Ostrava Seminar on Mathematical Physics
March 28, 2018

Spectral properties of periodic Schrödinger operators with δ'-potentials

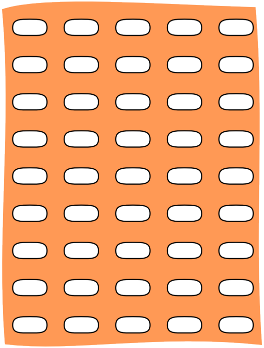
Andrii Khrabustovskyi

Graz University of Technology

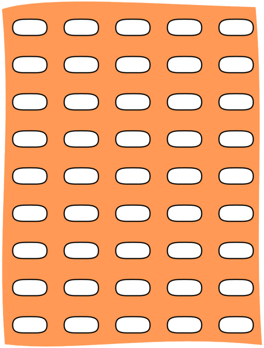
joint work with Pavel Exner
arXiv:1802.07522

Our goal in a nutshell

Construct periodic media with prescribed spectral properties

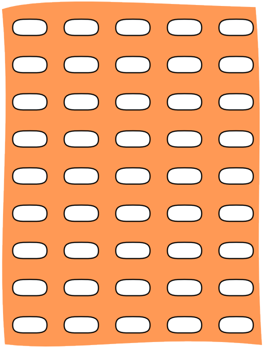


The problem we are going to study relates to the spectral theory of elliptic self-adjoint differential operators with **periodic** coefficients in unbounded domains with some **periodic** structure.



The problem we are going to study relates to the spectral theory of elliptic self-adjoint differential operators with **periodic** coefficients in unbounded domains with some **periodic** structure.

It is known that the spectra of such operators have a band structure, i.e. the spectrum is a locally finite union of compact intervals called **bands**.



The problem we are going to study relates to the spectral theory of elliptic self-adjoint differential operators with **periodic** coefficients in unbounded domains with some **periodic** structure.

It is known that the spectra of such operators have a band structure, i.e. the spectrum is a locally finite union of compact intervals called **bands**.

The interval (a, b) is called a **gap** in the spectrum $\sigma(\mathcal{A})$ of the operator $\mathcal{A}$ if

$$(a, b) \cap \sigma(\mathcal{A}) = \varnothing \text{ and } a, b \in \sigma(\mathcal{A}).$$

Example: Periodic Schrödinger operator on the line

We consider the following operator acting in $\mathsf{L}^2(\mathbb{R})$:

$$\mathcal{A}u = -u'' + Vu, \quad \mathrm{dom}(\mathcal{A}) = \mathsf{H}^2(\mathbb{R}).$$

The potential $V \in \mathsf{C}(\mathbb{R})$ satisfies $V(x+1) = V(x),\ \forall x.$

Example: Periodic Schrödinger operator on the line

We consider the following operator acting in $\mathsf{L}^2(\mathbb{R})$:

$$\mathcal{A}u = -u'' + Vu, \quad \mathrm{dom}(\mathcal{A}) = \mathsf{H}^2(\mathbb{R}).$$

The potential $V \in \mathsf{C}(\mathbb{R})$ satisfies $V(x+1) = V(x),\ \forall x.$

To describe $\sigma(\mathcal{A})$ we inspect auxiliary operators $\mathcal{A}_\theta$, $\theta \in [0, 2\pi]$:

$$\mathcal{A}_\theta \text{ acts in } \mathsf{L}^2(0,1), \quad \mathcal{A}_\theta u = -u'' + Vu,$$
$$\mathrm{dom}(\mathcal{A}_\theta) = \left\{u \in \mathsf{H}^2(0,1):\ u(1) = u(0)e^{i\theta},\ u'(1) = u'(0)e^{i\theta}\right\}.$$

Example: Periodic Schrödinger operator on the line

We consider the following operator acting in $\mathsf{L}^2(\mathbb{R})$:

$$\mathcal{A}u = -u'' + Vu, \quad \mathrm{dom}(\mathcal{A}) = \mathsf{H}^2(\mathbb{R}).$$

The potential $V \in \mathsf{C}(\mathbb{R})$ satisfies $V(x+1) = V(x),\ \forall x.$

To describe $\sigma(\mathcal{A})$ we inspect auxiliary operators $\mathcal{A}_\theta$, $\theta \in [0, 2\pi]$:

$$\mathcal{A}_\theta \text{ acts in } \mathsf{L}^2(0,1), \quad \mathcal{A}_\theta u = -u'' + Vu,$$

$$\mathrm{dom}(\mathcal{A}_\theta) = \left\{u \in \mathsf{H}^2(0,1):\ u(1) = u(0)e^{i\theta},\ u'(1) = u'(0)e^{i\theta}\right\}.$$

- $\sigma(\mathcal{A}_\theta) = \{\lambda_k(\theta), k \in \mathbb{N}\}, \lambda_k(\theta) \underset{k\to\infty}{\nearrow} \infty.$

Example: Periodic Schrödinger operator on the line

We consider the following operator acting in $\mathsf{L}^2(\mathbb{R})$:

$$\mathcal{A}u = -u'' + Vu, \quad \mathrm{dom}(\mathcal{A}) = \mathsf{H}^2(\mathbb{R}).$$

The potential $V \in \mathsf{C}(\mathbb{R})$ satisfies $V(x+1) = V(x),\ \forall x.$

To describe $\sigma(\mathcal{A})$ we inspect auxiliary operators $\mathcal{A}_\theta$, $\theta \in [0, 2\pi]$:

$$\mathcal{A}_\theta \text{ acts in } \mathsf{L}^2(0,1), \quad \mathcal{A}_\theta u = -u'' + Vu,$$

$$\mathrm{dom}(\mathcal{A}_\theta) = \left\{u \in \mathsf{H}^2(0,1):\ u(1) = u(0)e^{i\theta},\ u'(1) = u'(0)e^{i\theta}\right\}.$$

- $\sigma(\mathcal{A}_\theta) = \{\lambda_k(\theta), k \in \mathbb{N}\}, \lambda_k(\theta) \underset{k\to\infty}{\nearrow} \infty.$
- For fixed k, the function $\lambda_k(\theta)$ is continuous.

Example: Periodic Schrödinger operator on the line

We consider the following operator acting in $\mathsf{L}^2(\mathbb{R})$:

$$\mathcal{A}u = -u'' + Vu, \quad \mathrm{dom}(\mathcal{A}) = \mathsf{H}^2(\mathbb{R}).$$

The potential $V \in \mathsf{C}(\mathbb{R})$ satisfies $V(x+1) = V(x),\ \forall x.$

To describe $\sigma(\mathcal{A})$ we inspect auxiliary operators $\mathcal{A}_\theta$, $\theta \in [0, 2\pi]$:

$$\mathcal{A}_\theta \text{ acts in } \mathsf{L}^2(0,1), \quad \mathcal{A}_\theta u = -u'' + Vu,$$

$$\mathrm{dom}(\mathcal{A}_\theta) = \left\{u \in \mathsf{H}^2(0,1) :\ u(1) = u(0)e^{i\theta},\ u'(1) = u'(0)e^{i\theta}\right\}.$$

- $\sigma(\mathcal{A}_\theta) = \{\lambda_k(\theta), k \in \mathbb{N}\}$, $\lambda_k(\theta) \underset{k\to\infty}{\nearrow} \infty.$
- For fixed k, the function $\lambda_k(\theta)$ is continuous.
- The range of the function $\lambda_k(\theta)$ is a compact interval.

Example: Periodic Schrödinger operator on the line

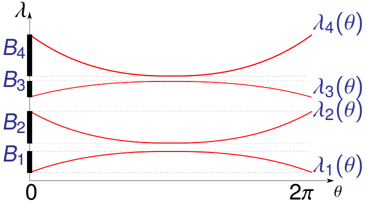
We consider the following operator acting in $\mathsf{L}^2(\mathbb{R})$:

$$\mathcal{A}u = -u'' + Vu, \quad \mathrm{dom}(\mathcal{A}) = \mathsf{H}^2(\mathbb{R}).$$

The potential $V \in \mathsf{C}(\mathbb{R})$ satisfies $V(x+1) = V(x),\ \forall x.$

To describe $\sigma(\mathcal{A})$ we inspect auxiliary operators $\mathcal{A}_\theta$, $\theta \in [0, 2\pi]$:

$$\mathcal{A}_\theta \text{ acts in } \mathsf{L}^2(0,1), \quad \mathcal{A}_\theta u = -u'' + Vu,$$
$$\mathrm{dom}(\mathcal{A}_\theta) = \left\{u \in \mathsf{H}^2(0,1):\ u(1) = u(0)e^{i\theta},\ u'(1) = u'(0)e^{i\theta}\right\}.$$



Example: Periodic Schrödinger operator on the line

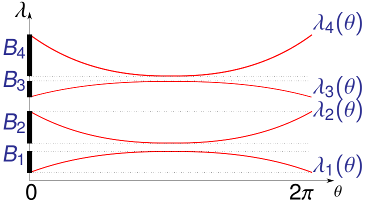
We consider the following operator acting in $\mathsf{L}^2(\mathbb{R})$:

$$\mathcal{A}u = -u'' + Vu, \quad \mathrm{dom}(\mathcal{A}) = \mathsf{H}^2(\mathbb{R}).$$

The potential $V \in \mathsf{C}(\mathbb{R})$ satisfies $V(x+1) = V(x),\ \forall x.$

To describe $\sigma(\mathcal{A})$ we inspect auxiliary operators $\mathcal{A}_\theta$, $\theta \in [0, 2\pi]$:

$$\mathcal{A}_\theta \text{ acts in } \mathsf{L}^2(0,1), \quad \mathcal{A}_\theta u = -u'' + Vu,$$
$$\mathrm{dom}(\mathcal{A}_\theta) = \left\{ u \in \mathsf{H}^2(0,1):\ u(1) = u(0)e^{i\theta},\ u'(1) = u'(0)e^{i\theta} \right\}.$$



Floquet-Bloch decomposition

$$\sigma(\mathcal{A}) = \bigcup_{k=1}^{\infty} B_k,$$

where $B_k = \{\lambda_k(\theta), \theta \in [0, 2\pi]\}$.

In general the presence of gaps in the spectrum is not guaranteed, e.g. $\sigma(-\Delta_{\mathbb{R}^n}) = [0, \infty)$.

In general the presence of gaps in the spectrum is not guaranteed, e.g. $\sigma(-\Delta_{\mathbb{R}^n}) = [0, \infty)$.

Example: periodic waveguides with gaps

Post (2003); Pankrashkin (2010); Nazarov (2009, 2010); Nazarov–Ruotsalainen–Taskinen (2010); Borisov (2015)

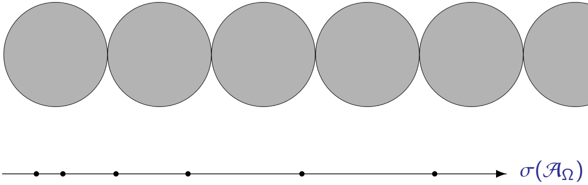
We denote by $\mathcal{A}_\Omega$ the Laplacian in the domain $\Omega \subset \mathbb{R}^n$ subject to Neumann or Dirichlet boundary conditions.

In general the presence of gaps in the spectrum is not guaranteed, e.g. $\sigma(-\Delta_{\mathbb{R}^n}) = [0, \infty)$.

Example: periodic waveguides with gaps

Post (2003); Pankrashkin (2010); Nazarov (2009, 2010); Nazarov–Ruotsalainen–Taskinen (2010); Borisov (2015)

We denote by $\mathcal{A}_\Omega$ the Laplacian in the domain $\Omega \subset \mathbb{R}^n$ subject to Neumann or Dirichlet boundary conditions.

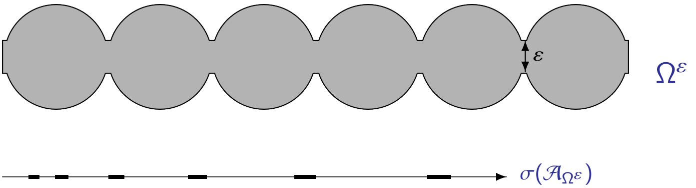


In general the presence of gaps in the spectrum is not guaranteed, e.g. $\sigma(-\Delta_{\mathbb{R}^n}) = [0, \infty)$.

Example: periodic waveguides with gaps

Post (2003); Pankrashkin (2010); Nazarov (2009, 2010); Nazarov–Ruotsalainen–Taskinen (2010); Borisov (2015)

We denote by $\mathcal{A}_\Omega$ the Laplacian in the domain $\Omega \subset \mathbb{R}^n$ subject to Neumann or Dirichlet boundary conditions.



Problem 1

For a given class $\mathcal{P}$ of periodic differential operators, construct an operator $\mathcal{A} \in \mathcal{P}$ with at least one gap in the spectrum

Problem 1

For a given class $\mathcal{P}$ of periodic differential operators, construct an operator $\mathcal{A} \in \mathcal{P}$ with at least one gap in the spectrum

The class $\mathcal{P}$ may consist of

- periodic Schrödinger operators, $-\Delta + V$
- periodic elliptic operators, $-\mathrm{div}(a(x)\nabla)$
- Laplace-Beltrami operators on periodic manifolds
- Laplacians acting in domains with periodic geometry
- periodic Maxwell operators
- ...and so on and so forth

Problem 1

For a given class $\mathcal{P}$ of periodic differential operators, construct an operator $\mathcal{A} \in \mathcal{P}$ with at least one gap in the spectrum

The class $\mathcal{P}$ may consist of

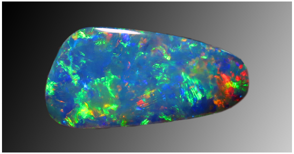
- periodic Schrödinger operators, $-\Delta + V$
- periodic elliptic operators, $-\mathrm{div}(a(x)\nabla)$
- Laplace-Beltrami operators on periodic manifolds
- Laplacians acting in domains with periodic geometry
- periodic Maxwell operators
- ...and so on and so forth

R. Hempel, O. Post, **Spectral gaps for periodic elliptic operators with high contrast: an overview**
Progress in analysis, Vol. I, II, 577587, World Sci. Publ., 2003.

P. Kuchment, **The mathematics of photonic crystals**
Ch. 7 in “Mathematical modeling in optical science”, 207–272, Frontiers Appl. Math., 22, SIAM, Philadelphia, PA, 2001.

Possible applications: photonic crystals

Photonic crystals are periodic nanostructures that have been attracting much attention in recent years. Their characteristic feature is that the electromagnetic waves of certain frequencies fail to propagate in them, which is caused by gaps in the spectrum of the corresponding Maxwell operators (or related scalar operators).



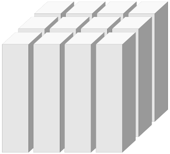
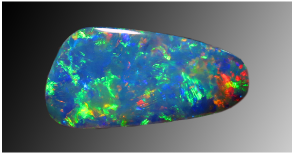


Figure 1: Opal – an example of natural photonic crystal (photo: https://en.wikipedia.org/wiki/Opal).

Figure 2: The dielectric constant equals 1 on the vertical columns, and it equals $\gg 1$ in the rest of the media.

Possible applications: photonic crystals

Photonic crystals are periodic nanostructures that have been attracting much attention in recent years. Their characteristic feature is that the electromagnetic waves of certain frequencies fail to propagate in them, which is caused by gaps in the spectrum of the corresponding Maxwell operators (or related scalar operators).



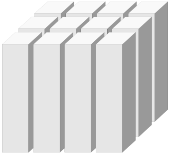


Figure 1: Opal – an example of natural photonic crystal (photo: https://en.wikipedia.org/wiki/Opal).

Figure 2: The dielectric constant equals 1 on the vertical columns, and it equals $\gg 1$ in the rest of the media.

W. Dörfler, A. Lechleiter, M. Plum, G. Schneider, C. Wieners,
Photonic Crystals. Mathematical Analysis and Numerical Approximation,
Springer, Berlin, 2011.

Problem 2

For a given class $\mathcal{P}$ of periodic differential operators, construct an operator $\mathcal{A} \in \mathcal{P}$ having gaps which are close (in some natural sense) to **predefined** intervals

- Laplace-Beltrami operators on periodic Riemannian manifolds
  [A. K., J. Differ. Equations 252(3) (2012)]
- Scalar elliptic operators in divergence form
  [A. K., Asympt. Analysis 82(1-2) (2013)]
- Laplacians posed in noncompact periodic domains
  [A. K., J. Math. Phys. 55(12) (2014)]
- Periodic quantum graphs
  [D. Barseghyan, A. K., J. Phys. A 48(25) (2015)]

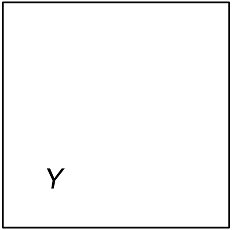
Problem 2

For a given class $\mathcal{P}$ of periodic differential operators, construct an operator $\mathcal{A} \in \mathcal{P}$ having gaps which are close (in some natural sense) to **predefined** intervals

- Laplace-Beltrami operators on periodic Riemannian manifolds
  [A. K., J. Differ. Equations 252(3) (2012)]
- Scalar elliptic operators in divergence form
  [A. K., Asympt. Analysis 82(1-2) (2013)]
- Laplacians posed in noncompact periodic domains
  [A. K., J. Math. Phys. 55(12) (2014)]
- Periodic quantum graphs
  [D. Barseghyan, A. K., J. Phys. A 48(25) (2015)]
- **Periodic Schrödinger operators with singular potentials**

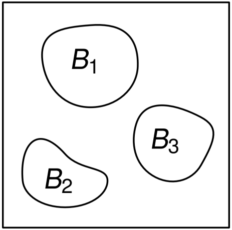
- $m \in \mathbb{N}$

- $m \in \mathbb{N}$
- $Y := (0,1)^n$, $n \geq 2$



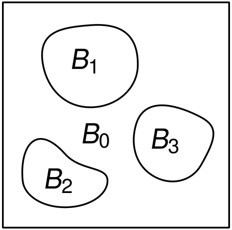
- $m \in \mathbb{N}$
- $Y := (0,1)^n$, $n \geq 2$
- B_j, $j = 1, \dots, m$ be domains satisfying

$$\overline{B_{j_1}} \cap \overline{B_{j_2}} = \varnothing, \quad \bigcup_{j=1}^{m} \overline{B_j} \subset Y$$



- $m \in \mathbb{N}$
- $Y := (0,1)^n$, $n \geq 2$
- B_j, $j = 1, \dots, m$ be domains satisfying

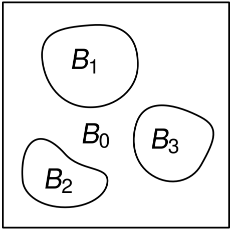
$$\overline{B_{j_1}} \cap \overline{B_{j_2}} = \varnothing, \quad \bigcup_{j=1}^{m} \overline{B_j} \subset Y$$



- $B_0 := Y \setminus \bigcup_{j=1}^{m} \overline{B_j}$

- $m \in \mathbb{N}$
- $Y := (0,1)^n$, $n \geq 2$
- B_j, $j = 1, \dots, m$ be domains satisfying

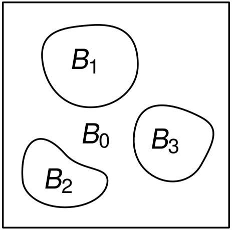
$$\overline{B_{j_1}} \cap \overline{B_{j_2}} = \varnothing, \quad \bigcup_{j=1}^{m} \overline{B_j} \subset Y$$



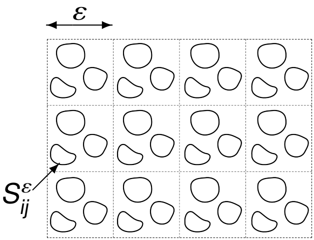
- $B_0 := Y \setminus \bigcup_{j=1}^{m} \overline{B_j}$
- $\varepsilon > 0$ - a small parameter

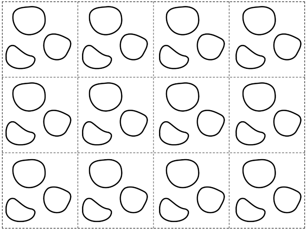
- $m \in \mathbb{N}$
- $Y := (0,1)^n$, $n \geq 2$
- B_j, $j = 1, \dots, m$ be domains satisfying

$$\overline{B_{j_1}} \cap \overline{B_{j_2}} = \varnothing, \quad \bigcup_{j=1}^{m} \overline{B_j} \subset Y$$



- $B_0 := Y \setminus \bigcup_{j=1}^{m} \overline{B_j}$
- $\varepsilon > 0$ - a small parameter
- $S_{ij}^\varepsilon := \varepsilon(\partial B_j + i)$, $i \in \mathbb{Z}^n$, $j = 1, \dots, m$





By $\mathcal{A}^\varepsilon$ we denote the operator acting in the space $\mathsf{L}^2(\mathbb{R}^n)$,

- $\mathcal{A}^\varepsilon u = -\Delta u$
- $u \in \mathrm{dom}(\mathcal{A}^\varepsilon)$ satisfies the following interface conditions on S_{ij}^ε:

$$\left(\frac{\partial u}{\partial n}\right)_{ij}^{+} = \left(\frac{\partial u}{\partial n}\right)_{ij}^{-}, \quad q_j \varepsilon^{-1} \left(\frac{\partial u}{\partial n}\right)_{ij}^{\pm} + \left((u)_{ij}^{-} - (u)_{ij}^{+}\right) = 0, \qquad (*)$$

where $q_j > 0$.

For $j = 1, \ldots, m$ we set:

$$a_j := q_j^{-1} |\partial B_j| |B_j|^{-1}.$$

It is assumed that the numbers a_j are pairwise non-equivalent. We renumber them in the ascending order: $a_j < a_{j+1}$, $j = 1, \ldots, m+1$.

For $j = 1, \dots, m$ we set:

$$a_j := q_j^{-1} |\partial B_j| |B_j|^{-1}.$$

It is assumed that the numbers a_j are pairwise non-equivalent. We renumber them in the ascending order: $a_j < a_{j+1}$, $j = 1, \dots, m+1$.

We consider the following equation (with unknown $\lambda \in \mathbb{C}$):

$$\mathcal{F}(\lambda) = 0, \text{ where } \mathcal{F}(\lambda) := 1 + \frac{1}{|B_0|} \sum_{i=1}^{m} \frac{q_j^{-1} |\partial B_j|}{\lambda - q_j^{-1} |\partial B_j| |B_j|^{-1}}.$$

It has exactly m roots b_j satisfying (after appropriate renumbering)

$$a_j < b_j < a_{j+1}, \ j = 1, \dots, m-1, \quad a_m < b_m < \infty.$$

Theorem 1

Let $L > 0$ be an arbitrary number. Then the spectrum of the operator $\mathcal{A}^\varepsilon$ in $[0, L]$ has the following structure for ε small enough:

$$\sigma(\mathcal{A}^\varepsilon) \cap [0, L] = [0, L] \setminus \bigcup_{j=1}^{m} \big(a_j(\varepsilon), b_j(\varepsilon)\big),$$

where the endpoints of the intervals $\big(a_j(\varepsilon), b_j(\varepsilon)\big)$ satisfy the following relations as $\varepsilon \to 0$,

$$0 \le a_j - a_j(\varepsilon) = \mathcal{O}(\varepsilon), \quad 0 \le b_j - b_j(\varepsilon) = \mathcal{O}(\varepsilon).$$

Let (α_j, β_j), $j = 1, \dots, m$ be arbitrary intervals satisfying

$$0 < \alpha_1, \quad \alpha_j < \beta_j < \alpha_{j+1},\ j = \overline{1, m-1}, \quad \alpha_m < \beta_m < \infty.$$

Let (α_j,β_j), $j=1,\dots,m$ be arbitrary intervals satisfying

$$0<\alpha_1,\quad \alpha_j<\beta_j<\alpha_{j+1},\ j=\overline{1,m-1},\quad \alpha_m<\beta_m<\infty.$$

Theorem 2

One has

$$a_j=\alpha_j,\quad b_j=\beta_j,\quad j=1,\dots,m$$

provided the domains B_j and the numbers q_j satisfy

$$\frac{|B_j|}{|B_0|}=\frac{\beta_j-\alpha_j}{\alpha_j}\prod_{i=\overline{1,m}|i\neq j}\left(\frac{\beta_i-\alpha_j}{\alpha_i-\alpha_j}\right),\quad q_j=\frac{|B_j|}{\alpha_j|\partial B_j|},\quad j=1,\dots,m.$$

We denote $Y^\varepsilon := \varepsilon Y$ (the period cell of $\mathcal{A}^\varepsilon$), $S_j^\varepsilon = \varepsilon \partial B_j$.

We denote $Y^\varepsilon := \varepsilon Y$ (the period cell of $\mathcal{A}^\varepsilon$), $S_j^\varepsilon = \varepsilon \partial B_j$.

The spectrum of $\mathcal{A}^\varepsilon$ is a locally finite union of compact intervals:

$$\sigma(\mathcal{A}^\varepsilon) = \bigcup_{k=1}^{\infty} L_k(\varepsilon)$$

We denote $Y^\varepsilon := \varepsilon Y$ (the period cell of $\mathcal{A}^\varepsilon$), $S_j^\varepsilon = \varepsilon \partial B_j$.

The spectrum of $\mathcal{A}^\varepsilon$ is a locally finite union of compact intervals:

$$\sigma(\mathcal{A}^\varepsilon) = \bigcup_{k=1}^{\infty} L_k(\varepsilon)$$

By

$$\lambda_k^N(\varepsilon), \quad \lambda_k^D(\varepsilon), \quad \lambda_k^{per}(\varepsilon), \quad \lambda_k^{antiper}(\varepsilon)$$

we denote the k-th eigenvalues of the operator in $\mathsf{L}^2(Y^\varepsilon)$ being defined by

- the operation $-\Delta$,
- interface conditions $(*)$ on S_j^ε,
- Neumann, Dirichlet, periodic, antiperiodic conditions on ∂Y^ε.

We denote $Y^\varepsilon := \varepsilon Y$ (the period cell of $\mathcal{A}^\varepsilon$), $S_j^\varepsilon = \varepsilon \partial B_j$.

The spectrum of $\mathcal{A}^\varepsilon$ is a locally finite union of compact intervals:

$$\sigma(\mathcal{A}^\varepsilon) = \bigcup_{k=1}^{\infty} L_k(\varepsilon)$$

By

$$\lambda_k^N(\varepsilon), \quad \lambda_k^D(\varepsilon), \quad \lambda_k^{per}(\varepsilon), \quad \lambda_k^{antiper}(\varepsilon)$$

we denote the k-th eigenvalues of the operator in $\mathsf{L}^2(Y^\varepsilon)$ being defined by

- the operation $-\Delta$,
- interface conditions $(*)$ on S_j^ε,
- Neumann, Dirichlet, periodic, antiperiodic conditions on ∂Y^ε.

One has the following enclosures:

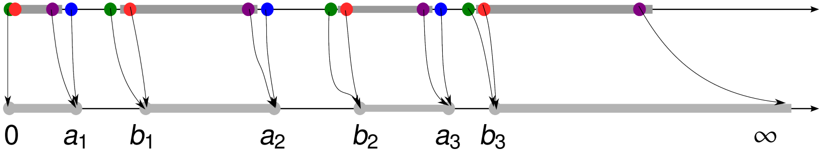
$$[\lambda_k^{per}(\varepsilon), \lambda_k^{antiper}(\varepsilon)] \subset L_k(\varepsilon) \subset [\lambda_k^N(\varepsilon), \lambda_k^D(\varepsilon)]$$

$$\lim_{\varepsilon\to 0}\lambda_k^D(\varepsilon)=a_k \quad \text{if } k=\overline{1,m};\quad \lim_{\varepsilon\to 0}\lambda_{m+1}^D(\varepsilon)=\infty$$

$$\lim_{\varepsilon\to 0}\lambda_k^{antiper}(\varepsilon)=a_k \quad \text{if } k=\overline{1,m};\quad \lim_{\varepsilon\to 0}\lambda_{m+1}^{antiper}(\varepsilon)=\infty$$

$$\lim_{\varepsilon\to 0}\lambda_k^D(\varepsilon)=a_k \quad \text{if } k=\overline{1,m};\quad \lim_{\varepsilon\to 0}\lambda_{m+1}^D(\varepsilon)=\infty$$
$$\lim_{\varepsilon\to 0}\lambda_k^{antiper}(\varepsilon)=a_k \quad \text{if } k=\overline{1,m};\quad \lim_{\varepsilon\to 0}\lambda_{m+1}^{antiper}(\varepsilon)=\infty$$

$$\lim_{\varepsilon\to 0}\lambda_k^N(\varepsilon)=b_{k-1} \quad \text{if } k=\overline{2,m+1};\quad \lim_{\varepsilon\to 0}\lambda_{m+2}^N(\varepsilon)=\infty$$
$$\lim_{\varepsilon\to 0}\lambda_k^{per}(\varepsilon)=b_{k-1} \quad \text{if } k=\overline{2,m+1};\quad \lim_{\varepsilon\to 0}\lambda_{m+2}^{per}(\varepsilon)=\infty$$



$$\lim_{\varepsilon\to 0}\lambda_k^D(\varepsilon)=a_k \quad \text{if } k=\overline{1,m}; \quad \lim_{\varepsilon\to 0}\lambda_{m+1}^D(\varepsilon)=\infty$$

$$\lim_{\varepsilon\to 0}\lambda_k^{antiper}(\varepsilon)=a_k \quad \text{if } k=\overline{1,m}; \quad \lim_{\varepsilon\to 0}\lambda_{m+1}^{antiper}(\varepsilon)=\infty$$

$$\lim_{\varepsilon\to 0}\lambda_k^N(\varepsilon)=b_{k-1} \quad \text{if } k=\overline{2,m+1}; \quad \lim_{\varepsilon\to 0}\lambda_{m+2}^N(\varepsilon)=\infty$$

$$\lim_{\varepsilon\to 0}\lambda_k^{per}(\varepsilon)=b_{k-1} \quad \text{if } k=\overline{2,m+1}; \quad \lim_{\varepsilon\to 0}\lambda_{m+2}^{per}(\varepsilon)=\infty$$

Thank you for your attention!