A look on some results about Camassa-Holm type equations

Igor Leite Freire

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Mathematical Institute in Opava
Silesian University in Opava
Centro de Matemática, Computação e Cognição
Universidade Federal do ABC

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- to show that the CH and other similar equations can be derived using arguments from symmetries and conserved quantities;
- discuss some weak solutions of these equations.

Glossary

Consider an equation $\Delta(x, t, u, u_{(1)}, \cdots, u_{(n)}) = 0$.

Infinitesimal generator (of point symmetry)

Operator $X = \tau(x,t,u)\partial_t + \xi(x,t,u)\partial_x + \eta(x,t,u)\partial_u$ satisfying the condition $X^{(n)}\Delta = 0$ when $\Delta = 0$.

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Recursion operator

An operator $\mathcal{R}$ with the following property: if $X_Q = Q\partial_u$ is an evolutionary symmetry of $\Delta = 0$, then $X_{\mathcal{R}Q} = (\mathcal{R}Q)\partial_u$ is also another evolutionary symmetry.

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Conserved vector and conservation law

A vector field $C = (C^0, C^1)$, depending on x, t, u and derivatives of u, is called conserved vector for the equation $\Delta = 0$ if its space-time divergence $D_t C^0 + D_x C^1 = 0$ (conservation law) when $\Delta = 0$ and all of its differential consequences.

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☞ If the domain of the equation is the entire real line, and if $C^1 \to 0$ when $|x| \to \infty$, then the quantity

$$H = \int_{\mathbb{R}} C^0 dx$$

is constant (constant of motion).

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1834 (J. S. Russell): solitary waves

A wave described by a function $u(x, t) = \phi(x - ct)$ such that $u(t, x) \to u_\pm$ as $x - ct \to \pm\infty$, where $u_\pm$ are constants. In Russell's observation there was a particular solitary wave like a pulse propagating in a channel with water.

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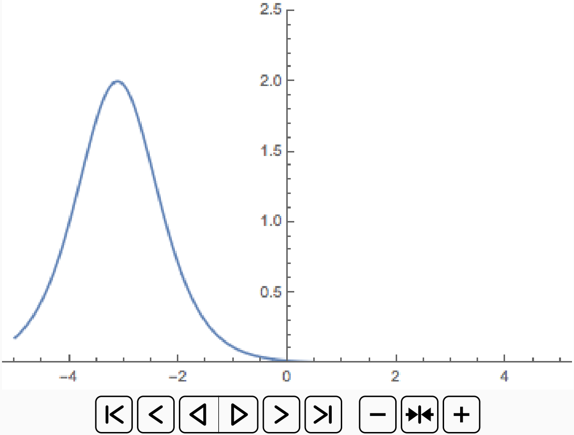
1895 (D. Korteweg, G. de Vries): the KdV equation

An equation that might explain the wave observed by Russel:

$u_t - 6uu_x + u_{xxx} = 0, \qquad u = u(x,t).$

A solitary wave: 1-soliton of KdV equation



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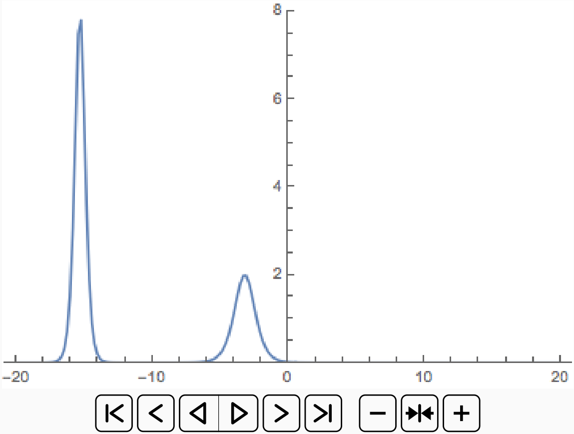
- They can interact with other solitons, and emerge from the collision unchanged, except for a phase shift.

2-soliton



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Comment about solutions

The solitons of the KdV equation are smooth!

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These structures are related to what today is called integrability.

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An equation is said to be integrable if it has at least one of the following strucutres: bi-Hamiltonian structure, a Lax pair, or infinitely many generalized symmetries.

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☞ Usually, but not always, "superposition of soliton-like solutions" is an indicative of integrable equations.

Camassa-Holm's work

R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Let., vol. 71, 1661–1664, (1993)

$m_t + 2\,u_x\,m + u\,m_x = 0, \quad m = u - u_{xx}.$

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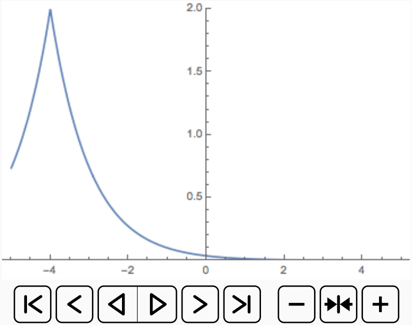
- a "sort of soliton" given by $u(x,t) = ce^{-|x-ct|}$;
- a "sort of superposition" of these solutions, given by

$$u(x,t) = \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|}.$$


2.0
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$$q_j'(t) = \sum_{k=1}^{N} p_k(t)\, e^{-|q_j(t)-q_k(t)|},$$

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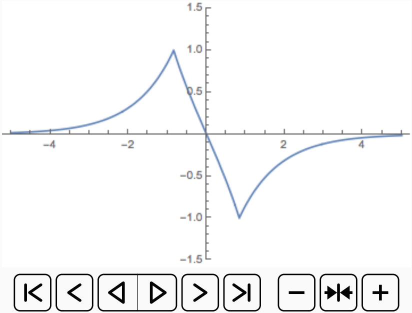

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☞ Only the DP ($b = 3$) and CH ($b = 2$) equations are integrable.

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Symmetries of a family of equations including the $b-$ equation (although it had not been discovered yet!):

$X_1 = \partial_x,\ X_2 = \partial_t,\ X_3 = u\partial_u - t\partial_t.$

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Y. Bozhkov, I. L. Freire and N. H. Ibragimov, Group analysis of the Novikov equation, Comp. Appl. Math., 33, 193–202, (2014).

Symmetries of the Novikov equation:

$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = 2u\partial_u - t\partial_t, \quad X_\pm = e^{\pm x}\partial_x \pm e^{\pm x}u\partial_u.$

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Sobolev space $H^1(\mathbb{R})$

It is the space of functions (including distributions) endowed with the norm defined above.

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☞P. L. da Silva and I. L. Freire, An equation unifying both Camassa-Holm and Novikov equations, Discrete Contin. Dyn. Syst., Suppl., 304–311, (2015).

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Answers

Other integrable members?

No. See M. Hay, A. N.W. Hone, V. S. Novikov and J. P. Wang, Remarks on certain two-component systems with peakon solutions, arXiv:1805.03323, (2018).

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S. C. Anco, P. L. da Silva and I. L. Freire, A family of wave-breaking equations generalizing the Camassa-Holm and Novikov equations, J. Math. Phys., 56, 091506, (2015).

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Observe that the Sobolev norm is conserved if $b = (p+1)c$**, that is, we have the equation** $u_t - u_{txx} + au^pu_{xx} - (p+1)cu^{p-1}u_xu_{xx} - cu^pu_{xxx} = 0.$

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Weak formulation for travelling waves

A weak solution of the ODE is a function satisfying the integral equation, for any test function ψ

$$0 = \int_{\mathbb{R}} (v(\psi'' - \psi)\phi' + (a\psi - c\psi'')\phi^p \phi')dz$$

$$+\frac{1}{2}\int_{\mathbb{R}} (b - 3pc)\psi'\phi^{p-1}(\phi')^2 + (p-1)(b - pc)\psi\phi^{p-2}(\phi')^3)dz.$$

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If the equation has the 1-peakon solution $u(x,t) = (v/c)^{1/p} e^{-|x-vt|}$ and also has the Sobolev norm $\|u\|_{H^1}$ as a conserved quantity, then, after a scaling in t, we have $u_t - u_{txx} + (p+2)u^p u_{xx} = (p+1)u^{p-1} u_x u_{xx} + u^p u_{xxx}$ or its equivalent form $m_t + (p+1)u^{p-1} u_x m + u^p m_x = 0$.

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- Substituting these quantities into $m_t + (p+1)u^{p-1}u_x m + u^p m_x = 0$ and integrating against test functions, we have the following:

Multipeakon solution

Theorem

The equation

$$m_t + (p+1)u^{p-1}u_x m + u^p m_x = 0$$

admits $u(x,t) = \sum_{i=1}^{N} p_i(t)e^{-|x-q_i(t)|}$ as a multipeakon solution if the functions $p_i, q_i,\ i = 1, \dots, N$, satisfy the following dynamical system:

$$p_i' = p_i \sum_{i_1,\dots,i_b=1}^{N} \operatorname{sign}(q_i - q_{i_1}) p_{i_1} \dots p_{i_b} e^{-|q_j - q_{i_1}| - \dots - |q_j - q_{i_b}|},$$

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What could be done?

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What could be done?

We can try to have some information for the case in which we have 2 peakons.

2-peakons: what can be done

Let us consider a solution given by
$u(x,t) = p_1(t)e^{-|x-q_1(t)|} + p_2(t)e^{-|x-q_2(t)|}$.

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Qualitative analysis of the dynamical system

$$q_1' = A_1^p,\ q_2' = A_2^p,$$

$$A_1 = (H + p_1^2 - p_2^2)/(2p_1),\ A_2 = (H - p_1^2 + p_2^2)/(2p_1),$$

$$p_1' = \frac{1}{2}\text{sign}(q_1 - q_2)A_1^{p-1}(H - p_1^2 - p_2^2),$$

$$p_2' = -\frac{1}{2}\text{sign}(q_1 - q_2)A_2^{p-1}(H - p_1^2 - p_2^2).$$

Some work in progress

Our old friend: $u_t - u_{txx} + au^p u_{xx} - bu^{p-1} u_x u_{xx} - cu^p u_{xxx} = 0.$

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Kink solution: $u(x,t) = \sum_{j=1}^{N} c_j \mathbf{sign}(x - q_j(t))(e^{-|x-q_j(t)|} - 1).$

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- Example of 2-kink solutions with $p = 1$ (B. Xia and Z. Qiao, Physics Letters A, 377(2013)2340–2342)

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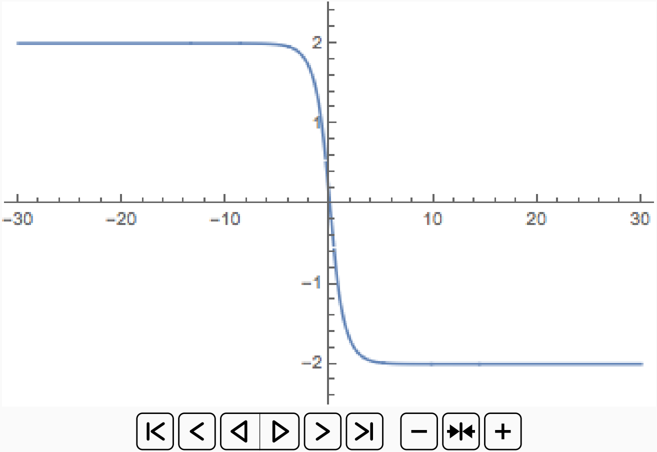
$$\begin{aligned} u(x,t) &= \text{sign}(x - \tfrac{1}{2})\ln(e^{2t}+1)(e^{-|x-\frac{1}{2}\ln(e^{2t}+1)|} - 1) \\ &\quad + \text{sign}(x + \tfrac{1}{2})\ln(e^{2t}+1)(e^{-|x+\frac{1}{2}\ln(e^{2t}+1)|} - 1). \end{aligned}$$

- For $p > 1$ we hope to report some results soon!

Simulation of the solution



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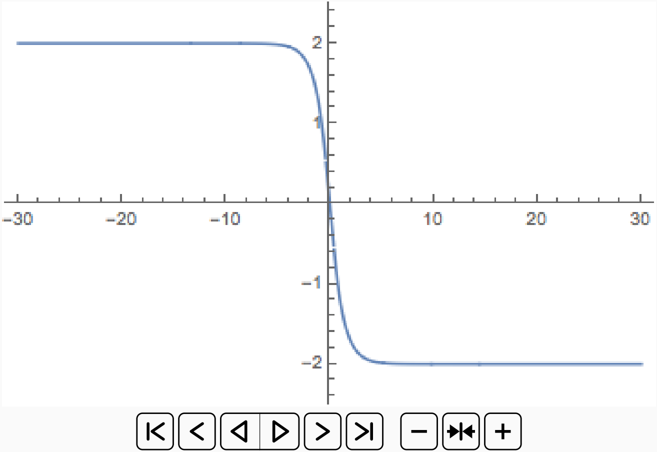

Thank you! ☺