On realizability of Gauss diagrams and construction of meanders

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Basic Concepts and Definitions

Classically, a knot

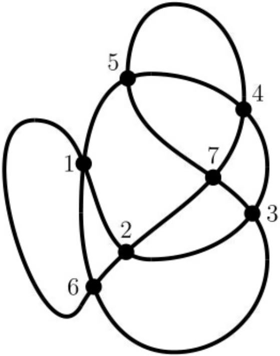
is defined as an embedding of the circle S^1 into $\mathbb{R}^3$, or equivalently into the 3-sphere S^3, i.e., a knot is a closed curve embedded on $\mathbb{R}^3$ (or S^3) without intersecting itself, up to ambient isotopy.

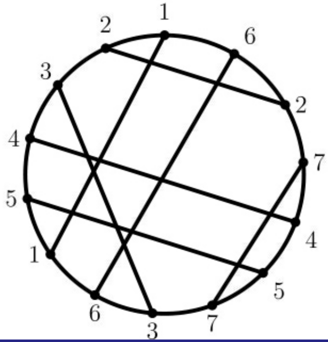
The projection of a knot onto a 2-manifold is considered with all multiple points are transversal double with will be call crossing points (or shortly crossings). Such a projection is called the shadow or plane curves.

Gauss Diagrams

A generic immersion of a circle to a plane is characterized by its Gauss diagram.

The Gauss diagram is the immersing circle with the preimages of each double point connected with a chord.

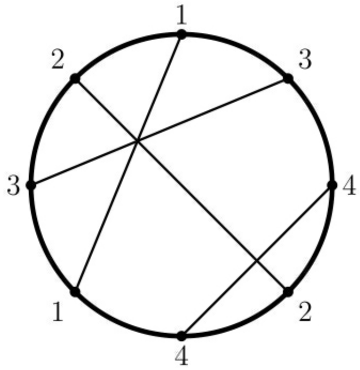


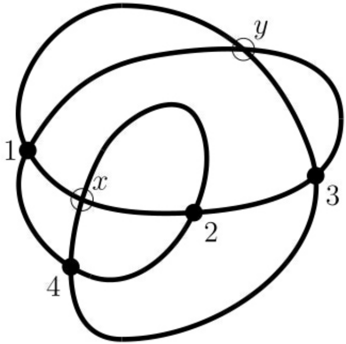


Virtual plane curves

If a Gauss diagram can be realized by a plane curve

we say that it is realizable and it can be realizable by a virtual plane curve otherwise.





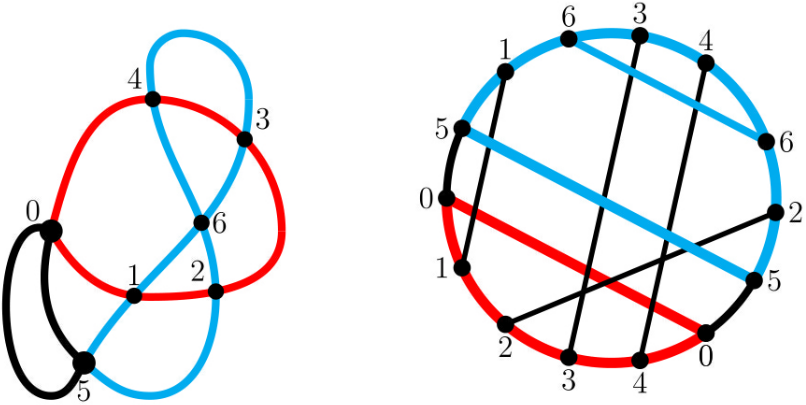
The Sketch of the Main Idea

We suggest an approach, which satisfies the above principle. We use the fact that every Gauss diagram $\mathfrak{G}$ defines a (virtual) plane curve $\mathscr{C}(\mathfrak{G})$ and the following simple ideas:

(1) For every chord of a Gauss diagram $\mathfrak{G}$, we can associate a closed path along the curve $\mathscr{C}(\mathfrak{G})$.
(2) For every two non-intersecting chords of a Gauss diagram $\mathfrak{G}$, we can associate two closed paths along the curve $\mathscr{C}(\mathfrak{G})$ such that every chord crosses both of those chords correspondences to the point of intersection of the paths.
(3) If a Gauss diagram $\mathfrak{G}$ is realizable (say by a plane curve $\mathscr{C}(\mathfrak{G})$), then for every closed path (say) $\mathscr{P}$ along $\mathscr{C}(\mathfrak{G})$ we can associate a coloring another part of $\mathscr{C}(\mathfrak{G})$ into two colors (roughly speaking we get "inner" and "outer" sides of $\mathscr{P}$ cf. Jordan curve Theorem). If a Gauss diagram is not realizable then it defines a virtual plane curve $\mathscr{C}(\mathfrak{G})$. We shall show that there exists a closed path along $\mathscr{C}(\mathfrak{G})$ for which we cannot associate a well-defined coloring of $\mathscr{C}(\mathfrak{G})$, i.e., $\mathscr{C}(\mathfrak{G})$ contains a path is colored into two colors.

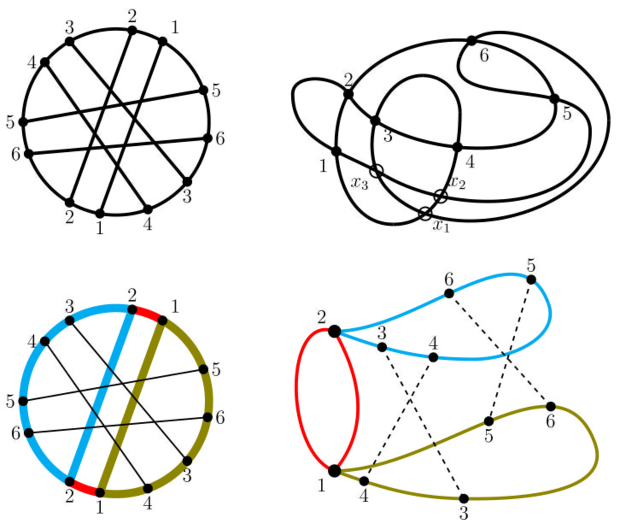
A chord with a chosen arc = a closed path along the plane curve

The colored chords with colored arcs = the colored paths[1]



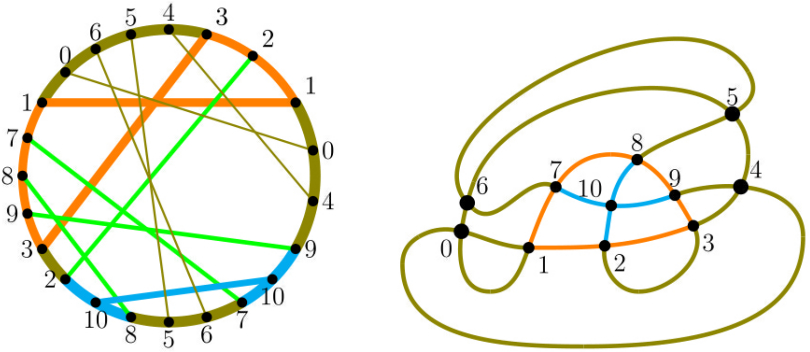
[1] The chord 6 correspondences to the self-intersection point 6 of the cyan loop

The plane curve can be obtained by attaching the cyan loop to the olive loop by the points 3, 4, 5, 6, and thus the olive loop has to have "new" crossings (= self-intersections)

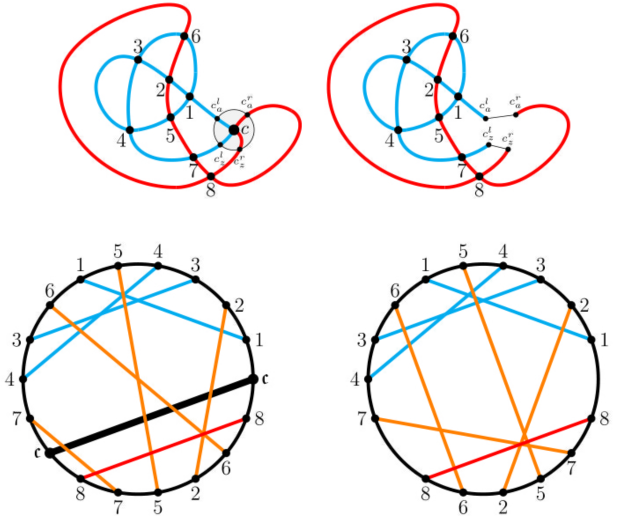


Partition of Gauss diagrams (=plane curves)

The $\mathscr{X}$-contour $\mathscr{X}(1,3)$ (= orange loop) divides the plane curve into two colored parts

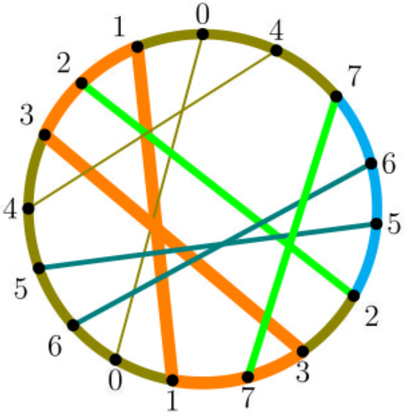


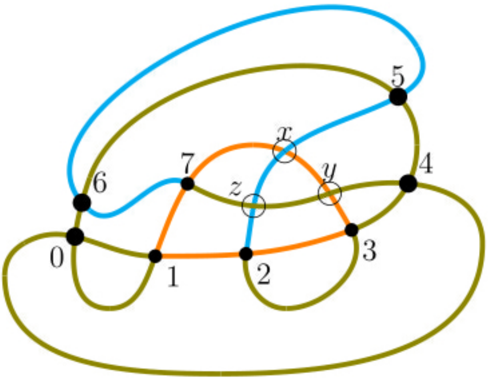
Conway smoothing



Existing of colorful chords means non realizability of Gauss diagrams

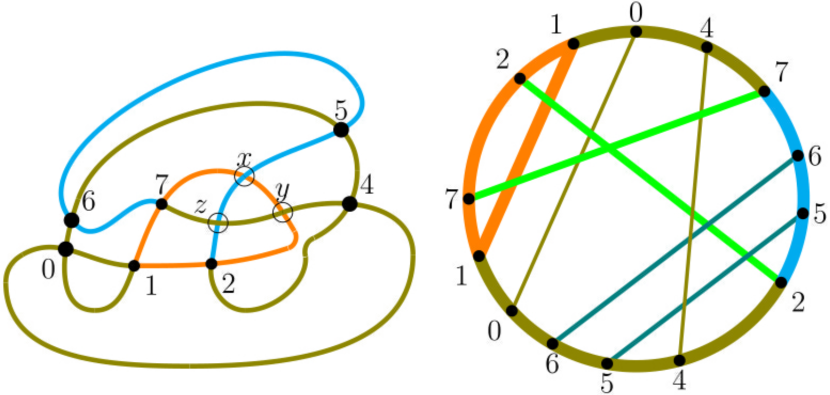
This Gauss diagram satisfies even condition but is non-realizable. There are colorful chords (e.g. the chord with endpoints 5).





Example

The Gauss diagram does not satisfy the even condition; both the chords 1, 6 are crossed by only one chord 2.



The Main Theorem

Theorem

A Gauss diagram $\mathfrak{G}$ is realizable if and only if the following conditions hold:

(1) the number of all chords that cross a both of non-intersecting chords and every chord is even (including zero),

(2) for every chord $\mathfrak{c} \in \mathfrak{G}$ the Gauss diagram $\widehat{\mathfrak{G}}_{\mathfrak{c}}$ (= Conway's smoothing the chord $\mathfrak{c}$) also satisfies the above condition.

Matrices of Gauss diagrams

Definition

Given a Gauss diagram $\mathfrak{G}$ contains n chords, say, $\mathfrak{c}_1, \ldots, \mathfrak{c}_\mathfrak{n}$, introduce the following $n \times n$ matrix $M(\mathfrak{G}) := (m_{ij})_{1 \leq i,j \leq n}$;

(0) $m_{ii} = 0$, $1 \leq i \leq n$,

(1) $m_{ij} = 1$ iff the chords $\mathfrak{c}_i$, $\mathfrak{c}_j$ are intersecting chords,

(2) $m_{ij} = 0$ iff the chords $\mathfrak{c}_i$, $\mathfrak{c}_j$ are not intersecting chords.

Scalar product of strings

Let $\{m_i = (m_{i1}, \ldots, m_{in})\}_{1 \leq i \leq n}$ be the strings of $M(\mathfrak{G})$. Set

$$\langle m_i, m_j \rangle := m_{i1}m_{j1} + \cdots + m_{in}m_{jn}$$

Reformulation of the Main Theorem

Let a Gauss diagram $\mathfrak{G}$ has n chords $\mathfrak{c}_1, \dots, \mathfrak{c}_n$.

Consider its matrix $M(\mathfrak{G})$. The Gauss diagram $\mathfrak{G}$ is realizable if and only if;

(1) $\langle m_i, m_i \rangle \equiv 0 \bmod (2)$, $1 \le i \le n$,

(2) $\langle m_i, m_j \rangle \equiv 0 \bmod (2)$, if the chords $\mathfrak{c}_i$, $\mathfrak{c}_j$ are not intersecting,

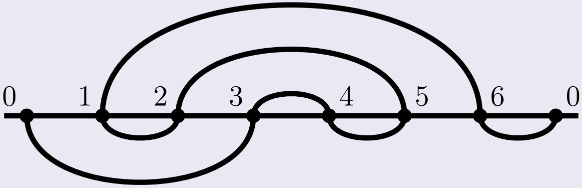
(3) $\langle m_i, m_j \rangle + \langle m_i, m_k \rangle + \langle m_j, m_k \rangle \equiv 1 \bmod (2)$, whenever the chords $\mathfrak{c}_i, \mathfrak{c}_j, \mathfrak{c}_k$ are intersecting.

Meanders

Definition

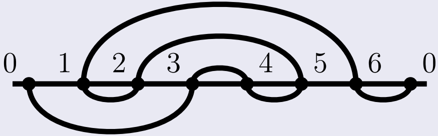
Given an even number of points on a line. A plane curve contains all the points and has no self-intersecting points is called meander.

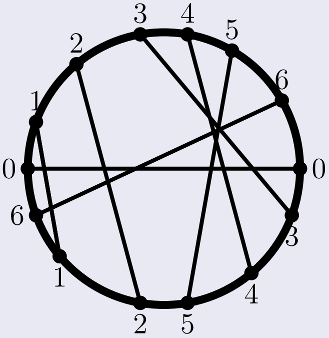
Example



Gauss diagrams of the meander

Example





Braids

Braid group B_n

is generated by $\sigma_1, \ldots, \sigma_{n-1}$, and the following relations among them

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i, \ \text{if } |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}. \end{cases}$$

Braids gives a permutation

One obvious invariant of an isotopy of a braid is the permutation it induces on the order of the strands: given a braid B, the strands define a map p(B) from the top set of endpoints to the bottom set of endpoints, which we interpret as a permutation of $\{1, \ldots, n\}$. In this way we get a homomorphism

$$p : B_n \to \mathfrak{S}_n,$$

where $\mathfrak{S}_n$ is the symmetric group. The generator σ_i is mapped to the transposition $s_i = (i, i+1)$. We denote by $S_n = \{s_1, \ldots, s_{n-1}\}$ the set of generators for the symmetric group $\mathfrak{S}_n$.

Thurston's generators of B_n

We want to define an inverse map $p^{-1} : \mathfrak{S}_n \to Br_n$.

Definition

Let $S = \{s_1, \ldots, s_{n-1}\}$ be the set of generators for $\mathfrak{S}_n$. Each permutation π gives rise to a total order relation $\leq_\pi$ on $\{1, \ldots, n\}$ with $i \leq_\pi j$ if $\pi(i) < \pi(j)$. We set

$$R_\pi := \{(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, n\} | i < j, \pi(i) > \pi(j)\}.$$

We then put $p^{-1}(\pi) := R_\pi$.

Example

Let us consider the permutation $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 6 & 1 & 5 & 3 \end{pmatrix}$, we have

$$\begin{cases} 1 < 2, \\ \pi(1) > \pi(2), \end{cases} \quad \begin{cases} 1 < 4, \\ \pi(1) > \pi(4), \end{cases} \quad \begin{cases} 1 < 6, \\ \pi(1) > \pi(6), \end{cases}$$

$$\begin{cases} 2 < 4, \\ \pi(2) > \pi(4), \end{cases} \quad \begin{cases} 3 < 4, \\ \pi(3) > \pi(4), \end{cases} \quad \begin{cases} 3 < 5, \\ \pi(3) > \pi(5), \end{cases}$$

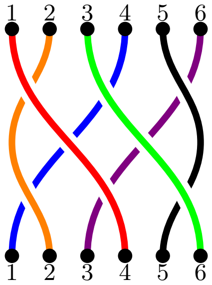
$$\begin{cases} 3 < 6, \\ \pi(3) > \pi(6), \end{cases} \quad \begin{cases} 5 < 6, \\ \pi(5) > \pi(6). \end{cases}$$

thus we get

$$\mathrm{R}_\pi = \{(1,2),(1,4),(1,6),(2,4),(3,4),(3,5),(3,6),(5,6)\}.$$

Example; $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 6 & 1 & 5 & 3 \end{pmatrix}$

The $\mathrm{R}_\pi = \{(1,2),(1,4),(1,6),(2,4),(3,4),(3,5),(3,6),(5,6)\}$ correspondences to the following braid



Thurston proved

Lemma[2]

A set R of pairs (i, j), with $i < j$, comes from some permutation if and only if the following two conditions are satisfied:

(1) if $(i, j) \in R$ and $(j, k) \in R$, then $(i, k) \in R$,

(2) if $(i, k) \in R$, then $(i, j) \in R$ or $(j, k) \in R$ for every j with $i < j < k$.

[2] see Lemma 9.1.6 in the book D.B.A. Epstein, I.W. Cannon, D.E. Holt, S.V.F. Levy, M.S. Paterson and W.P. Thurston, Word Processing in Groups, Jones and Bartlett Publishers, INC., 1992.

Algorithm

Given $N \times N$ matrix, $S_1 = 1, 3, \dots, N-1$, $S_2 = \{2, 4, \dots, N\}$, here N is an even number.

(0) Chose and D-fill the string with number 0,

(1) Chose and D-fill a string with an odd number n,

(2) Get equalities for non-filled cells,

(3) Take a string with an odd number $n \in S_1$,

(4) D-fill the string and get equalities for non-filled cells; taking into account all obtained equalities check the conditions,

(5) IF there are no contradictions then PRINT n and put $S_1 := S_1 \setminus \{n\}$ and GO TO (6), ELSE GO TO (3) and don't take this string,

(6) Take a string with an even number $n \in S_2$,

(7) D-fill the string and get equalities for non-filled cells; taking into account all obtained equalities check the conditions,

(8) IF there are no contradictions then PRINT n and put $S_2 := S_2 \setminus \{n\}$

Example

Choose 0

	0	1	2	3	4	5	6	7	8	
0	0	1	1	1	1	1	1	1	1	0
1	1	0								1
2	1		0							2
3	1			0						3
4	1				0					4
5	1					0				5
6	1						0			6
7	1							0		7
8	1								0	8
	0	1	2	3	4	5	6	7	8	

Example

Choose 5

to 5

	0	1	2	3	4	5	6	7	8
0	0	1	1	1	1	1	1	1	1
1	1	0				0			
2	1		0			0			
3	1			0		0			
4	1				0	0			
5	1	0	0	0	0	0	1	1	1
6	1					1	0		
7	1					1		0	
8	1					1			0
	0	1	2	3	4	5	6	7	8

Example

We get

	0	1	2	3	4	5	6	7	8	
0	0	1	1	1	1	1	1	1	1	0
1	1	0				0				1
2	1		0			0				2
3	1			0		0				3
4	1				0	0				4
5	1	0	0	0	0	0	1	1	1	5
6	1					1	0	a	a	6
7	1					1	a	0	a	7
8	1					1	a	a	0	8
	0	1	2	3	4	5	6	7	8	

Example

Choose 2

	0	1	2	3	4	5	6	7	8
0	0	1	1	1	1	1	1	1	1
1	1	0	0			0			
2	1	0	0	1	1	0	1	1	1
3	1		1	0		0			
4	1		1		0	0			
5	1	0	0	0	0	0	1	1	1
6	1		1			1	0	a	a
7	1		1			1	a	0	a
8	1		1			1	a	a	0
	0	1	2	3	4	5	6	7	8

Example

We get

	0	1	2	3	4	5	6	7	8	
0	0	1	1	1	1	1	1	1	1	0
1	1	0	0			0				1
2	1	0	0	1	1	0	1	1	1	2
3	1		1	0		0	b	c	d	3
4	1		1		0	0	b	c	d	4
5	1	0	0	0	0	0	1	1	1	5
6	1		1	b	b	1	0	a	a	6
7	1		1	c	c	1	a	0	a	7
8	1		1	d	d	1	a	a	0	8
	0	1	2	3	4	5	6	7	8	

Example

We then obtain

	0	1	2	3	4	5	6	7	8	
0	0	1	1	1	1	1	1	1	1	0
1	1	0	0			0	1	1	1	1
2	1	0	0	1	1	0	1	1	1	2
3	1		1	0		0	b	c	d	3
4	1		1		0	0	b	c	d	4
5	1	0	0	0	0	0	1	1	1	5
6	1	1	1	b	b	1	0	a	a	6
7	1	1	1	c	c	1	a	0	a	7
8	1	1	1	d	d	1	a	a	0	8
	0	1	2	3	4	5	6	7	8	

Example

We thus have

	0	1	2	3	4	5	6	7	8	
0	0	1	1	1	1	1	1	1	1	0
1	1	0	0	0	0	0	1	1	1	1
2	1	0	0	1	1	0	1	1	1	2
3	1	0	1	0		0	b	c	d	3
4	1	0	1		0	0	b	c	d	4
5	1	0	0	0	0	0	1	1	1	5
6	1	1	1	b	b	1	0	a	a	6
7	1	1	1	c	c	1	a	0	a	7
8	1	1	1	d	d	1	a	a	0	8
	0	1	2	3	4	5	6	7	8	

Example

Choose 3

	0	1	2	3	4	5	6	7	8	
0	0	1	1	1	1	1	1	1	1	0
1	1	0	0	0	0	0	1	1	1	1
2	1	0	0	1	1	0	1	1	1	2
3	1	0	1	0	1	0	0	0	0	3
4	1	0	1	1	0	0	b=0	c=0	d=0	4
5	1	0	0	0	0	0	1	1	1	5
6	1	1	1	0	b=0	1	0	a	a	6
7	1	1	1	0	c=0	1	a	0	a	7
8	1	1	1	0	d=0	1	a	a	0	8
	0	1	2	3	4	5	6	7	8	

Example

We get

	0	1	2	3	4	5	6	7	8	
0	0	1	1	1	1	1	1	1	1	0
1	1	0	0	0	0	0	1	1	1	1
2	1	0	0	1	1	0	1	1	1	2
3	1	0	1	0	1	0	0	0	0	3
4	1	0	1	1	0	0	0	0	0	4
5	1	0	0	0	0	0	1	1	1	5
6	1	1	1	0	0	1	0	a	a	6
7	1	1	1	0	0	1	a	0	a	7
8	1	1	1	0	0	1	a	a	0	8
	0	1	2	3	4	5	6	7	8	