Rota—Baxter operators

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Rota—Baxter operator

Definition 1

Let A be an algebra, R be a linear map on A
R is called Rota—Baxter operator (RB-operator) if for all $x, y \in A$

$$R(x)R(y) = R\big(R(x)y + xR(y) + \lambda xy\big)$$

where $\lambda \in F$ is a fixed scalar from the ground field (weight of R)

A is called Rota—Baxter algebra (RB-algebra)

The operators 0 and $-\lambda\,\mathrm{id}$ are called *trivial* RB-operators

G. Baxter, 1960
G.-C. Rota, P. Cartier, 1960–70s
A.A. Belavin, V.G. Drinfel'd, 1982, M.A. Semenov-Tyan-Shanskii, 1983: connection with Yang—Baxter equation
Li Guo, 2012, An Introduction to Rota—Baxter Algebra (monograph)

Examples of Rota—Baxter operators

Example 1

A: an algebra of continuous functions on $\mathbb{R}$,

$\Rightarrow \quad R(f)(x) = \int\limits_0^x f(t)\,dt$ is an RB-operator of weight 0

Example 2

A: the polynomial algebra $F[x]$

$\Rightarrow \quad R(x^n) = \frac{x^{n+1}}{n+1}$ is an RB-operator of weight 0

Example 3

A: algebra, d: invertible derivation on A $(d(xy) = d(x)y + xd(y))$

$\Rightarrow \quad d^{-1}$ is an RB-operator of weight 0

Example 4

Let A be an associative algebra, a is an idempotent

$\Rightarrow \quad l_a\colon x \to ax$ is an RB-operator of weight -1

Examples of Rota—Baxter operators (cont.)

Example 5

A: $\mathbb{Z}_2$-graded algebra $A = A_0 \oplus A_1$ with abelian odd part
$\Rightarrow \quad R\colon A_0 \to A_1 \to (0)$ is an RB-operator of weight 0

Example 6

$A = A_1 \dot{+} A_2$: a direct sum (as vector space) of subalgebras

$$R(x_1 + x_2) = -\lambda x_2, \quad x_1 \in A_1,\ x_2 \in A_2,$$

is an RB-operator of weight λ (splitting RB-operator)

R: an RB-operator of weight $\lambda \neq 0$
$\Rightarrow \quad \lambda^{-1}R$ is an RB-operator of weight 1

A: an algebra, R: an RB-operator on A, ψ: (anti)automorphism of A
$\Rightarrow \quad \psi^{-1}R\psi$ is an RB-operator on A of the same weight

Classical Yang—Baxter equation

Let $\mathfrak{g}$ be a semisimple f.d. Lie algebra over $\mathbb{C}$. For $r = \sum a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$, introduce CYBE as

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0,$$

where $r_{12} = \sum a_i \otimes b_i \otimes 1 \in (U(\mathfrak{g}))^{\otimes 3}$, $r_{13} = \sum a_i \otimes 1 \otimes b_i$ and $r_{23} = \ldots$

J.B. McGuire, C.N. Yang, R.J. Baxter, A.B. Zamolodchikov, L.D. Faddeev

Let r be a skew-symmetric ($\sum a_i \otimes b_i = -\sum b_i \otimes a_i$) solution of CYBE
A linear map $R\colon \mathfrak{g} \to \mathfrak{g}$ defined as $R(x) = \sum \langle a_i, x\rangle b_i$
is an RB-operator on $\mathfrak{g}$ of weight 0 $\quad$ ($\langle\cdot,\cdot\rangle$ = Killing form on $\mathfrak{g}$)
[M. Goncharov, 2017]: similar for nonzero weight and simple f.d. Lie algebra

Example 7

Up to conjugation and scalar multiple unique skew-symmetric solution of CYBE on $\mathrm{sl}_2(\mathbb{C})$ is $e \otimes h - h \otimes e$. It corresponds to the RB-operator $R(e) = 0$, $R(f) = 4h$, $R(h) = -8e$

Associative Yang—Baxter equation

Let A be an associative algebra, $r = \sum a_i \otimes b_i \in A \otimes A$
Associative Yang—Baxter equation [V.N. Zhelyabin, 1998]:

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = 0$$

Let r be a solution of AYBE. A linear map $P_r \colon A \to A$ defined as $P_r(x) = \sum a_i x b_i$ is an RB-operator on A of weight 0 ([M. Aguiar, 2000])

$M_n(\mathbb{C})$: r — solution of AYBE $\Longleftrightarrow$ P_r — RB-operator of weight 0 (V.G., P. Kolesnikov)

Example 8 [M. Aguiar, 2000]

Up to conjugation, transpose and scalar multiple all nonzero solutions of AYBE on $M_2(\mathbb{C})$ are
(a) $(e_{11} + e_{22}) \otimes e_{12}$, (b) $e_{12} \otimes e_{12}$, (c) $e_{22} \otimes e_{12}$, (d) $e_{11} \otimes e_{12} - e_{12} \otimes e_{11}$

[V.V. Sokolov, 2013]: classification of all skew-symmetric solutions of AYBE on $M_3(\mathbb{C})$ (8 cases)

Modified Yang—Baxter equation

Let A be an algebra, R a linear map on A, introduce MYBE [M.A. Semenov-Tyan-Shansky, 1983]

$$R(x)R(y) - R(R(x)y + xR(y)) = -xy$$

R is a solution of MYBE $\iff$ $R + \mathrm{id}$ is an RB-operator of the weight -2

Connection with:
Hamiltonian systems, quantum groups, Poisson structures (algebra, double algebra, Lie group)

RB-operators: classification

Classification of all RB-operators on:

- $\mathrm{sl}_2(\mathbb{C})$ (nonzero weight) — Y. Pan, Q. Liu, C. Bai, L. Guo, 2012
- $\mathrm{sl}_2(\mathbb{C})$ (weight zero) — J. Pei, C. Bai, and L. Guo, 2014
  (see also P. Kolesnikov, 2015)
- $M_2(\mathbb{C})$ (weight zero) — X. Tang, Y. Zhang, and Q. Sun, 2014
- $\mathrm{sl}_3(\mathbb{C})$ (nonzero weight) — E. Konovalova, 2008
- Witt and Virasoro algebras (partly) — X. Gao et al., 2016
- $k[x]$ (partly) — H. Yu, 2016

X. Li, D. Hou, and C. Bai, 2007, RB-operators on pre-Lie algebras:

a) It is hard and less practicable to extend what we have done in section 5 and 6 [*Classification of all RB-operators of weight zero on pre-Lie algebras of dimension 2 and 3 (partly)*] to other cases since the Rota—Baxter relation involves the nonlinear quadratic equations.

b) Whether we can give a meaningful “classification rules” so that the classification of Rota—Baxter operators can be more “interesting”?

List of varieties and algebras

Associative algebras: $(x, y, z) := (xy)z - x(yz) = 0$

- Matrix algebra $M_n(F)$ of order n
- Grassmann algebra: $\mathrm{Gr}_n = \mathrm{As}\langle 1, e_1, \ldots, e_n \mid e_i e_j = -e_j e_i \rangle$

Alternative algebras: $(x, x, z) = 0$ and $(x, y, y) = 0$

- octonion algebra
- the matrix Cayley—Dickson algebra $C(F) = M_2(F) \oplus vM_2(F)$,

$$a(vb) = v(\bar{a}b), \quad (vb)a = v(ab), \quad (va)(vb) = b\bar{a}, \quad a, b \in M_2(F)$$

Jordan algebras: $xy = yx$ and $(x^2, y, x) = 0$

- a Jordan algebra $J(f) = F \cdot 1 \oplus V$ of a nondegenerate bilinear symmetric form f acting on V,

$$(\alpha \cdot 1 + v)(\beta \cdot 1 + u) = (\alpha\beta + f(v, u)) \cdot 1 + (\alpha u + \beta v)$$

- 27-dimensional Albert algebra $J = H_3(C(F))$, $A \circ B = AB + BA$

Properties of RB-operators of nonzero weight

Theorem 1 [P. Benito, 2010]

All RB-operators on a quadratic division algebra are trivial (0 or $-\lambda\,\mathrm{id}$)

A is called *algebraic* if any element generates in A a f.d. subalgebra
A is called *power-associative* if any element generates in A an associative subalgebra

Theorem 2

Given an algebraic power-associative algebra A without zero divisors and RB-operator R of weight zero on A, we have $R = 0$

Theorem 3

All RB-operators of nonzero weight on
a) odd dimensional simple Jordan algebra of a bilinear form,
b) the Grassmann algebra
are splitting

RB-operators on matrix algebra of nonzero weight

An RB-operator R on $M_n(\mathbb{C})$ of nonzero weight is called *diagonal*, if $\psi^{-1}R\psi(D_n) \subseteq D_n$ for some $\psi \in \mathrm{Aut}(M_n(\mathbb{C}))$

Theorem 4

For identity matrix I_n, there exists $\psi \in \mathrm{Aut}(M_n(\mathbb{C}))$ s.t. $\psi^{-1}R\psi(I_n) \in D_n$

Corollary

All RB-operators of nonzero weight on $M_2(\mathbb{C})$ and $M_3(\mathbb{C})$ are diagonal

Question 1. Are all RB-operators on $M_n(\mathbb{C})$ of nonzero weight diagonal?
Question 2. To classify all diagonal RB-operators on $M_n(\mathbb{C})$

Properties of RB-operators of weight zero

Lemma 1

A: unital power-associative algebra, R: RB-operator on A of weight 0
F: field of characteristic zero. Also, $(R(1))^k = 0$
a) If A is associative (alternative), then $R^{2k} = 0$
b) If A is Jordan, then $R^{3k-1} = 0$

Theorem 5

Let A be unital associative (alternative, Jordan) algebraic algebra over a field of characteristic zero. There exists N s.t. $R^N = 0$ for any RB-operator R on A of weight zero

A: algebra, define rb-index of A as

$$\mathrm{rb}(A) = \min\{n \in \mathbb{N} \mid R^n = 0 \text{ for any RB-operator of weight 0 on } A\}$$

rb-index

- $C(F)$: Cayley—Dickson matrix algebra $\Rightarrow$ $\boxed{\mathrm{rb}(C(F)) = 3}$ ✓
- $J(f)$: a Jordan algebra of a bilinear form $\Rightarrow$ $\boxed{\mathrm{rb}(J(f)) \leq 3}$ ✓

Moreover, if F is an algebraically closed field with $\mathrm{char}\, F \neq 2$, then

$$\mathrm{rb}(J(f)) = \begin{cases} 2, & \dim(J(f)) = 3 \\ 3, & \dim(J(f)) > 3 \end{cases} \quad \text{(for simple } J(f))$$

- A: Albert algebra over algebraically closed F $\Rightarrow$ $\boxed{5 \leq \mathrm{rb}(A) \leq 8}$?
- $\mathrm{Gr}_n = \langle 1, e_1, \ldots, e_n \rangle_{\mathrm{alg}}$: Grassmann algebra $\Rightarrow$ $\boxed{\mathrm{rb}(\mathrm{Gr}_n) \leq n+1}$?

Theorem 7

Let F be a field of characteristic 0, then $\boxed{\mathrm{rb}(M_n(F)) = 2n-1}$ ✓

Conjecture 1

Let A be a semisimple finite-dimensional associative (alternative) algebra over $\mathbb{C}$. If $A = A_1 \oplus \ldots \oplus A_k$ for simple A_i, then $\mathrm{rb}(A) = \max\limits_{i=1,\ldots,k} \{\mathrm{rb}(A_i)\}$

Postalgebras

Post-Lie algebra [B. Vallette, 2008]: the bracket $[,]$ is Lie and

$$(x\cdot y)\cdot z - x\cdot(y\cdot z) - (y\cdot x)\cdot z + y\cdot(x\cdot z) = [y,x]\cdot z,$$
$$x\cdot[y,z] = [x\cdot y, z] + [y, x\cdot z].$$

Postassociative algebra (*dendriform trialgebra*) [J.-L. Loday, 2004]: $\cdot \in \mathrm{As}$

$$(x \prec y) \prec z = x \prec (y \succ z + y \prec z + y\cdot z),$$
$$x \succ (y \succ z) = (x \succ y + x \prec y + x\cdot y) \succ z,$$
$$x \succ (y\cdot z) = (x \succ y)\cdot z, \quad (x \prec y)\cdot z = x\cdot(y \succ z),$$
$$(x\cdot y) \prec z = x\cdot(y \prec z), \quad (x \succ y) \prec z = x \succ (y \prec z).$$

For any variety: [C. Bai et al, 2013; V.G., P. Kolesnikov, 2013]

$$\mathrm{postVar} = \mathrm{postLie} \bullet \mathrm{Var}$$

Embedding of postalgebras into RB-algebras

Theorem 8 [K. Ebrahimi-Fard, 02'; J.-L. Loday, 07'; C. Bai et al, 13']

A: a Rota—Baxter algebra of a variety Var and weight 1
With respect to the operations

$$x \succ y = R(x)y, \quad x \prec y = xR(y), \quad x \cdot y = xy$$

A is a post-Var-algebra

E.g., in associative case: $R(x)yR(z) = (x \succ y) \prec z = x \succ (y \prec z)$

$$R(x)R(y)z = R(R(x)y + xR(y) + xy)z,$$

$$x \succ (y \succ z) = (x \succ y + x \prec y + x \cdot y) \succ z.$$

Theorem 9 [V.G., P. Kolesnikov, 2013]

Any post-Var-algebra could be injectively embedded into its universal enveloping RB-algebra of the variety Var and weight 1

Universal enveloping Rota—Baxter algebras of postalgebras were constructed in associative, commutative, and Lie cases (V.G., 2017)

Post-Lie algebra structures

Definition is due to [D. Burde et al., 2012] (equivalent to post-Lie algebra)

$\mathfrak{g} = \langle V, [\,,]\rangle$ and $\mathfrak{n} = \langle V, \{\,,\}\rangle$: Lie brackets on a vector space V
A *post-Lie algebra structure* on $(\mathfrak{g}, \mathfrak{n})$ is a bilinear product $x \cdot y$ satisfying

$$[x, y] = x \cdot y - y \cdot x + \{x, y\} \quad (1)$$

$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z) \quad (2)$$

$$x \cdot \{y, z\} = \{x \cdot y, z\} + \{y, x \cdot z\} \quad (3)$$

R: an RB-operator of weight 1 on a Lie algebra $\mathfrak{n} = \langle V, \{\,,\}\rangle$
$\Rightarrow$ $x \cdot y = \{R(x), y\}$ is a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ ($[,]$ is defined by (1))

A Lie algebra L is *complete* if all derivations on L are inner and $Z(L) = (0)$

Statement [C. Bai, L. Guo, X. Ni, 2010]

Let $\mathfrak{n}$ be a complete Lie algebra. Then all PA-structures on $(\mathfrak{g}, \mathfrak{n})$ are defined by RB-operators of weight 1 on $\mathfrak{n}$.

Known results about PA-structures

All Lie algebras are finite-dimensional over $\mathbb{C}$

- $\mathfrak{g}$ is nilpotent $\Rightarrow$ $\mathfrak{n}$ is solvable [D. Burde et al., 2012]
- $\mathfrak{g}$ is semisimple $\Rightarrow$ $\mathfrak{n}$ is not solvable [D. Burde, K. Dekimpe, 2013]
- $\mathfrak{g}$ is simple $\Rightarrow$ PA-structure is *trivial* [D. Burde, K. Dekimpe, 2016]

(either $x \cdot y = 0$, $[x, y] = \{x, y\}$ or $x \cdot y = [x, y] = -\{x, y\}$)

A Lie algebra L is *perfect* if $[L, L] = L$
A Lie algebra L is *unimodular* if all adjoint operators have trace zero

- $\mathfrak{n}$ is non-nilpotent solvable $\Rightarrow$ $\mathfrak{g}$ is not perfect [D. Burde et al., 2012]
- $\mathfrak{n}$ is semisimple $\Rightarrow$ $\mathfrak{g}$ is not solvable unimodular
- $\mathfrak{n}$ is semisimple $\Rightarrow$ $\mathfrak{g}$ can be non-nilpotent solvable

both: [D. Burde, K. Dekimpe, 2013]

Homomorphic nature of RB-operators

R: an RB-operator of weight 1 on $\mathfrak{n}$

$\Rightarrow \quad x \cdot y = \{R(x), y\}$ is a PA-structure on $(\mathfrak{g}, \mathfrak{n})$

Moreover, R is the RB-operator of weight 1 on $\mathfrak{g}$

Denote by $\mathfrak{g}_i$ be the Lie algebra structure on V defined by

$$[x, y]_0 = \{x, y\},$$
$$[x, y]_{i+1} = [R(x), y]_i + [x, R(y)]_i + [x, y]_i,$$

for all $i \geq 0$. Then R defines a PA-structure on each pair $(\mathfrak{g}_{i+1}, \mathfrak{g}_i)$

Both R and $R + \mathrm{id}$ are Lie algebra homomorphisms from $\mathfrak{g}_{i+1}$ to $\mathfrak{g}_i$

$\Rightarrow \quad$ we obtain a composition of homomorphisms

$$\mathfrak{g}_i \xrightarrow[R+\mathrm{id}]{R} \mathfrak{g}_{i-1} \xrightarrow[R+\mathrm{id}]{R} \cdots \xrightarrow[R+\mathrm{id}]{R} \mathfrak{g}_0 = \mathfrak{n}$$

New results about PA-structures

[D. Burde et al., 2012]: PA-structures with $\mathfrak{n} = \mathrm{sl}_2(\mathbb{C})$ exist on $(\mathfrak{g}, \mathfrak{n})$ $\Leftrightarrow$ $\mathfrak{g} \cong \mathrm{sl}_2(\mathbb{C})$ or $\mathfrak{g} \cong \mathfrak{r}_{3,\lambda}(\mathbb{C})$, $\lambda \neq -1$ ($[e_1, e_2] = e_2$, $[e_1, e_3] = \lambda e_3$)

[D. Burde, V.G., 2018]: classification of all $\mathfrak{g}$ which appear in all PA-structures with $\mathfrak{n} = \mathrm{sl}_2(\mathbb{C}) \oplus \mathrm{sl}_2(\mathbb{C})$

Theorem 10 [D. Burde, V.G., 2018]

Let $x \cdot y$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ over $\mathbb{C}$, where $\mathfrak{n}$ is simple and $\mathfrak{g}$ is semisimple. Then $\mathfrak{g}$ is also simple, and the PA-structure is trivial

Proof of Theorem 10

Step 1. $x \cdot y = \{R(x), y\}$ for an RB-operator R of weight 1 on $\mathfrak{n}$
We know $\ker(R), \ker(R+\mathrm{id}) \lhd \mathfrak{g}$ and $\ker(R) \cap \ker(R+\mathrm{id}) = (0)$
Assume $\mathfrak{g} \ncong \mathfrak{n}$, so both kernels are nonzero and

$$\mathfrak{g} = \ker(R) \oplus \ker(R+\mathrm{id}) \oplus \mathfrak{s}$$

Step 2. $\mathfrak{n} = \mathrm{Im}(R) + \mathrm{Im}(R+\mathrm{id})$ since $x = R(-x) + (R+\mathrm{id})(x)$, so

$$\mathrm{Im}(R) \cong \ker(R+\mathrm{id}) \oplus \mathfrak{s}, \quad \mathrm{Im}(R+\mathrm{id}) \cong \ker(R) \oplus \mathfrak{s}$$

This yields a semisimple decomposition

$$\mathfrak{n} = (\ker(R+\mathrm{id}) \oplus \mathfrak{s}) + (\ker(R) \oplus \mathfrak{s})$$

If $\mathfrak{s} \neq (0)$, both summands aren't simple, contradiction to [A. Onishchik, 69']

Step 3. $\mathfrak{s} = (0)$, then $\mathfrak{n} = \mathrm{Im}(R) \dotplus \mathrm{Im}(R+\mathrm{id})$
[J.L. Koszul, 1977] implies that $\mathfrak{n} \cong \mathrm{Im}(R) \oplus \mathrm{Im}(R+\mathrm{id})$, but $\mathfrak{n}$ is simple

Lemma 3 [D. Burde, V.G., 2018]

Let $\mathfrak{g} = \mathfrak{s}_1 + \mathfrak{s}_2$ be the vector space sum of two complex semisimple subalgebras of $\mathfrak{g}$. Then $\mathfrak{g}$ is semisimple

Theorem 11 [D. Burde, V.G., 2018]

Suppose there is a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ over $\mathbb{C}$, where $\mathfrak{g}$ is semisimple and $\mathfrak{n}$ is complete. Then $\mathfrak{n}$ must be semisimple

Theorem 12 [D. Burde, V.G., 2018]

Let $x \cdot y = \{R(x), y\}$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ over $\mathbb{C}$ defined by an RB-operator R of weight 1 on $\mathfrak{n}$. Assume $\mathfrak{n}$ and $\mathfrak{g}_k$, $k = \dim(V)$, are semisimple. Then all $\mathfrak{g}_i$ are isomorphic to $\mathfrak{n}$

Conjecture 2. Suppose we have a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ over $\mathbb{C}$, where $\mathfrak{g}$ and $\mathfrak{n}$ are semisimple. Then $\mathfrak{g} \cong \mathfrak{n}$

Thank You for Your attention!