Finsler metrics with three dimensional projective symmetry algebra in dimension 2

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1. **Introduction to (Projective) Finsler geometry**
2. Main theorem and proof

   Main theorem: Every (fiber-global, C^∞) Finsler metric F on a two dimensional manifold with $\dim \mathfrak{p} = 3$ is locally projectively related to
   - either a Randers metric $F = \sqrt{g} + \beta$ with $\mathfrak{p}(F) = \mathfrak{iso}(g)$
   - or to a Riemannian metric.

3. What I want to do next

What is a Finsler metric?

Setting:

- M smooth manifold with local coordinates (x^i)
- TM tangent bundle with local but fiber global coordinates (x^i, ξ^j)
- All objects are assumed C^∞ and defined locally on M, but *fiber global*.

Definition: A **Finsler metric** is a smooth collection of norms for each T_xM. More explicitly a function $F : TM \to \mathbb{R}_{\geq 0}$ with properties

(a) *(Regularity)* F is C^∞ on $TM\backslash 0 := \bigcup_{x\in M} T_xM\backslash\{0\}$

(b) *(Homogenity)* $F(\lambda\xi) = \lambda F(\xi)$ for all $\lambda > 0$ and $\xi \in TM$

(c) *(Strict convexity)* $g_{ij}(x,\xi) := \frac{\partial^2(\frac{1}{2}F^2)}{\partial\xi^i\partial\xi^j}(x,\xi)$ is positive definite $\forall(x,\xi) \in TM\backslash 0$

- F measures *length of vectors* and *length of curves* $\mathcal{L}(c) = \int_0^1 F(\dot{c}(t))dt$.
- Hence induces a *system of geodesics* - we will study F only by its geodesics.

Bernhard Riemann
Habilitationsvortrag (1854)

Paul Finsler
Studied variational problems for arbitrary metrics (1918)

Ludwig Berwald
Riemann and Berwald curvature (1926)

The fundamental form g_{ij}

Definition: A **Finsler metric** is a function $F : TM \to \mathbb{R}_{\geq 0}$ with properties

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(b) *(Homogenity)* $F(\lambda\xi) = \lambda F(\xi)$ for all $\lambda > 0$ and $\xi \in TM$

(c) *(Strict convexity)* $g_{ij}(x,\xi) := \frac{\partial^2(\frac{1}{2}F^2)}{\partial\xi^i\partial\xi^j}(x,\xi)$ is positive definite $\forall(x,\xi) \in TM\backslash 0$

The matrix $g = (g_{ij}) = (\frac{\partial^2(\frac{1}{2}F^2)}{\partial\xi^i\partial\xi^j})$ is called **fundamental form**.

- For each (x,ξ), it is $g(x,\xi)$ an inner product on T_xM.
- Notation for the inverse matrix $g^{-1}(x,\xi) = (g^{ij}(x,\xi))$

In Finsler geometry, all objects are *homogenuous* in ξ.

- **Euler theorem:** If $f(\xi)$ is k-homogeneous, then $\begin{cases} f_{\xi^i}(\xi) \text{ is } (k-1)\text{-homogeneous} \\ f_{\xi^i}(\xi)\xi^i = k\cdot f(\xi) \end{cases}$.

$\Rightarrow g_{ij}(x,\xi) = g_{ij}(x,\lambda\xi)$ for $\lambda > 0$

$\Rightarrow g_{ij}(x,\xi)\xi^i\xi^j = F^2(x,\xi)$ (recover F from g)

Finsler geometry generalizes Riemannian geometry

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(b) *(Homogenity)* $F(\lambda \xi) = \lambda F(\xi)$ for all $\lambda > 0$ and $\xi \in TM$

(c) *(Strict convexity)* $g_{ij}(x, \xi) := \frac{\partial^2(\frac{1}{2}F^2)}{\partial \xi^i \partial \xi^j}(x, \xi)$ is positive definite $\forall (x, \xi) \in TM \backslash 0$

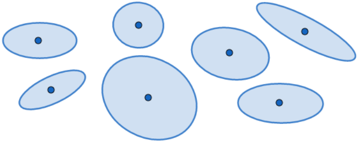
Famous example: Riemannian metrics

$$F(x, \xi) = \sqrt{g_{ij}(x) \xi^i \xi^j} \qquad (g_{ij}(x)) \text{ positive definite matrix}$$

- Fundmantal form $g_{ij}(x, \xi) = g_{ij}(x)$
- The norm on $T_x M$ is induced by a inner product.

How to visualize Finsler metrics? By its **indicatrices** $\Omega_x = \{\xi \in T_x M \mid F(\xi) = 1\}$.

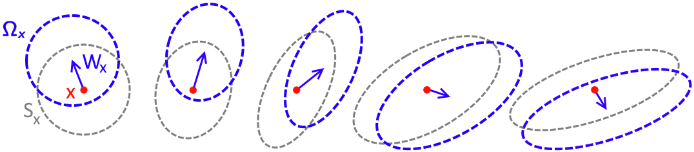
- F Riemannian $\Rightarrow$ Every Ω_x is an ellipse symmetric wrt. the origin.



Motivational example: Windy surfaces and Randers metrics

Take a surface with Riemannian metric g and a wind vector field W.

- Without wind, in an infinitesimal time unit one can move from x to $S_x = \{\xi \in T_xM \mid g_x(\xi,\xi) = 1\}$.
- With the wind W, in an infinitesimal time unit one can move from x to $S_x + W_x$.



- **Fact:** There is a Finsler metric with indicatrix $\Omega_x = S_x + W_x$. (Zermelo navigation)

First non-Riemannian example: Randers metrics

$$F(x,\xi) = \underbrace{\sqrt{g_{ij}(x)\xi^i\xi^j}}_{\text{Riemannian metric } \sqrt{g}} + \underbrace{\beta_i(x)\xi^i}_{\text{1-form } \beta=\beta_i(x)dx^i}$$

- Indicatrices Ω_x in each T_xM are shifted ellipses (as in the wind example)
- $F(x,\xi) = F(x,-\xi)$ if and only if $\beta = 0$
- If β is small ($g^{ij}\beta_i\beta_j < 1$), then Ω_x encloses the origin and F is a Finsler metric.

General Finsler metrics can be very complicated

Definition: A **Finsler metric** is a function $F : TM \to \mathbb{R}_{\geq 0}$ with properties

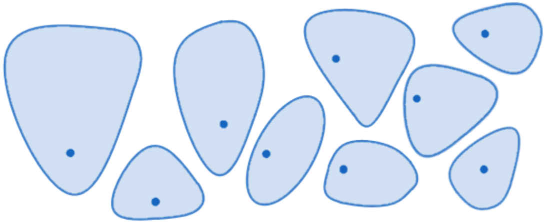
(a) *(Regularity)* F is C^∞ on $TM\backslash 0 := \bigcup_{x \in M} T_x M \backslash \{0\}$

(b) *(Homogenity)* $F(\lambda\xi) = \lambda F(\xi)$ for all $\lambda > 0$ and $\xi \in TM$

(c) *(Strict convexity)* $g_{ij}(x,\xi) := \frac{\partial^2(\frac{1}{2}F^2)}{\partial\xi^i \partial\xi^j}(x,\xi)$ is positive definite $\forall (x,\xi) \in TM\backslash 0$

Property (c) $\Leftrightarrow$ Indicatrices Ω_x enclose a *strictly convex* body

- For a general Finsler metric, Ω_x can be any strictly convex body



- *Riemannian* and *Randers* metrics are rather easy Finsler metrics

Euler-Lagrange equations and Finsler Geodesics

Euler-Lagrange equation: For a *Lagrangian* $L : TM \to \mathbb{R}$ the extremals of the functional $\int_0^1 L(\dot{c}(t))dt$ are the solutions $c(t)$ of the ODEs

$$E_i(L) := \frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \xi^i}\right) = 0.$$

Definition: Geodesics of a Finsler metric F are the solutions of $E_i(\frac{1}{2}F^2) = 0$

- $(\frac{F^2}{2})_{x^i} - (\frac{F^2}{2})_{\xi^i x^j}\dot{c}^j - (\frac{F^2}{2})_{\xi^i \xi^j}\ddot{c}^j = 0 \quad \overset{g^{mj}}{\Leftrightarrow} \quad \ddot{c}^m + 2\frac{g^{mj}}{4}\left(2\frac{\partial g_{jk}}{\partial x^l} - \frac{\partial g_{kl}}{\partial x^j}\right)\dot{c}^k\dot{c}^l = 0$

Definition: A **spray** is a vector field S on $TM\backslash 0$ of the form

$$S|_{(x,\xi)} = \xi^i \partial_{x^i} - 2G^i(x,\xi)\partial_{\xi^i} \qquad \text{with } \forall \lambda > 0 : G^i(x,\lambda\xi) = \lambda^2 G^i(x,\xi).$$

- (Projections of) Integral curves $\overset{\text{1-to-1}}{\leftrightarrow}$ Solutions of $\ddot{c} + 2G^i(c,\dot{c}) = 0$.
- For initial value $(x,\xi) \in TM\backslash 0$ there is a unique integral curve c ... and the curve for initial value $(x,\lambda\xi)$ is the curve $c(\lambda t)$ for $\lambda > 0$.

Definition: Geodesic spray $G^i(x,\xi) = \frac{g^{ij}}{4}\left(2\frac{\partial g_{jk}}{\partial x^l} - \frac{\partial g_{kl}}{\partial x^j}\right)\xi^k\xi^l$

- Curves are exactly geodesics of F

$E_i(F)$ vs. $E_i(\frac{1}{2}F^2)$

Why don't we define geodesics as the solutions of $E_i(F) = 0$?

- Main difference: F 1-homogeneous, $\frac{F^2}{2}$ 2-homogeneous (in ξ)
- Does *not* define a spray, since the Hessian of F is singular: $\frac{\partial^2 F}{\partial \xi^i \partial \xi^j}(x, \xi)\xi^i = 0$
- *No distinguished parametrization*:
  $c(t)$ solution of $E_i(F) = 0 \Rightarrow c(\varphi(s))$ with $\varphi' > 0$ is solution of $E_i(F) = 0$

However:

- Solutions of $E_i(\frac{1}{2}F^2) = 0$ are exactly the reparametrizations to F-arc length of solutions of $E_i(F) = 0$

Projectively related sprays and path structures

Definition: Two sprays are **projectively related** if (the projections to M of) their curves coincide as oriented point sets.

To *quotient by this equivalence relation*, we go

- from *sprays* (vector fields on TM)
- to *path structures* (1-dim. distributions on the unit sphere bundle SM).

Consider $S_xM = (T_xM \backslash 0)/\mathbb{R}_+$ with projection $\pi : TM\backslash 0 \to SM$.

- $\ell \in S_xM$ is an oriented direction/ray on M

Definition: The **path structure** $P(S)$ of a spray S is the family $\ell \in SM$

$$P_\ell := \left\langle d\pi_{(x,\xi)}(S_{(x,\xi)}) \mid (x,\xi) \in \pi^{-1}(\ell) \right\rangle \quad \subseteq T_\ell(SM).$$

A *path structure* is just *collection of unparametrized curves* on M, s.t. for each point and direction there is exactly one.
Its *curves* (whose lift to SM is tangent to P) are oriented reparametr. of curves of S.

Lemma: Two sprays $S, \tilde{S}$ are projectively related,

1. if and only if $\tilde{S} = S - 2h\mathbf{V}$, for some $h : TM \to \mathbb{R}$ and $\mathbf{V} = \xi^i \partial_{\xi^i}$
2. if and only if $\tilde{P} = P$.

Path structures in microlocal coordinates for dim $M = 2$

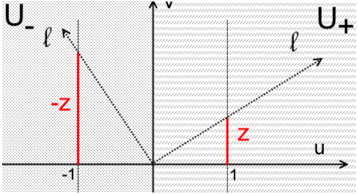
Definition: The **path structure** $P(S)$ of a spray S is the family $\ell \in SM$

$$P_\ell := \langle d\pi_{(x,\xi)}(S_{(x,\xi)}) \mid (x,\xi) \in \pi^{-1}(\ell) \rangle \quad \subseteq T_\ell(SM).$$

Let dim $M = 2$ and (x, y) local coordinates on M, (x, y, u, v) on TM.

We use two charts (x, y, z) for SM:

$$U_+ = \{[(x, y, u, v)] \in SM \mid u > 0\}$$
$$U_- = \{[(x, y, u, v)] \in SM \mid u < 0\}$$
$$\varphi_+ : U_+ \to \mathbb{R}^3$$
$$\varphi_- : U_- \to \mathbb{R}^3$$
$$(x, y, u, v) \mapsto (x, y, \tfrac{v}{u})$$



- U_+ and U_- cover SM up to vertical directions
- By continuity knowing P in the two charts is the same as knowing P on SM
- In U_+ coordinates, every path structure P is of the form

$$P_{(x,y,z)} = \langle \partial_x + z\partial_y + f_+(x, y, z)\partial_z \rangle .$$

- Then the curves with $\dot{x} > 0$ of P parametrized by x are given by

$$y'' = f_+(x, y, y')$$

- Path structure $P \longleftrightarrow$ Two ODEs $y'' = f_\pm(x, y, y')$

Definition/Lemma: Let S be a spray and $X \in \mathfrak{X}(M)$ a vector field.

- X is an **affine symmetry** if its flow maps *parametrized* curves of S to such.
  $\Leftrightarrow$ $\mathcal{L}_X S = 0$
- X is an **projective symmetry** if its flow maps *unparametrized* curves to such.
  $\Leftrightarrow$ $\mathcal{L}_X S = f \cdot \mathbf{V}$, where $\mathbf{V} = \xi^i \partial_{\xi^i}$
  $\Leftrightarrow$ $\mathcal{L}_X P \subseteq P$, i.e. for all $Z \in \mathfrak{X}(SM)$ with $Z_\ell \in P_\ell$ we have $\mathcal{L}_X Z \in P$:

Lemma: The set of $\left\{ \begin{array}{l} \text{affine} \\ \text{projective} \end{array} \right.$ symmetries forms a Lie algebra $\left\{ \begin{array}{l} \mathfrak{s}(S) \\ \mathfrak{p}(S) \end{array} \right.$.

- Clearly $\mathfrak{s}(S) \subseteq \mathfrak{p}(S)$.

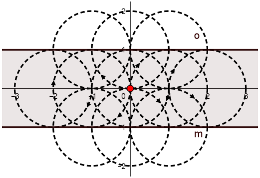
Example (Circles of radius 1)

On $\mathbb{R}^2$ consider the spray $S_{(x,y,u,v)} = u\partial_x + v\partial_y + \sqrt{u^2+v^2}\,(v\partial_u - u\partial_v)$.

In microlocal coordinates its path structure P is given by

$$(U_+) \quad y'' = -\left((y')^2+1\right)^{3/2} \qquad (U_-) \quad y'' = \left((y')^2+1\right)^{3/2}$$

The general solution is $y(x) = \mp\sqrt{1-(x-a)^2} + b$, i.e. negative oriented circles.



Projective symmetry algebra: $\mathfrak{p} = \mathbb{R}^2 + \mathfrak{so}(2) = \langle \partial_x, \partial_y, -y\partial_x + x\partial_y \rangle$

- Lift of a vector field to SM in U_+ coordinates
  $$\check{X} = a\partial_x + b\partial_y + c\partial_z \text{ with } c = b_x + (b_y - a_x)z - a_y z^2$$
- In U_+, $P = \langle Z \rangle = \langle \partial_x + z\partial_y + f(x,y,z) \rangle$ with $f(x,y,z) = -(z^2+1)^{3/2}$
- $X \in \mathfrak{p} \Leftrightarrow [\check{X}, Z] = \lambda Z \quad \Leftrightarrow af_x + bf_y + cf_z = (c_z - a_x - za_y)f + c_x + zc_y$

Hence this is a **path structure with** $\dim \mathfrak{p} = 3$.

- Is it the geodesic structure of some Finsler function?

Projective Finsler Geometry and Lie Problem

Definition: Let $F, \tilde{F}$ be Finsler metrics.

1. F and $\tilde{F}$ are **projectively related**, if their geodesic sprays $S, \tilde{S}$ are, i.e. if $P = \tilde{P}$.
2. The $\begin{cases} \textbf{affine} \\ \textbf{projective} \end{cases}$ **symmetry algebra** $\begin{cases} \mathfrak{s}(F) \\ \mathfrak{p}(F) \end{cases}$ is the one of the geodesic spray.
3. $X \in \mathfrak{X}(M)$ is a **Killing vector field** if $\mathcal{L}_X F = 0$.
   They form a Lie algebra $\mathfrak{iso}(F)$.

Problem (Sophus Lie 1882):
Describe (Finsler) metrics F on surfaces with $\dim \mathfrak{p}(F) \geq 2$.

Fact (Cartan/Tresse): If $\dim \mathfrak{p} > 3$, then $\dim \mathfrak{p} = 8$ and F is *projectively flat*.

- i.e. there are local coordinates where all unparametrized geodesics are straight.
- Projectively flat metrics were studied a lot, next one should consider $\dim \mathfrak{p} = 3$

Main theorem: Every (fiber-global, C^∞) Finsler metric F on a two dimensional manifold with $\dim \mathfrak{p} = 3$ is locally projectively related to

- either a Randers metric $F = \sqrt{g} + \beta$ with $\mathfrak{p}(F) = \mathfrak{iso}(g)$
- or to a Riemannian metric.

1. Introduction to (Projective) Finsler geometry
2. **Main theorem and proof**

> **Main theorem:** Every (fiber-global, C^∞) Finsler metric F on a two dimensional manifold with $\dim \mathfrak{p} = 3$ is locally projectively related to
> - either a Randers metric $F = \sqrt{g} + \beta$ with $\mathfrak{p}(F) = \mathfrak{iso}(g)$
> - or to a Riemannian metric.

3. What I want to do next

Idea of the proof

For a Finsler metric F we have

$$\begin{array}{cl} F & \text{Finsler metric} \\ \downarrow & \\ S(F) & \text{geodesic spray} \\ \downarrow & \\ P(F) & \text{geodesic (path) structure} \\ \downarrow & \\ \mathfrak{p}(F) & \text{projective symmetry algebra} \end{array}$$

To proof the theorem, we go backwards. In $\mathbb{R}^2$, *locally, up to coordinate change...*

Step 1. Find all possible 3-dimensional algebras of vector fields.

Step 2. Find for them all possible path structures.

Step 3. For each path structure, find a Finsler metric.

Then if F is Finsler metric with $\dim \mathfrak{p} = 3$, there are coordinates where

- $\mathfrak{p}$ is as in Step 1
- hence P is as in Step 2
- hence F is **projectively related to a metric** from Step 3.

Step 1: List of 3-dimensional algebras of vector fields.

Let $\mathfrak{g}, \tilde{\mathfrak{g}}$ be Lie algebras of vector fields on $\mathbb{R}^n$.
The **isotropy subalgebra** in a point $p \in \mathbb{R}^n$ is $\mathfrak{g}_p := \{X \in \mathfrak{g} \mid X_p = 0\}$.
We call $\mathfrak{g}$ **transitive** at p, if $\{X_p \mid X \in \mathfrak{g}\}$ has full dimension n.

Fact: Suppose $\mathfrak{g}, \tilde{\mathfrak{g}}$ are transitive at 0. Then $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ *differ by a coordinate change* around 0 if and only if *there is a Lie algebra isomorphism* $\mathfrak{g} \to \tilde{\mathfrak{g}}$ *mapping* $\mathfrak{g}_0$ *to* $\tilde{\mathfrak{g}}_0$.

- Make a list of pairs $(\mathfrak{g}, \mathfrak{h})$ of three dimensional Lie algebras $\mathfrak{g}$ with one dimensional subalgebra $\mathfrak{h}$.
  - 16 non-isomorphic pairs (two with parameter)
- If $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$, the pair $(\mathfrak{g},\mathfrak{h})$ can not be realized as a vector field algebra.
- Realize the remaining pairs.
  - 10 algebras of vector fields (two with parameter)

Result: For every vector field algebra $\mathfrak{g}$ around a transitive point there are coordinates where $\mathfrak{g}$ is as in one of the 10 cases.

- One example case: $\mathfrak{g} = \langle \partial_x, \partial_y, -y\partial_x + x\partial_y \rangle$, $\mathfrak{g}_0 = \langle -y\partial_x + x\partial_y \rangle$

Step 2: List of path structures with $\dim \mathfrak{p} = 3$

- Assume P with $\dim \mathfrak{p} = 3$. If $\mathfrak{p}$ is transitive in a point $x \in M$, we can assume that $\mathfrak{p} = \langle X_1, X_2, X_3 \rangle$ is from the constructed list by a coordinate change.
- Microlocal PDEs in U_+ on $P = \langle Z \rangle = \langle \partial_x + z\partial_y + f(x,y,z) \rangle$
  - Lifts to SM $\check{X}_i = a_i\partial_x + b_i\partial_y + c_i\partial_z$
  - $X \in \mathfrak{p} \Leftrightarrow [\check{X}, Z] = \lambda Z \quad \Leftrightarrow af_x + bf_y + cf_z = (c_z - a_x - za_y)f + c_x + zc_y$
  - If $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ is regular, f and P are determined by a initial value $f(x_0, y_0, z_0)$.
- Sort out cases with $\dim \mathfrak{p} = 8$ and not fiber global extandable.
- Five cases remain, two with parameter:

Theorem: If P is a fiber global path structure with $\dim \mathfrak{p} = 3$, then there are local coordinates where P is one of the following:

(P1)	Circles of radius 1 in $\mathbb{R}^2$	$\mathfrak{p} = \mathbb{R}^2 + \mathfrak{so}(2)$
(P2)	$\underbrace{\text{'Circles' of radius } R}_{\text{will be explained later}}$ in S^2	$\mathfrak{p} = \mathfrak{so}(3)$
(P3)	'Circles' of radius R in H^2	$\mathfrak{p} = \mathfrak{sl}(2)$
(P4)	Origin centered ellipses with area 1 in $\mathbb{R}^2$	$\mathfrak{p} = \mathfrak{sl}(2)$
(P5)	Origin centered hyperbolas with 'fixed area' in $\mathbb{R}^2$	$\mathfrak{p} = \mathfrak{sl}(2)$

General question (Projective Finsler Metrization):
Given a path structure P,

- is there a Finsler metric whose geodesic structure is P?
- how can one describe all such Finsler metrics?

In $\dim M \geq 3$ exist path structures that are not even microlocally metrizable.

In $\dim M = 2$, it is not clear weither every path structure is locally metrizable.

- Every *reversible* path structure is locally metrizable. (Alvarez-Pavia/Berck)
- Every path structure is *microlocally* metrizable. (Sonin/Darboux/Matsumoto)
- If $\dim \mathfrak{p} > 3$, then P is projectively flat and locally metrizable.
- If $\dim \mathfrak{p} = 3$, then P is locally metrizable (Main theorem).
- **Expectation:** In dimension two, every path structure is locally metrizable.
- We will see that the circle example is *not* globally metrizable.

Step 3: Metrization of the path structures (P4) and (P5)

(P4)	Origin centered ellipses with area 1 in $\mathbb{R}^2$	$\mathfrak{p} = \mathfrak{sl}(2)$
(P5)	Origin centered hyperbolas with 'fixed area' in $\mathbb{R}^2$	$\mathfrak{p} = \mathfrak{sl}(2)$

(P4) and (P5) are reversible and the not vertical curves are given by

$$(P4)\ y'' = (xy' - y)^3 \qquad (P5)\ y'' = -(xy' - y)^3$$

Hence they could be Riemannian. Indeed, with techniques from

- R. Bryant, G. Manno, V. Matveev, *A solution of a problem of Sophus Lie: normal forms of two-dimensional metrics admitting two projective vector fields*

one can find *Riemannian metrics* for both cases.

After a coordinate change, the following metrics have this path structures:

$$(F4)\ \sqrt{e^{3x}dx^2 + e^x dy^2} \qquad (F5)\ \sqrt{\frac{1}{1-e^{-y}}dx^2 + \frac{e^{3y}}{(1-e^y)^2}dy^2}$$

Step 3: Metrize (P1),(P2),(P3) / Magnetic geodesics

- Take Riemannian metric g and a volume form Ω.
- The **Lorentz force** is the bundle map $J : TM \to TM$ with $\Omega(u, v) = g(u, Jv)$.
- The **magnetic geodesics** are the solutions of $\nabla_{\dot{c}}\dot{c} \overset{(*)}{=} J(\dot{c})$.
  - Magnetic geodesics are parametrized by g-arc length.
  - In local coordinates $\Omega = k\sqrt{\det g}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $J = g^{-1}\Omega$.

Construct a Finsler metric, whose geodesics are the magnetic geodesics with $g \equiv 1$.

- Choose a 1-form β s.t. $d\beta = \Omega$.
- Then $E_i(g + 2\beta) = 0$ is equivalent to equation $(*)$.
- Now consider the Randers metric $F = \sqrt{g} + \beta$.
  - Every solution of $(*)$ with $g(\dot{c}, \dot{c}) = 1$ is a solution of $E_i(F) = 0$. Hence **every such curve is a geodesic of** $F = \sqrt{g} + \beta$ after reparametrization to F-arc length.
  - In coordinates $\beta = \beta_i dx^i$. If $g^{ij}\beta_i\beta_j < 1$, then F is a Finsler metric.
  - Every Killing vector field of g is a projective symmetry of F.

Step 3: Metrize (P1),(P2),(P3) / Construction of metrizations

Choose a Riemannian metric with $\dim \mathfrak{iso}(g) = 3$. Then $\dim \mathfrak{p}(F) \geq 3$.

(P1) $\mathbb{R}^2$ with $\mathfrak{iso} = \mathbb{R}^2 + \mathfrak{so}(2)$

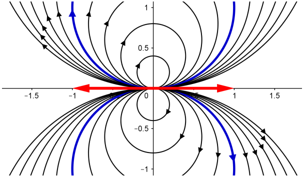
(P2) S^2 with $\mathfrak{iso} = \mathfrak{so}(3)$

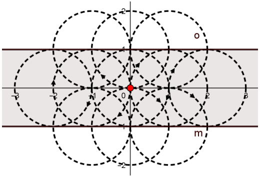
(P3) H^2 with $\mathfrak{iso} = \mathfrak{sl}(2)$

This *gives exactly the path structures* (P1), (P2), (P3). $\Rightarrow \dim \mathfrak{p} = 3 \Rightarrow \mathfrak{p} = \mathfrak{iso}(g)$.

Example $\mathbb{R}^2$**:** $g = \mathrm{Id}, \Omega = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}, \beta = adx + bdy$

- Magnetic geodesics $\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} \overset{(*)}{=} k \begin{pmatrix} \dot{y} \\ -\dot{x} \end{pmatrix}$ are for $k = 1$ negative oriented circles of all radii
- $d\beta = (b_x - a_y) dx \wedge dy \overset{!}{=} k dx \wedge dy$, for example $\beta = -ky\ dx$
- $g^{ij}\beta_i\beta_j = k^2y^2 \overset{!}{<} 1$, hence F is a Finsler metric for $|y| < \frac{1}{|k|}$
- Finsler metric $F = \sqrt{dx^2 + dy^2} - y \cdot dx$ **gives path structure (P1)**





- (P1) is locally Finsler metrizable, *but not globally* ([Shen],Hopf-Rinow)

Summing up/End of the proof

- Let F be Finsler metric, P the geodesic structure, $\dim \mathfrak{p} = 3$ and $x \in M$.
- Then $(\mathfrak{p}, \mathfrak{p}_x)$ is isomorphic to a pair from Step 1.
- Hence in some local coordinates $\mathfrak{p}$ is as in Step 1 and P one of (P1)-(P5).
- Thus F is projectively related to one of the constructed metrics (F1)-(F5):

Main theorem: Every (fiber-global, C^∞) Finsler metric F on a two dimensional manifold with $\dim \mathfrak{p} = 3$ is locally projectively related to

- either a Randers metric $F = \sqrt{g} + \beta$ with $\mathfrak{p}(F) = \mathfrak{iso}(g)$
- or to a Riemannian metric.

In some local coordinates F is projectively related to one of:

(F1) $\sqrt{dx^2 + dy^2} - ky\ dx$

(F2) $\sqrt{\frac{dx^2+dy^2}{(x^2+y^2+1)^2}} + \frac{C}{2}\left(\frac{x}{(y^2+1)(x^2+y^2+1)} + \frac{\arctan\left(\frac{x}{\sqrt{y^2+1}}\right)}{(y^2+1)^{3/2}}\right) dy$, where $C \in \mathbb{R}_{>0}$

(F3) $\sqrt{\frac{1}{y^2}(dx^2 + dy^2)} - C\frac{x}{y^2}dy$, where $C \in \mathbb{R}_{>0}$

(F4) $\sqrt{e^{3x}dx^2 + e^x dy^2}$

(F5) $\sqrt{\frac{1}{1-e^{-y}}dx^2 + \frac{e^{3y}}{(1-e^y)^2}dy^2}$

1. Introduction to (Projective) Finsler geometry
2. Main theorem and proof
3. **What I want to do next**

What I want to do next:

- Describe **all** Finsler metrics with $\dim \mathfrak{p} = 3$ up to coordinate change.
  - One must find all Finsler metrics, that induce the path structures (P1)-(P5).

(P1)	Circles of radius 1 in $\mathbb{R}^2$	$\mathfrak{p} = \mathbb{R}^2 + \mathfrak{so}(2)$
(P2)	'Circles' of radius R in S^2	$\mathfrak{p} = \mathfrak{so}(3)$
(P3)	'Circles' of radius R in H^2	$\mathfrak{p} = \mathfrak{sl}(2)$
(P4)	Origin centered ellipses with area 1 in $\mathbb{R}^2$	$\mathfrak{p} = \mathfrak{sl}(2)$
(P5)	Origin centered hyperbolas with 'fixed area' in $\mathbb{R}^2$	$\mathfrak{p} = \mathfrak{sl}(2)$

- Consider the case $\dim \mathfrak{p} = 2$.

How to find all Finsler metrics that induce a given path structure?

- **Trivial freedom of adding a total derivative:**
  Let $g : M \to \mathbb{R}$ with total derivative $dg = g_x dx + g_y dy$.
  $\Rightarrow E_i(dg) = 0 \quad \Rightarrow F$ and $F + dg$ have the same path structure [1]

- **Convex cone property:** Let $F, \tilde{F}$ be Finsler metrics with path structure P.
  $\Rightarrow \forall \lambda, \mu > 0 : \lambda F + \mu \tilde{F}$ is a Finsler metric with path structure P.

- We can **describe all Randers metrics** $F = \sqrt{g} + \beta$ with $\dim \mathfrak{p} = 3$:

> **Fact (Shen/Yu/Matveev):** Two Randers metrics $F = \sqrt{g} + \beta$ and $\tilde{F} = \sqrt{\tilde{g}} + \tilde{\beta}$ with β not closed are projectively related if and only if there is $\lambda > 0$ with $g = \lambda^2 \tilde{g}$ and $\beta - \lambda \tilde{\beta}$ is closed.

  - For each of (P1-3) we have a Randers metrization $F_0 = \sqrt{g} + \beta$ with β not closed.
    $\Rightarrow$ Every other Randers metrization is of the form $\lambda F_0 + dg$.
  - For (P4,P5) we can describe all Riemannian metrizations. If $\sqrt{g} + \beta$ is a Randers metrization, then β is closed and g a Riemannian metrization.

[1] $F + dg$ might not be a Finsler metric!

How to find all Finsler metrics that induce a given path structure?

- **Classical approach (Sonin/Darboux/Matsumoto)** to *microlocal* metrization:
  - In U_+ coordinates, P is given as $\langle \partial_x + z\partial_y + f(x,y,z)\partial_z \rangle \leftrightarrow y'' = f(x,y,y')$.
  - By comparing Euler-Lagrange equations, $L(x,y,z) := F(x,y,1,z)$ fulfills the PDE

$$-L_y + L_{xz} + zL_{yz} + f \cdot L_{zz} = 0 \qquad (*)$$

  - Differentiate by z.
    Then $R(x,y,z) = \frac{\partial^2 L}{\partial z^2}(x,y,z)$ must fulfill the *linear first order PDE*

$$R_x + z \cdot R_y + f \cdot R_z + f_z \cdot R = 0. \qquad (**)$$

  - $(**)$ can be solved microlocally and general solution for $(*)$ can be given *implicitly*.
  - General F must be combination of this L and similar function for U_-

- Description of all **Lagragians with circle path structure (P1):**
  - S. Tabachnikov, *Remarks on magnetic flows and magnetic billiards, Finsler metrics and a magnetic analog of Hilbert's fourth problem*

$$L(x,y,r,\alpha) = r\Big(\int_0^{\alpha+\pi/2} \cos(\alpha-\phi) g(x+\cos\phi, y+\sin\phi) d\phi + a(x,y)\cos\alpha + b(x,y)\sin\alpha\Big)$$

    where g, a, b fullfil some additional property

- ...

Finsler metrics with $\dim \mathfrak{p} = 2$

- Because there are just two two dimensional Lie algebras, we can assume that

$$\text{(A)} \quad \mathfrak{p} = \langle \partial_x, \partial_y \rangle \qquad \text{or} \qquad \text{(NA)} \quad \mathfrak{p} = \langle \partial_y, x\partial_x + y\partial_y \rangle$$

- Any path strcture with that projective algebra is in U_+ coordinates

$$\text{(A)}\ \langle \partial_x + z\partial_y + f(z)\partial_z \rangle \qquad \text{or} \qquad \text{(NA)}\ \langle \partial_x + z\partial_y + \frac{h(z)}{x}\partial_z \rangle$$

$$\text{(A)}\ y'' = f(y') \qquad \text{or} \qquad \text{(NA)}\ y'' = \frac{h(y')}{x}\partial_z \rangle$$

- Given f (h) on U_+ and U_-, how can we construct at least one (fiber global) F with this path structure?
- Can we find a path structure, that is not fiber global metrizable?

Thank you for the attention!

Related literature:

Classification of 2nd order ODEs $y'' = f(x, y, y')$ (essential for Step 1 & 2):

- A. Tresse, *Sur les invariants différentiels des groupes continus de transformations*
- B. Doubrov, B. Komrakov, *The geometry of second-order ordinary differential equations*

Magnetic Geodesics (used to metrize (P1)-(P3))

- K. Burns, V. Matveev, *On the rigidity of magnetic systems with the same magnetic geodesics*

General Spray and Finsler geometry

- Z. Shen, *Differential geometry of spray and Finsler spaces*
- D. Bao, S.-S. Chern, Z. Shen, *An Introduction to Riemann-Finsler Geometry*

Solution of the Pseudo-Riemannian version of the Lie problem (Metrization of (P4),(P5))

- R. Bryant, G. Manno, V. Matveev, *A solution of a problem of Sophus Lie: normal forms of two-dimensional metrics admitting two projective vector fields*
- V. Matveev, *Two-dimensional metrics admitting precisely one projective vector field*

Finsler Metrization of Path Structures = Projective Finsler Metrization of Sprays

- J.C. Álvarez-Paiva, G. Berck *Finsler surfaces with prescribed geodesics*, arXiv:1002.0243
- N.J. Sonin, *About determining maximal and minimal properties*, 1886, translation by R. Ya. Matsyuk (Lepage Research Institute, Czech Republic)
- I. Bucataru, Z. Muzsnay *Projective metrizability and formal integrability*