Smilansky-Solomyak model with a δ'-interaction

Jiří Lipovský

University of Hradec Králové, Faculty of Science
jiri.lipovsky@uhk.cz

joint work with P. Exner

Ostrava, April 6, 2018

Univerzita Hradec Králové
Přírodovědecká fakulta

Smilansky-Solomyak model

- the usual way of constructing time-irreversible system is to couple the Hamiltonian with the bath of infinite degrees of freedom
- but infinitely many degrees of freedom are not necessary
- Uzy Smilansky proposed the model consisting of a quantum graph coupled with harmonic oscillators (a harmonic oscillator) and showed that if coupling is large enough, this system exhibits irreversible behaviour
- simplest model: Schrödinger operator on a line coupled with a δ condition with a harmonic oscillator – largely studied
- our model: Schrödinger operator on a line coupled with a δ' condition with a harmonic oscillator

Original Smilansky model

- the Hamiltonian formally written as

$$\mathbf{H}_\alpha = -\frac{\partial^2}{\partial x^2} + \frac{1}{2}\left(-\frac{\partial^2}{\partial y^2} + y^2\right) + \alpha y \delta(x)\,,$$

- precisely defined as a differential operator in $L^2(\mathbb{R}^2)$

$$\mathbf{H}_\alpha \Psi = -\frac{\partial^2 \Psi}{\partial x^2} + \frac{1}{2}\left(-\frac{\partial^2 \Psi}{\partial y^2} + y^2 \Psi\right)\,,$$

  with the domain consisting of functions satisfying

$$\frac{\partial \Psi}{\partial x}(0+, y) - \frac{\partial \Psi}{\partial x}(0-, y) = \alpha y \Psi(0, y) \quad \text{for} \quad y \in \mathbb{R}\,.$$

- swap $\alpha \to -\alpha$ is equivalent to the change $y \to -y$ and hence it does not influence the spectrum, we can assume only $\alpha > 0$

Spectral properties of the original Smilansky model

- the continuous spectrum covers the interval $(1/2, \infty)$ for $\alpha < \sqrt{2}$, covers the interval $(0, \infty)$ if $\alpha = \sqrt{2}$ and the whole real axis if $\alpha > \sqrt{2}$
- for $\alpha \in (0, \sqrt{2})$ the discrete spectrum is nonempty, simple and is contained in $(0, 1/2)$; for $\alpha > \sqrt{2}$ the point spectrum is empty
- the number of eigenvalues increases as $\alpha \to \sqrt{2}$:

$$N(\frac{1}{2}, \mathbf{H}_\alpha) \sim \frac{1}{4}\sqrt{\frac{1}{\sqrt{2}(\sqrt{2}-\alpha)}}$$

- for α large enough there is only one eigenvalue which behaves as

$$\varepsilon_1(\alpha) = \frac{1}{2} - \frac{\alpha^4}{64} + \mathcal{O}(\alpha^5)$$

Smilansky model with δ'-interaction

- the Hamiltonian formally written as

$$\mathbf{H}_\beta = -\frac{\partial^2}{\partial x^2} + \frac{1}{2}\left(-\frac{\partial^2}{\partial y^2} + y^2\right) + \frac{\beta}{y}\,\delta'(x)\,,$$

- precisely defined as a differential operator in $L^2(\mathbb{R}^2)$

$$\mathbf{H}_\beta \Psi(x,y) = -\frac{\partial^2 \Psi}{\partial x^2}(x,y) + \frac{1}{2}\left(-\frac{\partial^2 \Psi}{\partial y^2}(x,y) + y^2 \Psi(x,y)\right)$$

  with the domain consisting of functions in
  $\Psi \in H^2((0,\infty)\times\mathbb{R}) \oplus H^2((-\infty,0)\times\mathbb{R})$ satisfying

$$\Psi(0+,y) - \Psi(0-,y) = \frac{\beta}{y}\frac{\partial \Psi}{\partial x}(0+,y)\,, \qquad (1)$$

$$\frac{\partial \Psi}{\partial x}(0+,y) = \frac{\partial \Psi}{\partial x}(0-,y)\,. \qquad (2)$$

- again, swap $\beta \to -\beta$ is equivalent to the change $y \to -y$ and hence it does not influence the spectrum, we can assume only $\beta > 0$

Spectral properties of the Smilansky model with δ'

Theorem 1 (absolutely continuous spectrum of the operators $\mathbf{H}_0$ and $\mathbf{H}_\beta$)

The spectrum of operator $\mathbf{H}_0$ is purely absolutely continuous, $\sigma(\mathbf{H}_0) = [\frac{1}{2}, \infty)$ with $\mathfrak{m}_{\mathrm{ac}}(E, \mathbf{H}_0) = 2n$ for $E \in (n - \frac{1}{2}, n + \frac{1}{2})$, $n \in \mathbb{N}$.
For $\beta > 2\sqrt{2}$ the absolutely continuous spectrum of $\mathbf{H}_\beta$ coincides with the spectrum of $\mathbf{H}_0$. For $\beta \leq 2\sqrt{2}$ there is a new branch of continuous spectrum added to the spectrum of $\mathbf{H}_0$. For $\beta = 2\sqrt{2}$ we have $\sigma(\mathbf{H}_\beta) = [0, \infty)$ and for $\beta < 2\sqrt{2}$ the spectrum covers the whole real line.

- $\mathfrak{m}_{\mathrm{ac}}$ denote the multiplicity function of the absolutely continuous spectra

Theorem 2 (discrete spectrum of the operator $\mathbf{H}_\beta$ for $\beta \in (2\sqrt{2}, \infty)$)

Assume $\beta \in (2\sqrt{2}, \infty)$, then the discrete spectrum of $\mathbf{H}_\beta$ is nonempty and lies in the interval $(0, \frac{1}{2})$. The number of eigenvalues is approximately given by

$$\frac{1}{4\sqrt{2\left(\frac{\beta}{2\sqrt{2}} - 1\right)}} \quad \text{as} \quad \beta \to 2\sqrt{2}+ \ .$$

Theorem 3 (discrete spectrum of the operator $\mathbf{H}_\beta$ for large β)

For large enough β there is a single eigenvalue which asymptotically behaves as

$$\Lambda_1 = \frac{1}{2} - \frac{4}{\beta^4} + \mathcal{O}\left(\beta^{-5}\right) \ .$$

The quadratic form

- the quadratic form $\mathbf{a}_\beta[\Psi] = \mathbf{a}_0[\Psi] + \frac{1}{\beta}\mathbf{b}[\Psi]$

$$\begin{aligned} \mathbf{a}_0[\Psi] &= \int_{\mathbb{R}^2} \left(\left|\frac{\partial \Psi}{\partial x}\right|^2 + \frac{1}{2}\left|\frac{\partial \Psi}{\partial y}\right|^2 + \frac{1}{2} y^2 |\Psi|^2 \right) \mathrm{d}x\mathrm{d}y\,, \\ \mathbf{b}[\Psi] &= \int_{\mathbb{R}} y\,|\Psi(0+,y) - \Psi(0-,y)|^2\,\mathrm{d}y \end{aligned}$$

is associated with the operator $\mathbf{H}_\beta$. The domain $D = \mathrm{dom}\,\mathbf{a}_0$ of the form $\mathbf{a}_0$ is

$$D = \left\{ \Psi \in H^1((0,\infty)\times\mathbb{R}) \oplus H^1((-\infty,0)\times\mathbb{R})\,;\mathbf{a}_0[\Psi] < \infty \right\}$$

Bound on the quadratic form

Theorem 4

If $\beta \geq 2\sqrt{2}$ *it holds*

$$\mathbf{a}_\beta[\Psi] \geq \frac{1}{2}\left(1 - \frac{2\sqrt{2}}{\beta}\right)\|\Psi\|^2\,.$$

Lemma 5

For complex numbers c, d *it holds* $2|\mathrm{Re}\,(\bar{\mathrm{c}}\mathrm{d})| \leq |c|^2 + |d|^2$.

Lemma 6

It holds

$$\gamma(|\psi(0+)|^2 + |\psi(0-)|^2) \le \int_{\mathbb{R}} \left(|\psi'(x)|^2 + \gamma^2 |\psi(x)|^2\right) \, \mathrm{d}x$$
$$\forall \psi \in H^1((0,\infty)) \oplus H^1((-\infty,0)), \quad \gamma > 0,$$

with the equality attained on the subspace generated by

$$\tilde{\psi}_\gamma(x) = \frac{\operatorname{sgn} x}{\sqrt{2\gamma}} \mathrm{e}^{-\gamma|x|}, \quad \gamma > 0. \tag{3}$$

Proof of Lemma 6.

- we have

$$0 \le \int_{-\infty}^{0} |\psi'(x) - \gamma\psi(x)|^2 \, \mathrm{d}x = \int_{-\infty}^{0} (|\psi'(x)|^2 + \gamma^2 |\psi(x)|^2) \, \mathrm{d}x$$
$$- \gamma \int_{-\infty}^{0} (\bar{\psi}'(x)\psi(x) + \psi'(x)\bar{\psi}(x)) \, \mathrm{d}x$$
$$= \int_{-\infty}^{0} (|\psi'(x)|^2 + \gamma^2 |\psi(x)|^2) \, \mathrm{d}x - \gamma [|\psi(x)|^2]_{-\infty}^{0-} ,$$

  and therefore

$$\int_{-\infty}^{0} (|\psi'(x)|^2 + \gamma^2 |\psi(x)|^2) \, \mathrm{d}x \ge \gamma |\psi(0-)|^2 .$$

- similarly, $0 \le \int_0^\infty |\psi'(x) + \gamma\psi(x)|^2 \, \mathrm{d}x$ implies $\int_0^\infty (|\psi'(x)|^2 + \gamma^2 |\psi(x)|^2) \, \mathrm{d}x \ge \gamma |\psi(0+)|^2$, and combining both inequalities one obtains the result.

□

Proof of Theorem 4.

- we use separation of variables and the expansion of Ψ in the harmonic oscillator basis, i.e. Hermite functions in the variable y normalized in $L^2(\mathbb{R})$

$$\Psi(x,y) = \sum_{n\in\mathbb{N}_0} \psi_n(x)\chi_n(y)\,, \tag{4}$$

- we insert this into the form $\mathbf{a}_0$

$$\mathbf{a}_0[\Psi] = \sum_{n\in\mathbb{N}_0} \int_{\mathbb{R}} \left(|\psi_n'(x)|^2 + \left(n+\frac{1}{2}\right)|\psi_n(x)|^2\right) \mathrm{d}x \tag{5}$$

- we use twice expansion (4) and the relation

$$\sqrt{n+1}\chi_{n+1}(y) - \sqrt{2}y\chi_n(y) + \sqrt{n}\chi_{n-1}(y) = 0\,, \quad n\in\mathbb{N}_0\,, \tag{6}$$

Proof of Theorem 4 (continued).

- we obtain

$$\mathbf{b}[\Psi] = \frac{1}{\sqrt{2}} \int_{\mathbb{R}} \sum_{m\in\mathbb{N}_0} \sum_{n\in\mathbb{N}_0} (\bar{\psi}_m(0+) - \bar{\psi}_m(0-))\bar{\chi}_m(y)$$

$$(\psi_n(0+) - \psi_n(0-)) \left[\sqrt{n+1}\chi_{n+1}(y) + \sqrt{n}\chi_{n-1}(y)\right] \mathrm{d}y$$

$$= \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{N}_0} [(\bar{\psi}_{n+1}(0+) - \bar{\psi}_{n+1}(0-))\sqrt{n+1}$$

$$+ (\bar{\psi}_{n-1}(0+) - \bar{\psi}_{n-1}(0-))\sqrt{n}](\psi_n(0+) - \psi_n(0-)) =$$

$$= \frac{2}{\sqrt{2}} \sum_{n\in\mathbb{N}} \sqrt{n}\,\mathrm{Re}\,[(\bar{\psi}_n(0+) - \bar{\psi}_n(0-))(\psi_{n-1}(0+) - \psi_{n-1}(0-))]\,. \tag{7}$$

- we employed the Hermite functions orthonormality here and in the last line we have changed the summation index, $n+1 \to n$, in the first part of the sum

Proof of Theorem 4 (continued).

- it follows from Lemma 5 that

$$|\mathbf{b}[\Psi]| \leq \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{N}} \sqrt{n}(|\psi_n(0+) - \psi_n(0-)|^2 + \\ + |\psi_{n-1}(0+) - \psi_{n-1}(0-)|^2)\,.$$

- changing the summation index in the second part of the sum we get

$$|\mathbf{b}[\Psi]| \leq \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{N}_0} (\sqrt{n} + \sqrt{n+1})|\psi_n(0+) - \psi_n(0-)|^2 \\ \leq \sum_{n\in\mathbb{N}_0} \sqrt{2n+1}|\psi_n(0+) - \psi_n(0-)|^2\,,$$

where we have used the inequality
$\sqrt{n} + \sqrt{n+1} < \sqrt{2(2n+1)}$

Proof of Theorem 4 (continued).

- using subsequently Lemmata 5 and 6 we obtain

$$
\begin{aligned}
|\mathbf{b}[\Psi]| &\leq 2\sqrt{2} \sum_{n\in\mathbb{N}_0} \sqrt{n+\frac{1}{2}}\left(|\psi_n(0+)|^2 + |\psi_n(0-)|^2\right) \\
&\leq 2\sqrt{2} \sum_{n\in\mathbb{N}_0} \int_{\mathbb{R}} \left(|\psi_n'(x)|^2 + \left(n+\frac{1}{2}\right)|\psi_n(x)|^2\right)\,\mathrm{d}x \\
&= 2\sqrt{2}\,\mathbf{a}_0[\Psi]\,.
\end{aligned}
$$

- we use $\mathbf{a}_0[\Psi] \geq \frac{1}{2}\|\Psi\|^2$, which follows from (5)
- we obtain

$$
\begin{aligned}
\mathbf{a}_\beta[\Psi] = \mathbf{a}_0[\Psi] + \frac{1}{\beta}\mathbf{b}[\Psi] &\geq \left(1 - \frac{2\sqrt{2}}{\beta}\right)\mathbf{a}_0[\Psi] \\
&\geq \frac{1}{2}\left(1 - \frac{2\sqrt{2}}{\beta}\right)\|\Psi\|^2\,,
\end{aligned}
$$

Construction of the Jacobi operator

- we will rephrase the problem with using certain Jacobi operator
- we substitute to eq. (1) the Ansatz (4) for Ψ, multiply the equation by $\bar{\chi}_m(y)$, integrate with respect to y over $\mathbb{R}$. and use the orthonormality

$$\sum_{n\in\mathbb{N}_0} \int_{\mathbb{R}} \bar{\chi}_m(y) y (\psi_n(0+) - \psi_n(0-)) \chi_n(y)\,\mathrm{d}y =$$

$$= \beta \sum_{n\in\mathbb{N}_0} \int_{\mathbb{R}} \frac{\partial \psi_n}{\partial x}(0+) \bar{\chi}_m(y) \chi_n(y)\,\mathrm{d}y$$

- relation (6) then yields the condition

$$\beta \frac{\partial \psi_m}{\partial x}(0+) = \sum_{n\in\mathbb{N}_0} \frac{1}{\sqrt{2}} \int_{\mathbb{R}} (\psi_n(0+) - \psi_n(0-))\bar{\chi}_m(y) \left(\sqrt{n+1}\chi_{n+1}(y) + \sqrt{n}\chi_{n-1}(y)\right) \mathrm{d}y$$

$$= \frac{\sqrt{m}}{\sqrt{2}}(\psi_{m-1}(0+) - \psi_{m-1}(0-)) + \frac{\sqrt{m+1}}{\sqrt{2}}(\psi_{m+1}(0+) - \psi_{m+1}(0-))\,, \qquad (8)$$

- on the other hand, the condition (2) implies

$$\frac{\partial \psi_n}{\partial x}(0+) = \frac{\partial \psi_n}{\partial x}(0-) \qquad (9)$$

- consider now the eigenvalue problem for the operator $\mathbf{H}_\beta$, which is equivalent to the set of equations

$$-\phi_n''(x) + (n + \frac{1}{2} - \Lambda)\phi_n(x) = 0\,, \quad x = 0\,, \quad n \in \mathbb{N}_0 \quad (10)$$

under the matching conditions (8) and (9), for $\phi_n \restriction \mathbb{R}_\pm \in H^2(\mathbb{R}_\pm)$ where Λ is the sought eigenvalue.
- we define $\zeta_n(\Lambda) = \sqrt{n + \frac{1}{2} - \Lambda}$ taking the branch of the square root which is analytic in $\mathbb{C}\backslash[n + \frac{1}{2}, \infty)$ and for number Λ from this set it holds

$$\mathrm{Re}\,\zeta_n(\Lambda) > 0\,, \quad \mathrm{Im}\,\zeta_n(\Lambda) \cdot \mathrm{Im}\,\Lambda < 0\,.$$

- solutions to the equation (10) in $L^2(\mathbb{R}_\pm)$ are

$$\phi_n(x, \Lambda) = k_1(\Lambda)\,\mathrm{e}^{-\zeta_n(\Lambda)x}\,, \quad x > 0\,,$$
$$\phi_n(x, \Lambda) = k_2(\Lambda)\,\mathrm{e}^{\zeta_n(\Lambda)x}\,, \quad x < 0\,,$$

where from (9) we have $k_1(\Lambda) = -k_2(\Lambda)$

- we use the normalization $\phi_n(x,\Lambda) = C_n \eta_n(x,\Lambda)$ with

$$\eta_n(x,\Lambda) := \pm \left(n+\tfrac{1}{2}\right)^{1/4} \mathrm{e}^{\mp\zeta_n(\Lambda)x}\,. \quad x \in \mathbb{R}_\pm\,.$$

- hence

$$\begin{aligned} \phi_n(0+,\Lambda) - \phi_n(0-,\Lambda) &= 2C_n \left(n+\tfrac{1}{2}\right)^{1/4}\,, \\ \frac{\partial \phi_n}{\partial x}(0+,\Lambda) &= -C_n \left(n+\tfrac{1}{2}\right)^{1/4} \zeta_n(\Lambda)\,. \end{aligned} \tag{11}$$

- substituting from here to eq. (8) we obtain the relation

$$\begin{aligned} (n+1)^{1/2} \left(n+\tfrac{3}{2}\right)^{1/4} C_{n+1} + 2\mu \left(n+\tfrac{1}{2}\right)^{1/4} \zeta_n(\Lambda) C_n + \\ + n^{1/2} \left(n-\tfrac{1}{2}\right)^{1/4} C_{n-1} = 0\,, \quad n \in \mathbb{N}_0 \end{aligned} \tag{12}$$

  with $\mu := \frac{\beta}{2\sqrt{2}}$
- this equation defines the same Jacobi operator $\mathcal{J}(\Lambda,\mu)$ as for δ-condition, only our parameter μ differs

Absolutely continuous spectrum of $\mathbf{H}_\beta$

- one can represent the resolvent using the Krein formula with the obtained Jacobi operator
- one can proceed similarly to the known case of δ-condition and prove the following theorems

Theorem 7 (absolutely continuous spectrum of $\mathbf{H}_\beta$)

$$\begin{aligned}\sigma_{\rm ac}(\mathbf{H}_\beta) &= \sigma_{\rm ac}(\mathbf{H}_0) \cup \sigma_{\rm ac}(\mathcal{J}_0(\beta/(2\sqrt{2})))\,,\\ \mathfrak{m}_{\rm ac}(E,\mathbf{H}_\beta) &= \mathfrak{m}_{\rm ac}(E,\mathbf{H}_0) + \mathfrak{m}_{\rm ac}(E,\mathcal{J}_0(\beta/(2\sqrt{2})))\,.\end{aligned}$$

where

$$\mathcal{J}_0(\mu) := D\mathcal{S} + \mathcal{S}^* D + 2\mu Y_0$$

with

$$\begin{aligned}\mathcal{D}\,,\mathcal{S} : \ell^2(\mathbb{N}_0) \mapsto \ell^2(\mathbb{N}_0)\,,\\ \mathcal{D}\{\omega_n\} : \{r_0, r_1, \dots\} \mapsto \{\omega r_0, \omega_1 r_1, \dots\}\,,\\ D := \mathcal{D}(d_n)\,, \quad Y_0 := \mathcal{D}\{n+1/2\}\,,\\ \mathcal{S} : \{r_0, r_1, \dots\} \mapsto \{0, r_0, r_1, \dots\}\,,\\ d_n := n^{1/2}(n+\frac{1}{2})^{1/4}(n-\frac{1}{2})^{1/4}\,.\end{aligned}$$

Theorem 8 (spectrum of $\mathcal{J}_0$)

$$\begin{aligned} \sigma(\mathcal{J}_0(\mu)) &= (-\infty,\infty) \quad \text{for} \quad \mu < 1\,, \\ \sigma(\mathcal{J}_0(1)) &= [0,\infty)\,, \\ \sigma_{\mathrm{ac}}(\mathcal{J}_0(\mu)) &= \emptyset \quad \text{for} \quad \mu > 1\,, \\ \mathfrak{m}_{\mathrm{ac}}(E,\mathcal{J}_0(\mu)) &= 1 \quad \text{a.e. on} \quad \sigma(\mathcal{J}_0(\mu))\,. \end{aligned}$$

- since we have $\mu = \frac{\beta}{2\sqrt{2}}$, these two theorem in combination with the well-known spectrum of $\mathbf{H}_0$ prove the claim of Theorem 1

Discrete spectrum of $\mathbf{H}_\beta$

Proof of Theorem 3.

- first we check that the spectrum on $(-\infty, \frac{1}{2})$ is non-empty using a variational argument
- the idea is to construct an element $\Psi^\varepsilon \in D$ such that $\mathbf{a}_\beta[\Psi^\varepsilon] < \frac{1}{2}\|\Psi^\varepsilon\|^2$
- consider functions ψ_0, ψ_1 satisfying the conditions

$$\psi_0(0+) - \psi_0(0-) = -C < 0\,, \qquad \psi_1(0+) - \psi_1(0-) = 1\,,$$

  and such that $\Psi = \{\psi_0, \psi_1, 0, 0, \dots\} \in D$
- we scale the first one, $\psi_0^\varepsilon(x) := \psi_0(\varepsilon x)$, and put $\Psi^\varepsilon := \{\psi_0^\varepsilon, \psi_1, 0, 0, \dots\}$ which belongs again to D

Proof of Theorem 3 (continued).

- from (5) and (7) we have

$$\mathbf{a}_\beta[\Psi^\varepsilon] - \frac{1}{2}\|\Psi^\varepsilon\|^2 = \int_{\mathbb{R}} \left(|\psi_0^{\varepsilon\prime}(x)|^2 + |\psi_1(x)|^2 + \right.$$
$$\left. + |\psi_1'(x)|^2\right)\,\mathrm{d}x - \frac{\sqrt{2}}{\beta}C =$$
$$= \int_{\mathbb{R}} \left(\varepsilon|\psi_0'(x)|^2 + |\psi_1(x)|^2 + |\psi_1'(x)|^2\right)\,\mathrm{d}x - \frac{\sqrt{2}}{\beta}C$$

- choosing ε small enough and C large enough one can achieve that the right-hand side of the last equation is negative, which means that the spectrum below $\frac{1}{2}$ is nonempty for any $\beta > 0$

Proof of Theorem 3 (continued).

- one can proof that $N_-(\frac{1}{2}, \mathbf{H}_\beta) = N_+(\mu, \mathbf{J}_0)$ or $N_-(\frac{1}{2}, \mathbf{H}_\beta) = N_+(\mu, \mathbf{J}_0) + 1$, where N_- (N_+ is the number of eigenvalues below (above) certain value)
- using the fact on the previous slide, and the fact that the eigenvalues of $\mathbf{J}_0$ have a single accumulation point at 1 (and consequently, there is a μ such that there are no eigenvalues of $\mathbf{J}_0$ larger than μ) we find that for β large enough the operator $\mathbf{H}_\beta$ has exactly one simple eigenvalue.
- the asymptotic expansion of this eigenvalue Λ_1 can be found by an argument similar to the original Smilansky model

Proof of Theorem 3 (continued).

- the system of equations (12) can be after substitution $Q_n = (n+\frac{1}{2})^{1/4} C_n$ rewritten as

$$\begin{aligned} Q_1 + 2\mu\sqrt{\tfrac{1}{2}-\Lambda_1}\, Q_0 &= 0\,, \qquad (13) \\ (n+1)^{1/2} Q_{n+1} + 2\mu\zeta_n(\Lambda_1) Q_n + n^{1/2} Q_{n-1} &= 0\,, \\ n \in \mathbb{N}\,. & \qquad (14) \end{aligned}$$

- we normalize $\|Q\| := \sum_{n=0}^{\infty} |Q_n|^2 = 1$, using then $\sqrt{n} \le \sqrt{n+\frac{1}{2}-\Lambda_1} = \zeta_n(\Lambda_1)$ and $\sqrt{n+1} \le \sqrt{2(n+\frac{1}{2}-\Lambda_1)}$ we obtain from (14) the estimate

$$|Q_n| \le \frac{1}{2\mu}|Q_{n-1}| + \frac{1}{\sqrt{2}\mu}|Q_{n+1}|\,. \qquad (15)$$

Proof of Theorem 3 (continued).

- in the analogy with Lemma 5 we have

$$|Q_n|^2 \leq \frac{1}{2\mu^2}|Q_{n-1}|^2 + \frac{1}{\mu^2}|Q_{n+1}|^2\,,$$

- hence

$$\sum_{n=1}^{\infty}|Q_n|^2 \leq \frac{1}{2\mu^2}\sum_{n=0}^{\infty}|Q_n|^2 + \frac{1}{\mu^2}\sum_{n=2}^{\infty}|Q_n|^2 \leq \frac{3}{2\mu^2}\,,$$

  where we have used the mentioned normalization

- from here it follows that

$$|Q_0| = \left(\sum_{n=0}^{\infty}|Q_n|^2 - \sum_{n=1}^{\infty}|Q_n|^2\right)^{1/2} = 1 + \mathcal{O}\left(\mu^{-2}\right)\,. \qquad (16)$$

- without loss of generality we may suppose that Q_0 is positive

Proof of Theorem 3 (continued).

- from (15) with $n = 2$ with the use of the normalization we obtain

$$|Q_2| \leq \frac{1}{2\mu} + \frac{1}{\sqrt{2}\mu} \quad \Rightarrow \quad Q_2 = \mathcal{O}\left(\mu^{-1}\right) .$$

- furthermore, from (14) and (16) we get

$$Q_1 = \frac{1}{2\mu} + \mathcal{O}\left(\mu^{-2}\right) .$$

- from (13) we obtain $\left(\frac{1}{2} - \Lambda_1\right)^{1/2} = -\frac{Q_1}{2\mu Q_0} = -\frac{1}{4\mu^2} + \mathcal{O}\left(\mu^{-3}\right)$, or equivalently

$$\frac{1}{2} - \Lambda_1 = \frac{1}{16\mu^4} + \mathcal{O}\left(\mu^{-5}\right) = \frac{4}{\beta^4} + \mathcal{O}\left(\beta^{-5}\right) ,$$

which concludes the proof.

□

Paper on which the talk was based

P. Exner, J. Lipovsky: Smilansky-Solomyak model with a δ'-interaction, *Phys. Lett. A* **382** (2018), 1207–1213.

Thank you for your attention!

Paper on which the talk was based

P. Exner, J. Lipovsky: Smilansky-Solomyak model with a δ'-interaction, *Phys. Lett. A* **382** (2018), 1207–1213.

Thank you for your attention!