^{-y+q}(x)$.



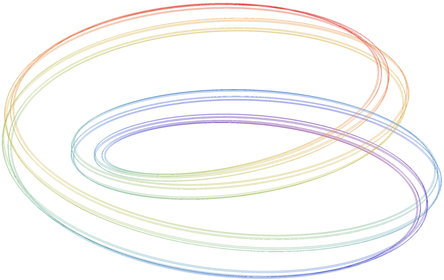
Suspension flows

The **suspension flow** defined by h is the continuous flow induced on X given by $\phi_t(x, s) = (x, s + t)/_\sim$. Since the orbits of h are dense, the flow orbits are dense in X, and so X is a continuum. For a generic choice of parameters t the flow ϕ_t is minimal.



Example

When h is the 2-adic odometer then the suspension is the 2-adic solenoid.



Outline

Contractions

- A map f is a **contraction** if there exists an $L < 1$ such that $d(f(x), f(y)) \leq Ld(x, y)$ for all $x, y \in X$;

Banach fixed point theorem (1922)

Every contraction on a complete metric space has a unique fixed point.

Local contractions

- A map f is a **local contraction** if for every $x \in X$ there exists an $L_x < 1$ and $q_x > 0$ such that $d(x, y) < q_x$ and $d(x, z) < q_x$ implies $d(f(y), f(z)) \leq L_x d(y, z)$;

Edelstein (1961)

For every local contraction f on a compact metric space X there exists an integer n such that f^n has a fixed point.

Weak local contractions

- A map f is a **weak local contraction** if for every $x \in X$ there exists an $r_x > 0$ such that $d(x, y) < r_x$ implies $d(f(x), f(y)) \leq d(x, y)$;
- A map f is a **local isometry** if for every $x \in X$ there exists an $R_x > 0$ such that $d(x, y) < R_x$ implies $d(f(x), f(y)) = d(x, y)$.

Edrei's conjecture (1952)

Suppose X is a compact metric space and $f : X \to X$ is a surjective weak local contraction. Then f is a local isometry.

Edrei's conjecture (1952)

Suppose X is a compact metric space and $f : X \to X$ is a weak local contraction. Then f is a local isometry.

Williams (1954)

4 examples constructed for which

- every point is a weak local contraction point,
- the map is not a local isometry at some points.

Definition

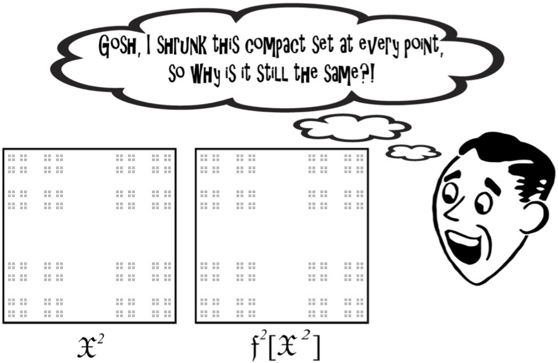
A map $f : X \to X$ is **locally radially shrinking** *(LRS) if for every $x \in X$ there exists an $\epsilon_x > 0$ such that $d(x, y) < \epsilon_x$ implies $d(f(x), f(y)) < d(x, y)$ for all $y \neq x$.*

Note: If a differentiable function f has LRS then $f'(x) < 1$ for every $x \in X$.

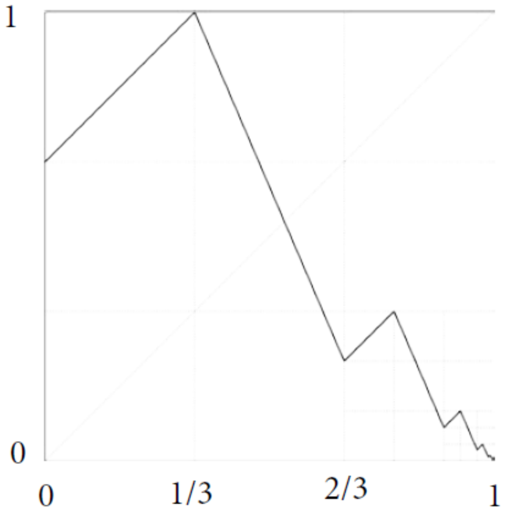
[Ciesielski, Jasinski, 2016]

There exists a minimal Cantor set homeomorphism $\mathfrak{f}$ that embeds in the real line $\mathbb{R}$ with **vanishing derivative everywhere**. Moreover, $\mathfrak{f}$ extends to a differentiable function $F : \mathbb{R} \to \mathbb{R}$

Note: The map $\mathfrak{f}$ above is the 2-adic odometer.



Picture by Ciesielski&Jasiński, **Canadian J. Math.** (2017)


1
0
0
1/3
2/3
1

[Ciesielski, Jasinski, 2016]

If X is an infinite compact metric space and a surjective $f : X \to X$ has the (LRS) property then there exists a perfect subset Y such that $f|Y$ is minimal.

Question

What Cantor set homeomorphisms can be embedded into $\mathbb{R}$ with vanishing derivative everywhere?

[Ciesielski, Jasinski, 2016]

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Outline

1. Introduction to minimality
2. Minimal systems meet Banach fixed point theorem
3. Minimality vs. LRS
4. Minimality and Cartesian Products

Piotr Oprocha, Kraków&Ostrava

Jiří Kupka, Ostrava

BORONSKI J.P.; KUPKA J.; OPROCHA P., *Edrei's Conjecture revisited*, **Annales Henri Poincaré** 19 (2018) 267–281

Let $\mathbf{s} = (s_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence of positive integers such that s_n divides s_{n+1}. For each $n \geq 1$ define $\pi_n \colon \mathbb{Z}_{s_{n+1}} \to \mathbb{Z}_{s_n}$ by the natural formula $\pi_n(m) = m \pmod{s_n}$ and let $G_\mathbf{s}$ denote the following inverse limit

$$G_\mathbf{s} = \varprojlim_n (\mathbb{Z}_{s_n}, \pi_n) = \left\{ x \in \prod_{i=1}^{\infty} \mathbb{Z}_{s_n} : x_n = \pi_n(x_{n+1}) \right\},$$

where each $\mathbb{Z}_{s_n}$ is given the discrete topology, and on $\prod_{i=1}^{\infty} \mathbb{Z}_{s_n}$ we have the Tychonoff product topology. On $G_\mathbf{s}$ we define $T_\mathbf{s} \colon G_\mathbf{s} \to G_\mathbf{s}$ by

$$T_\mathbf{s}(x)_n = x_n + 1 \pmod{s_n}.$$

Then $G_\mathbf{s}$ is a compact metrizable space and $T_\mathbf{s}$ is a homeomorphism, therefore $(G_\mathbf{s}, T_\mathbf{s})$ is a dynamical system (odometer).

IRAFM

All odometers are **equicontinuous**; i.e. for every $\varepsilon > 0$ there is $\delta > 0$ such that if $d(x, y) < \delta$ then $d(T^n(x), T^n(y)) < \varepsilon$ for every $n \geq 0$.

Theorem (B.,Kupka,Oprocha)

All odometers can be embedded into $\mathbb{R}$ *with derivative equal to* 0 *everywhere.*

We obtain the following immediate corollaries.

Lemma (Jarnik Theorem)

Let $X \subset \mathbb{R}$ be a perfect set and let $f \colon X \to \mathbb{R}$ be differentiable. Then there exists a differentiable extension $F \colon \mathbb{R} \to \mathbb{R}$ of f, that is $F|_X = f$.

Theorem (B.,Kupka,Oprocha)

Every odometer is conjugate to a homeomorphism $\mathfrak{f} : C \to C$ such that $\mathfrak{f}' \equiv 0$ and $\mathfrak{f}$ extends to a differentiable surjection $\bar{\mathfrak{f}} : \mathbb{R} \to \mathbb{R}$.

Corollary (B.,Kupka,Oprocha)

For every odometer (X, T) there exists an equivalent metric ρ such that T has (LRS) property with respect to ρ.

Corollary (B.,Kupka,Oprocha)

There exists a Cantor set $C \subseteq \mathbb{R}^2$ and a homeomorphism F such that F has the (LRS) property, $C = \bigcup_{i \in I} M_i$ where I is uncountable, $M_i \cap M_j = \varnothing$ for $i \neq j$ and (M_i, F) if minimal for every i.

Theorem (B.,Kupka,Oprocha)

There exists a minimal weakly mixing Cantor set homeomorphism $T : X \to X$ that embeds in $\mathbb{R}$ with vanishing derivative everywhere.

Theorem (B.,Kupka,Oprocha)

There exists a transitive, nonminimal and periodic point free Cantor set homeomorphism that embeds in $\mathbb{R}$ with vanishing derivative everywhere.

Theorem (B.,Kupka,Oprocha)

Every minimal dynamical system (X, T) with (LRS) property can be extended to a non-transitive dynamical system (Z, F) with (LRS) property.

Theorem (B.,Kupka,Oprocha)

There exists a Cantor set $W \subseteq \mathbb{R}^2$ and a nontransitive homeomorphism G with the (LRS) property such that the set of periodic points of G consists of a single fixed point.

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Question

Can every minimal Cantor set homeomorphism be embedded into $\mathbb{R}$ with vanishing derivative everywhere?

Outline

Piotr Oprocha, Kraków&Ostrava

Alex Clark, University of Leicester

Is minimality with respect to homeomorphisms preserved under Cartesian product in the class of compact spaces?

Observation

No product of a homeomorphism $h : X \to X$ with itself

$$(h, h) : X \times X \to X \times X$$

is minimal, by the fact that it keeps the diagonal

$$\Delta = \{(x, x) : x \in X\}$$

invariant.

Toy Models

1. If X is a **Cantor set** then $X \times X$ is a **Cantor set** as well, so minimality is preserved.
2. If $X = \mathbb{S}^1$ then $X \times X = \mathbb{T}^2$ and

$$H(x, y) = (x + \alpha, y + \beta)$$

is minimal if and only if $1, \alpha, \beta$ are $\mathbb{Q}$-independent.

Here we shall show that a **Cartesian power of a minimal spaces need not be minimal.**

Theorem

(B.,Clark,Oprocha)There exists a compact connected metric minimal space Y such that $Y \times Y$ is not minimal.

Almost Slovak space

A compact space X is an **almost Slovak space** if its homeomorphism group

$$\mathrm{H}(X) = \mathrm{H}_+(X) \cup \mathrm{H}_-(X),$$

with

$$\mathrm{H}_+(X) \cap \mathrm{H}_-(X) = \{\mathrm{id}_X\},$$

where $\mathrm{H}_+(X)$ is cyclic and generated by a minimal homeomorphism, and for every $g \in \mathrm{H}_-(X)$ we have $g^2 \in \mathrm{H}_+(X)$.

The Pseudo-circle

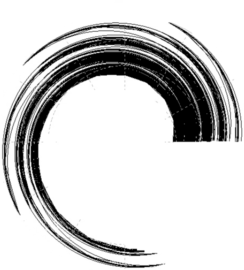
The *pseudo-arc* P is defined as the unique ***arc-like*** continuum that is ***hereditarily indecomposable***.

- P is ***arc-like*** means that for each $\epsilon > 0$ there exists a map $f_\epsilon : P \to [0,1]$ with $diam(f^{-1}(t)) < \epsilon$ for every t; alternatively, $P = \lim_{\leftarrow}\{[0,1], f_i\}$
- P is ***indecomposable*** means that it ***does not decompose*** into the union of ***two proper subcontinua***
- P is ***hereditarily indecomposable*** means that all subcontinua of P are indecomposable

The *pseudo-circle* is defined as the unique ***circle-like***, ***hereditarily indecomposable*** and **plane separating** continuum.

The Pseudo-arc

The pseudo-arc P may be considered as a very ***bad fractal***, as it is hereditarily equivalent, and so it has a self-similarity feature. **Hereditary equivalence** means that every subcontinuum of P is homeomorphic to P.



Note: No indecomposable continuum is a continuous image of $[0, 1]$**!**

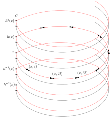
Pseudo-arcs and Pseudo-circles in Dynamics (select results)

- (1982) Handel: pseudo-circle as an attracting minimal set of a C^∞-smooth diffeomorphism of the plane
- (1996) Kennedy&Yorke: constructed a C^∞ diffeomorphism on a 7-manifold which has an invariant set with an uncontable number of pseudocircle components and is stable to C^1 perturbations
- (2010) Chéritat: pseudo-circle as a boundary of a Siegel disk of a holomorphic map
- (2014) B.&Oprocha: pseudo-circle as a Birkhoff-type attractor on the 2-torus
- (2016) Rempe-Gillen: pseudo-arc as a compactification Julia set component of entire functions of disjoint type

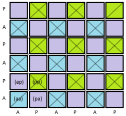
Outline of the proof:

Theorem. *(B.,Clark,Oprocha)There exists a compact connected metric minimal space Y such that $Y \times Y$ is not minimal.*

1. Start with a **minimal suspension flow homeomorphism**.



2. Perform a **"surgery"** inserting obstacles in place of one of the orbits.
3. The resulting space will have an **almost cyclic homeomorphism group** and will be **factorwise rigid**.



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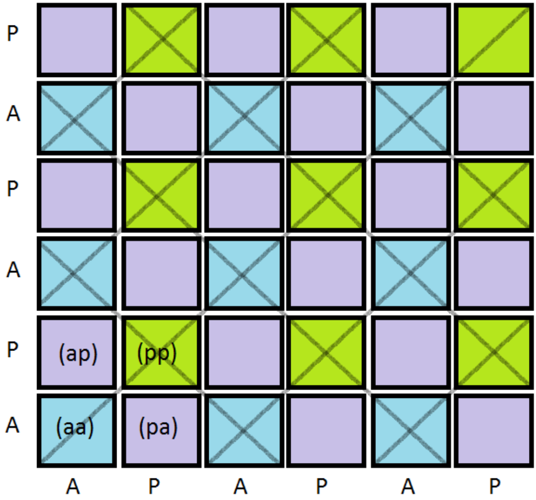

The square $W \times W$ of the special composant W

Thanksgiving

Thank You for Your Attention

Graphs

By a *graph* we mean a pair $G = (V, E)$ of finite sets, where $E \subset V \times V$ (V ... set of *vertices*, E ... set of *edges*).

The graphs we consider are always *edge surjective*, i.e. for every $v \in V$ there are $u, w \in V$ such that $(u, v), (v, w) \in E$.

Graph homomorphisms

A map $\phi: V_1 \to V_2$ is a *graph homomorphism* between graphs (V_1, E_1), (V_2, E_2) if for every $(u, v) \in E_1$ we have $(\phi(u), \phi(v)) \in E_2$.

A homomorphism ϕ is *bidirectional* if $(u, v), (u, v') \in E_1$ implies $\phi(v) = \phi(v')$ and $(w, u), (w', u) \in E_1$ implies $\phi(w) = \phi(w')$.

bd-covers

If ϕ is a bidirectional map between edge-surjective graps then we call it *bd-cover*.

Construction [Akin, Glasner, Weiss, 2008]

Let $\mathcal{G} = \langle \phi_i \rangle_{i=0}^{\infty}$ denote a sequence of bd-covers $\phi_i \colon (V_{i+1}, E_{i+1}) \to (V_i, E_i)$, and let

$$V_{\mathcal{G}} = \varprojlim (V_i, \phi_i) = \{x \in \Pi_{i=0}^{\infty} V_i : \phi_i(x_{i+1}) = x_i \text{ for all } i \geq 0\}$$

be the inverse limit defined by $\mathcal{G}$.

Set

$$E_{\mathcal{G}} = \{e \in V_{\mathcal{G}} \times V_{\mathcal{G}} : e_i \in E_i \text{ for each } i = 1, 2, \dots\}$$

As usual, V_i is endowed with discrete topology and $\mathbb{X} = \prod_{i=0}^{\infty} V_i$ is endowed with product topology.

Lemma (Shimomura, 2016)

Let $\mathcal{G} = \langle \phi_i \rangle$ be a sequence of bd-covers $\phi_i \colon (V_{i+1}, E_{i+1}) \to (V_i, E_i)$. Then $V_{\mathcal{G}}$ is a zero-dimensional compact metric space and the relation $E_{\mathcal{G}}$ defines a homeomorphism.