Finsler metrics with three dimensional projective symmetry algebra in dimension 2

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15 March 2018 - Ostrava Seminar on Mathematical Physics

19 March 2018 - Geometry Seminar, University of Debrecen

1. **Introduction to (Projective) Finsler geometry**
2. Main theorem and proof

> **Main theorem:** Every (fiber-global, C^∞) Finsler metric F on a two dimensional manifold with $\dim \mathfrak{p} = 3$ is locally projectively related to
> - either a Randers metric $F = \sqrt{g} + \beta$ with $\mathfrak{p}(F) = \mathfrak{iso}(g)$
> - or to a Riemannian metric.

3. What I want to do next

What is a Finsler metric?

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Definition: A **Finsler metric** is a smooth collection of norms for each $T_x M$.

More explicitly a function $F : TM \to \mathbb{R}_{\geq 0}$ with properties

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Bernhard Riemann
Habilitationsvortrag (1854)

Paul Finsler
Studied variational problems for arbitrary metrics (1918)

Ludwig Berwald
Riemann and Berwald curvature (1926)

The fundamental form g_{ij}

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- **Euler theorem:** If $f(\xi)$ is k-homogeneous, then $\begin{cases} f_{\xi^i}(\xi) \text{ is } (k-1)\text{-homogeneous} \\ f_{\xi^i}(\xi)\xi^i = k \cdot f(\xi) \end{cases}$.

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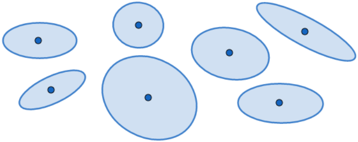

Motivational example: Windy surfaces and Randers metrics

Take a surface with Riemannian metric g and a wind vector field W.

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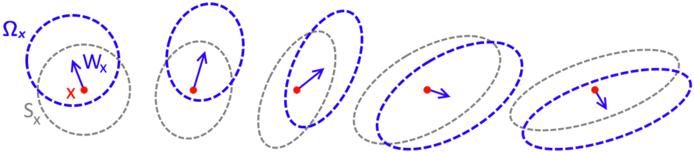

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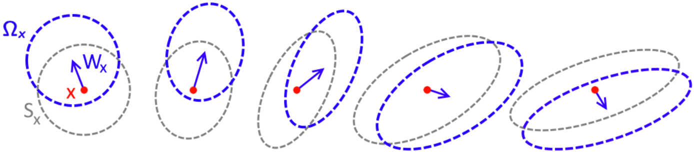

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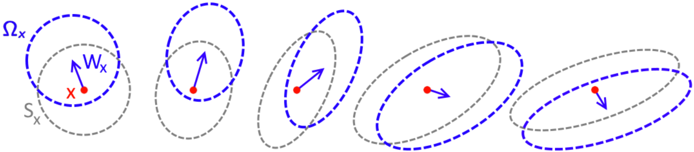

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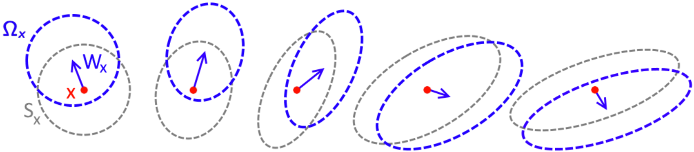

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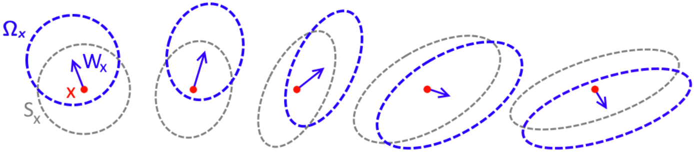

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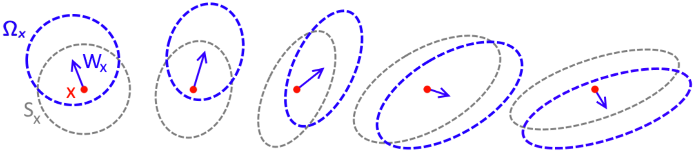

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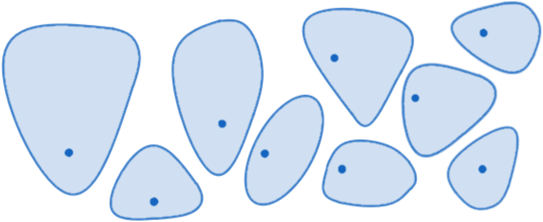
Property (c) $\Leftrightarrow$ Indicatrices Ω_x enclose a *strictly convex* body

- For a general Finsler metric, Ω_x can be any strictly convex body



- *Riemannian* and *Randers* metrics are rather easy Finsler metrics

General Finsler metrics can be very complicated

Definition: A **Finsler metric** is a function $F : TM \to \mathbb{R}_{\geq 0}$ with properties

(a) *(Regularity)* F is C^∞ on $TM\backslash 0 := \bigcup_{x \in M} T_x M \backslash \{0\}$

(b) *(Homogenity)* $F(\lambda \xi) = \lambda F(\xi)$ for all $\lambda > 0$ and $\xi \in TM$

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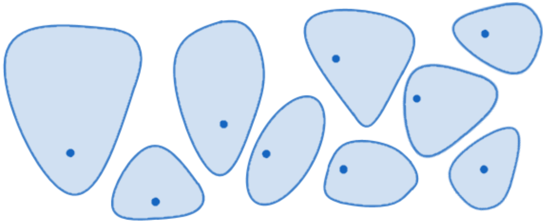

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Euler-Lagrange equation: For a *Lagrangian* $L : TM \to \mathbb{R}$ the extremals of the functional $\int_0^1 L(\dot{c}(t))dt$ are the solutions $c(t)$ of the ODEs

$$E_i(L) := \frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \xi^i}\right) = 0.$$

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Definition: Two sprays are **projectively related** if (the projections to M of) their curves coincide as oriented point sets.

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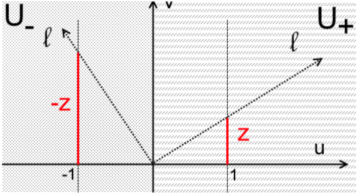

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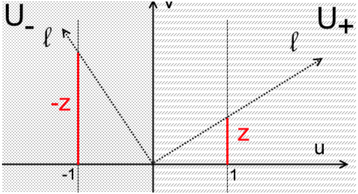

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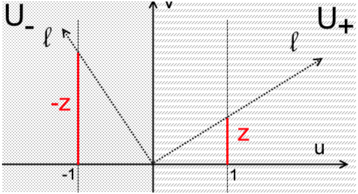

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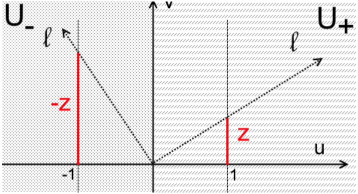

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Path structures in microlocal coordinates for dim $M = 2$

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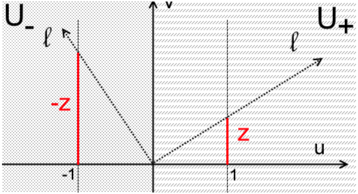

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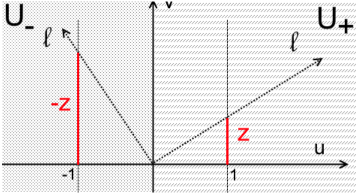

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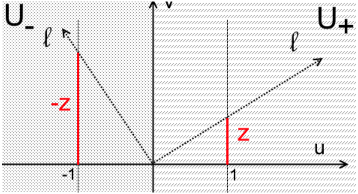

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Affine/Projective symmetries of a spray

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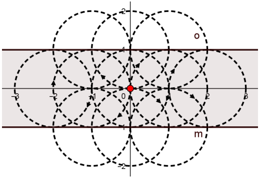

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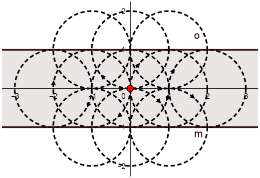

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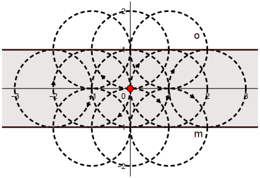

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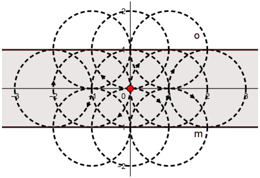

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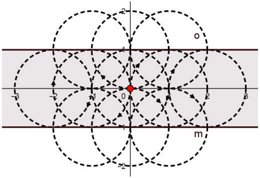

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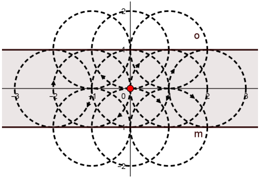

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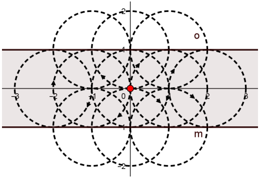

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Problem (Sophus Lie 1882):
Describe (Finsler) metrics F on surfaces with $\dim \mathfrak{p}(F) \geq 2$.

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1. Introduction to (Projective) Finsler geometry
2. **Main theorem and proof**

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3. What I want to do next

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$$\begin{array}{cl} F & \text{Finsler metric} \\ \downarrow & \\ S(F) & \text{geodesic spray} \\ \downarrow & \\ P(F) & \text{geodesic (path) structure} \\ \downarrow & \\ \mathfrak{p}(F) & \text{projective symmetry algebra} \end{array}$$

To proof the theorem, we go backwards. In $\mathbb{R}^2$, *locally, up to coordinate change...*

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We call $\mathfrak{g}$ **transitive** at p, if $\{X_p \mid X \in \mathfrak{g}\}$ has full dimension n.

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- Assume P with $\dim \mathfrak{p} = 3$. If $\mathfrak{p}$ is transitive in a point $x \in M$, we can assume that $\mathfrak{p} = \langle X_1, X_2, X_3 \rangle$ is from the constructed list by a coordinate change.
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(P1)	Circles of radius 1 in $\mathbb{R}^2$	$\mathfrak{p} = \mathbb{R}^2 + \mathfrak{so}(2)$
(P2)	'Circles' of radius R in S^2 (will be explained later)	$\mathfrak{p} = \mathfrak{so}(3)$
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- R. Bryant, G. Manno, V. Matveev, *A solution of a problem of Sophus Lie: normal forms of two-dimensional metrics admitting two projective vector fields*

one can find *Riemannian metrics* for both cases.

After a coordinate change, the following metrics have this path structures:

$$(F4)\ \sqrt{e^{3x}dx^2 + e^x dy^2} \qquad (F5)\ \sqrt{\frac{1}{1-e^{-y}}dx^2 + \frac{e^{3y}}{(1-e^y)^2}dy^2}$$

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$$(F4)\ \sqrt{e^{3x}dx^2 + e^x dy^2} \qquad (F5)\ \sqrt{\frac{1}{1-e^{-y}}dx^2 + \frac{e^{3y}}{(1-e^y)^2}dy^2}$$

Step 3: Metrize (P1),(P2),(P3) / Magnetic geodesics

- Take Riemannian metric g and a volume form Ω.
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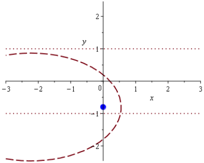

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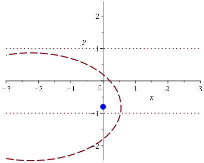

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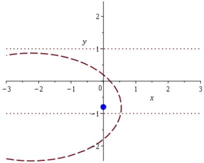

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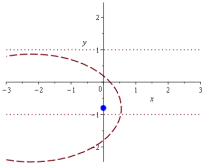

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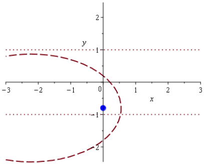

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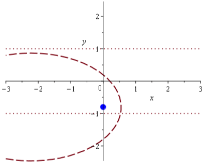

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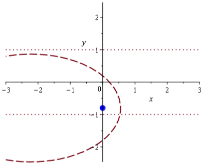

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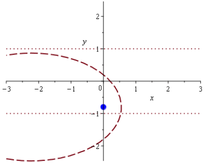

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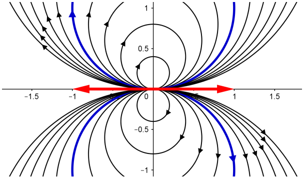

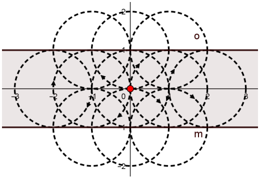

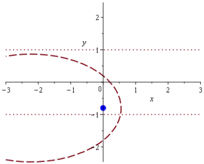

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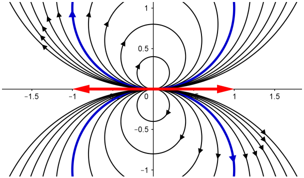

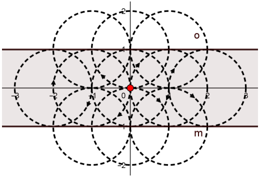

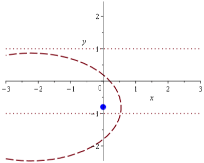

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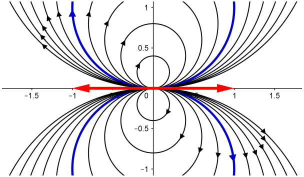

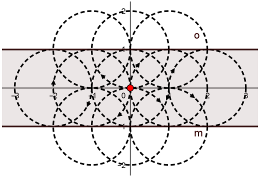

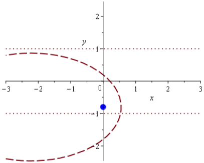

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(F4) $\sqrt{e^{3x}dx^2 + e^x dy^2}$

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(F2) $\sqrt{\frac{dx^2+dy^2}{(x^2+y^2+1)^2}} + \frac{C}{2}\left(\frac{x}{(y^2+1)(x^2+y^2+1)} + \frac{\arctan(\frac{x}{\sqrt{y^2+1}})}{(y^2+1)^{3/2}}\right) dy$, where $C \in \mathbb{R}_{>0}$

(F3) $\sqrt{\frac{1}{y^2}(dx^2 + dy^2)} - C\frac{x}{y^2}dy$, where $C \in \mathbb{R}_{>0}$

(F4) $\sqrt{e^{3x}dx^2 + e^x dy^2}$

(F5) $\sqrt{\frac{1}{1-e^{-y}}dx^2 + \frac{e^{3y}}{(1-e^y)^2}dy^2}$

Plan for my talk

1. Introduction to (Projective) Finsler geometry
2. Main theorem and proof
3. **What I want to do next**

What I want to do next:

- Describe **all** Finsler metrics with $\dim \mathfrak{p} = 3$ up to coordinate change.
  - One must find all Finsler metrics, that induce the path structures (P1)-(P5).

(P1)	Circles of radius 1 in $\mathbb{R}^2$	$\mathfrak{p} = \mathbb{R}^2 + \mathfrak{so}(2)$
(P2)	'Circles' of radius R in S^2	$\mathfrak{p} = \mathfrak{so}(3)$
(P3)	'Circles' of radius R in H^2	$\mathfrak{p} = \mathfrak{sl}(2)$
(P4)	Origin centered ellipses with area 1 in $\mathbb{R}^2$	$\mathfrak{p} = \mathfrak{sl}(2)$
(P5)	Origin centered hyperbolas with 'fixed area' in $\mathbb{R}^2$	$\mathfrak{p} = \mathfrak{sl}(2)$

- Consider the case $\dim \mathfrak{p} = 2$.

How to find all Finsler metrics that induce a given path structure?

- **Trivial freedom of adding a total derivative:**
  Let $g : M \to \mathbb{R}$ with total derivative $dg = g_x dx + g_y dy$.
  $\Rightarrow E_i(dg) = 0 \quad \Rightarrow F$ and $F + dg$ have the same path structure [1]

- **Convex cone property:** Let $F, \tilde{F}$ be Finsler metrics with path structure P.
  $\Rightarrow \forall \lambda, \mu > 0 : \lambda F + \mu \tilde{F}$ is a Finsler metric with path structure P.

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  Fact (Shen/Yu/Matveev): Two Randers metrics $F = \sqrt{g} + \beta$ and $\tilde{F} = \sqrt{\tilde{g}} + \tilde{\beta}$ with β not closed are projectively related if and only if there is $\lambda > 0$ with $g = \lambda^2 \tilde{g}$ and $\beta - \lambda\tilde{\beta}$ is closed.

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  - For (P4,P5) we can describe all Riemannian metrizations. If $\sqrt{g} + \beta$ is a Randers metrization, then β is closed and g a Riemannian metrization.

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  - In U_+ coordinates, P is given as $\langle \partial_x + z\partial_y + f(x,y,z)\partial_z \rangle \leftrightarrow y'' = f(x,y,y')$.
  - By comparing Euler-Lagrange equations, $L(x,y,z) := F(x,y,1,z)$ fulfills the PDE

$$-L_y + L_{xz} + zL_{yz} + f \cdot L_{zz} = 0 \qquad (*)$$

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  - S. Tabachnikov, *Remarks on magnetic flows and magnetic billiards, Finsler metrics and a magnetic analog of Hilbert's fourth problem*

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Finsler metrics with $\dim \mathfrak{p} = 2$

- Because there are just two two dimensional Lie algebras, we can assume that

$$\text{(A)} \quad \mathfrak{p} = \langle \partial_x, \partial_y \rangle \qquad \text{or} \qquad \text{(NA)} \quad \mathfrak{p} = \langle \partial_y, x\partial_x + y\partial_y \rangle$$

- Any path strcture with that projective algebra is in U_+ coordinates

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Thank you for the attention!

Related literature:

Classification of 2nd order ODEs $y'' = f(x, y, y')$ (essential for Step 1 & 2):

- A. Tresse, *Sur les invariants différentiels des groupes continus de transformations*
- B. Doubrov, B. Komrakov, *The geometry of second-order ordinary differential equations*

Magnetic Geodesics (used to metrize (P1)-(P3))

- K. Burns, V. Matveev, *On the rigidity of magnetic systems with the same magnetic geodesics*

General Spray and Finsler geometry

- Z. Shen, *Differential geometry of spray and Finsler spaces*
- D. Bao, S.-S. Chern, Z. Shen, *An Introduction to Riemann-Finsler Geometry*

Solution of the Pseudo-Riemannian version of the Lie problem (Metrization of (P4),(P5))

- R. Bryant, G. Manno, V. Matveev, *A solution of a problem of Sophus Lie: normal forms of two-dimensional metrics admitting two projective vector fields*
- V. Matveev, *Two-dimensional metrics admitting precisely one projective vector field*

Finsler Metrization of Path Structures = Projective Finsler Metrization of Sprays

- J.C. Álvarez-Paiva, G. Berck *Finsler surfaces with prescribed geodesics*, arXiv:1002.0243
- N.J. Sonin, *About determining maximal and minimal properties*, 1886, translation by R. Ya. Matsyuk (Lepage Research Institute, Czech Republic)
- I. Bucataru, Z. Muzsnay *Projective metrizability and formal integrability*