Combinatorial Calculation of Homology

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Basic Concepts and Definitions

As well known, any module M is a quotient $M = F_0/R_0$ of some free module F_0. The submodule R_0 is again a quotient $R_0 = F_1/R_1$ of a suitable free module F_1. Continuation of this process yields an exact sequence $0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots$ which will be called a free resolution of M.

(Co)chain Complex

Let Λ be an associative algebra with unit over some commutative ring R. A chain complex is a sequence

$$\mathscr{C}_\bullet : \cdots \to \mathscr{C}_{n+1} \xrightarrow{d_{n+1}} \mathscr{C}_n \xrightarrow{d_n} \mathscr{C}_{n-1} \xrightarrow{d_{n-1}} \mathscr{C}_{n-2} \to \cdots$$

of left(right) Λ-modules connected by Λ-homomorphisms such that $d_n \circ d_{n+1} = 0$ for all n. Λ-homomorphisms d_n are called boundary operators or differentials.

Basic Concepts and Definitions

A cochain complex

is a sequences

$$\mathscr{C}^\bullet : \cdots \leftarrow \mathscr{C}^{n+1} \xleftarrow{d^n} \mathscr{C}^n \xleftarrow{d^{n-1}} \mathscr{C}_{n-1} \leftarrow \cdots$$

of left(right) Λ-modules connected by Λ-homomorphisms such that $d^n \circ d^{n-1} = 0$ for all n.
A chain complex can be considered as a cochain complex by reversing the enumeration: $\mathscr{C}^n = \mathscr{C}_{-n}$, $d^n = d_{-n}$. This is why we will usually consider only chain complexes.
Setting $\mathscr{C}_\bullet := \bigoplus_{i\in\mathbb{Z}} \mathscr{C}_i$ we the get the homogenous homomorphism of degree 1, $d_\bullet : \mathscr{C}_\bullet \to \mathscr{C}_\bullet$.

Homology and Cohomology

Let $\mathscr{C}$ be a chain complex

$$\mathscr{C}_\bullet : \cdots \to \mathscr{C}_{n+1} \xrightarrow{d_{n+1}} \mathscr{C}_n \xrightarrow{d_n} \mathscr{C}_{n-1} \xrightarrow{d_{n-1}} \mathscr{C}_{n-2} \to \cdots$$

of Λ-modules. Since $d_n \circ d_{n+1} = 0$, we have $\mathrm{Im}(d_{n+1}) \subseteq \mathrm{Ker}(d_n)$.

A homology

of a chain complex is the Λ-module

$$H_n(\mathscr{C}_\bullet) := \mathrm{Ker}(d_n)/\mathrm{Im}(d_{n+1}).$$

A cohomology of a cochain complex is the Λ-module

$$H^n(\mathscr{C}^\bullet) := \mathrm{Ker}(d^n)/\mathrm{Im}(d_{n-1}).$$

The standard terminology is as follows: elements of $\mathscr{C}_n$ are called n-dimensional chains, elements of $\mathscr{C}^n$ are called n-dimensional cochains, elements of $\mathrm{Ker}(d_n)$ are called n-dimensional cycles, elements of $\mathrm{Ker}(d^n)$ are called n-dimensional cocycles, elements of $\mathrm{Im}(d_n)$ are called n-dimensional boundary, and elements of $\mathrm{Im}(d^{n-1})$ are called n-dimensional coboundary.

A complex is said to be acyclic if $H^n(\mathscr{C}) = 0$ for all n. It is easy to see that $H_n(\mathscr{C}) = 0$ means that the sequence $\mathscr{C}$ is exact at $\mathscr{C}_n$.

Resolution

Definition

A (left) resolution of M is a complex $(\mathrm{C}_\bullet, \mathrm{d}_\bullet)$ with $\mathrm{C}_\mathrm{i} = 0$ for $\mathrm{i} < 0$, together with a map $\varepsilon : \mathrm{C}_0 \to \mathrm{M}$ so that the (augmented) complex

$$0 \leftarrow \mathrm{M} \xleftarrow{\varepsilon} \mathrm{C}_0 \xleftarrow{\mathrm{d}_1} \mathrm{C}_1 \xleftarrow{\mathrm{d}_2} \mathrm{C}_2 \leftarrow \dots$$

is exact, i.e., $\mathrm{H}_\mathrm{n}(\mathrm{C}_\bullet) = 0$ for $\mathrm{n} > 0$ and $\varepsilon : \mathrm{H}_0(\mathrm{C}_\bullet) \cong \mathrm{M}$.

Examples

1. If $K = \mathbb{Z}$ (or any principal ideal domain) then submodules of a free module are free, hence any K-module M admits a free resolution
$$0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow 0.$$
In particular, the $\mathbb{Z}$-module $\mathbb{Z}_n$, $n > 1$ admits the resolution
$$0 \leftarrow \mathbb{Z}_n \xleftarrow{\text{mod }(n)} \mathbb{Z} \xleftarrow{\times n} \mathbb{Z} \leftarrow 0.$$
2. Let $K = \mathbb{Z}[x]/(x^2 - 1)$ and let χ be the image of x in K. Let $M = \mathbb{Z}$, regarded as a $\mathbb{Z}[x]/(x^2 - 1)$-module, with χ acting as the identity, i.e., M is the $\mathbb{Z}[x]/(x^2 - 1)$-module $K/(\chi - 1)$. Since $x^2 - 1 = (x - 1)(x + 1)$, it is clear that an element of K is annihilated by $\chi - 1$ (resp. $\chi + 1$) if and only if it is divisible by $\chi + 1$ (resp. $\chi - 1$). One therefore has a free resolution

The Bar Resolution for Algebras

The identity element 1_Λ gives a K-module map $\iota : K \to \Lambda$; its cokernel $\Lambda/\iota(K) = \Lambda/(K1_\Lambda)$ will be denoted as Λ/K, with elements the cosets $\lambda + K$. For each left Λ-module M construct the relatively free Λ-module

$$\mathscr{B}_n(\Lambda, M) := \Lambda \otimes_K \underbrace{(\Lambda/K) \otimes_K \cdots \otimes_K (\Lambda/K)}_{n} \otimes_K M.$$

As a K-module, it is spanned by elements which we write, with a vertical bar replacing "$\otimes_K$", as

$$\lambda[\lambda_1|\ldots|\lambda_n]m := \lambda \otimes_K [(\lambda_1 + K) \otimes_K \cdots \otimes_K (\lambda_n + K)] \otimes_K m;$$

in particular, $\mathscr{B}_0 = \{\lambda[\ ]m\}$. The left factor λ gives the left Λ-module structure of B_n, and $[\lambda_1|\ldots|\lambda]m$ without the operator will designate the corresponding element of $(\Lambda/K)^n \otimes_K M$. These elements are normalized, in the sense that $[\lambda_1|\ldots|\lambda_n]c = 0$

Further, define left Λ-module homomorphisms $\varepsilon : \mathscr{B}_0 \to \mathrm{M}$ and $\mathrm{d_n} : \mathscr{B}_\mathrm{n} \to \mathscr{B}_{\mathrm{n}-1}$ for $\mathrm{n} > 0$ by $\varepsilon(\lambda[\ ]\mathrm{m}) := \lambda\mathrm{m}$, and

$$\begin{aligned}
\mathrm{d_n}(\lambda[\lambda_1|\ldots|\lambda_\mathrm{n}]\mathrm{m}) \quad := \quad & \lambda\lambda_1[\lambda_2|\ldots|\lambda_\mathrm{n}]\mathrm{m} \\
& + \sum_{\mathrm{i}=1}^{\mathrm{n}-1}(-1)^\mathrm{i}\lambda[\lambda_1|\ldots|\lambda_\mathrm{i}\lambda_{\mathrm{i}+1}|\ldots|\lambda_\mathrm{n}]\mathrm{m} \\
& + (-1)^\mathrm{n}\lambda[\lambda_1|\ldots|\lambda_{\mathrm{n}-1}](\lambda_\mathrm{n}\mathrm{m}).
\end{aligned}$$

It is easy to verify that from associative low in Λ it follows that $\mathrm{d}_{\mathrm{i}+1} \circ \mathrm{d_i} = 0$, for every $\mathrm{i} \geq 0$. We thus get the complex $(\mathscr{B}_\bullet(\Lambda, \mathrm{M}), \mathrm{d}_\bullet)$,

$$0 \leftarrow \mathrm{M} \xleftarrow{\varepsilon} \mathscr{B}_0 \xleftarrow{\mathrm{d}_1} \mathscr{B}_1 \xleftarrow{\mathrm{d}_2} \mathscr{B}_2 \leftarrow \cdots .$$

Hochschild cohomology

nth Hochschild cohomology module of an K-algebra Λ with coefficients in a Λ-bimodule M is the K-module

$$\mathrm{H}^n(\Lambda, M) := \mathrm{H}^n(\mathrm{Hom}_K(\widetilde{\mathscr{B}}_n(\Lambda,\Lambda), M)), \qquad n = 0, 1, \dots;$$

where the elements of the complex are K-linear functions $f : \underbrace{\Lambda \times \cdots \times \Lambda}_{n} \to M$ such that $f(\lambda_1, \dots, \lambda_n) = 0$ whenever one λ_i belongs to K. The coboundary $\delta^n f$ is the function given as

$$\begin{aligned}\delta^n f(\lambda_1, \dots, \lambda_{n+1}) &= \lambda_1 f(\lambda_2, \dots, \lambda_{n+1}) \\ &\quad + \sum_{i=1}^{n} (-1)^n f(\lambda_1, \dots, \lambda_i\lambda_{i+1}, \dots, \lambda_{n+1}) \\ &\quad + (-1)^{n+1} f(\lambda_1, \dots, \lambda_n)\lambda_{n+1}.\end{aligned}$$

Invariants and Derivations

In particular, a zero-cochain is a constant $m \in M$; its coboundary is the function

$$\mathrm{ad}_m : \Lambda \to M$$

which is given by $\mathrm{ad}_m(\lambda) := m\lambda - \lambda m$, for every $\lambda \in \Lambda$. Hence

$$H^0(\Lambda, M) := \{m \in M : \lambda m = m\lambda, \text{for all } \lambda \in \Lambda\},$$

it is also called K-module of invariants.
Similarly, 1-cocycle is a K-module homomorphism $D : \Lambda \to M$ satisfying the identity

$$D(\lambda\lambda') = D(\lambda)\lambda' + \lambda D(\lambda'), \qquad \lambda, \lambda' \in \Lambda;$$

(=the Leibniz product rule) such a function D is called a derivation of Λ.

Let $M = \Lambda$.

As well known, every element $\lambda \in \Lambda$ determines the inner derivation ad_λ, $\mathrm{ad}_\lambda(a) = \lambda a - a\lambda$, $a \in \Lambda$. Denote by $\mathrm{ad}(\Lambda)$ the K-module of inner derivations of Λ.
Thus the K-module $HH^1(\Lambda) := \mathrm{Der}(\Lambda)/\,\mathrm{ad}(\Lambda)$ is the module of outer derivations of Λ.

Cohomology of Groups

Let G be a group. Consider $\mathbb{Z}[G]$ and let M be a $\mathbb{Z}[G]$-module. Next, let $\mathscr{P}_*$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$.

Cohomology of G with coefficients in M

is defined as follows

$$H^*(G, M) := H^*(\mathrm{Hom}_{\mathbb{Z}[G]}(\mathscr{P}_*, M))$$

For any group G we can always take $\mathscr{P}_\bullet$ to be the bar-resolution $\mathscr{B}_\bullet$. Namely, take $\mathscr{B}_n$ to be the free G-module with generators $[g_1|\dots|g_n]$ all n-tuples of elements $g_1 \neq 1_G, \dots, g_n \neq 1_G$ of G.

Operation on a generator with an $g \in G$ yields an element $g[g_1|\dots|g_n]$ in $\mathscr{B}_n$, so $\mathscr{B}_n$ may be described as the free abelian group generated by all $g[g_1|\dots|g_n]$. To give a meaning to every symbol $[g_1|\dots|g_n]$, set

$$[g_1|\dots|g_n] = 0$$

if any one $g_i = 1_G$.

In particular, $\mathscr{B}_0$ is the free module on one generator, denoted by $[\ ]$, so is isomorphic to $\mathbb{Z}[G]$, while $\varepsilon([\ ]) = 1$ is a G-module homomorphism $\varepsilon : \mathscr{B}_0 \to \mathbb{Z}$, with $\mathbb{Z}$ the trivial G-module.

Next, define G-module homomorphisms $d_n : \mathscr{B}_n \to \mathscr{B}_{n-1}$ for $n > 0$ by

$$\begin{aligned} d_n[g_1|\dots|g_n] &= g_1[g_2|\dots|g_n] \\ &+ \sum_{i=1}^{n-1}(-1)^i[g_1|\dots|g_ig_{i+1}|\dots|g_n] \\ &+ (-1)^n[g_1|\dots|g_{n-1}]; \end{aligned}$$

in particular

$$\begin{aligned} d[g] &= g[\ ] - [\ ], \\ d[a|b] &= a[b] - [ab] + [a]. \end{aligned}$$

Using $\mathrm{Hom}_{\mathbb{Z}[G]}(-, M)$, where M is a G-module, we get the following complex $(\mathscr{G}_\bullet, \partial_n)$,

$$\mathscr{G}_n := \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G]^{\otimes(n+1)}, M) \cong \mathrm{Hom}_{\mathbb{Z}}(G^{\times n}, M),$$

with

$$\begin{aligned} (\partial^n f)(g_1, \dots, g_{n+1}) &= g_1 f(g_2, \dots, g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \\ &+ (-1)^{n+1} f(g_1, \dots, g_n). \end{aligned}$$

Construction all groups G such that $G/\mathbb{Z} \cong \mathbb{Z}_n$

Let us construct all groups G such that $G/\mathbb{Z} \cong \mathbb{Z}_n$, where $n \in \mathbb{N}$, $n > 1$. We thus have

$$G = g_0\mathbb{Z} + g_1\mathbb{Z} + \cdots + g_{n-1}\mathbb{Z},$$

where $g_0 = 1_G$ and the homomorphism $\varphi : G \to \mathbb{Z}_n$ is such that $g_i \mapsto i$ for every $0 \le i \le n-1$. We thus can put that every element of G has a form (ζ, k), where $\zeta \in \mathbb{Z}$ and $k \in \mathbb{Z}_{n-1}$. Further, by $(g_i\mathbb{Z}) \cdot (g_j\mathbb{Z}) = g_ig_j\mathbb{Z}$, we get

$$(0, i) \cdot (0, k) = (f(i, k), i + k), \tag{1}$$

where $f(i, k) \in \mathbb{Z}$ and $i + k$ means the addition in $\mathbb{Z}_n$.

Construction all groups G such that $G/\mathbb{Z} \cong \mathbb{Z}_n$

In other words, we get a map $f : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}$ which determines the group G as follows

$$G = \langle \{(\zeta, i\}_{\zeta \in \mathbb{Z}, i \in \mathbb{Z}_n} | (\zeta, i) \cdot (\xi, k) := (\zeta + \xi + f(i,k), i + k) \rangle,$$

in particular, if $f \equiv 0$ we then get $G \cong \mathbb{Z} \oplus \mathbb{Z}_n$. We thus see that G looks like the $\mathbb{Z} \oplus \mathbb{Z}_n$, "perturbed" by f. Next, note that the convention of taking 1_G as the representative of $\mathbb{Z}$ in G yields, from (1),

$$f(i, 0) = f(0, j) = 0, \tag{2}$$

for all $i, j \in \mathbb{Z}_n$.

However, in general, an arbitrary map $f : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}$ does not define G. Indeed, if we define a product in G as above then by $[(\zeta, i) \cdot (\xi, j)] \cdot (\vartheta, k) = (\zeta, i) \cdot [(\xi, j) \cdot (\vartheta, k))]$ in G, we get

$$\begin{aligned}[(\zeta, i) \cdot (\xi, j)] \cdot (\vartheta, k) &= (\zeta + \xi + f(i, j), i + j) \cdot (\vartheta, k) \\ &= (\zeta + \xi + \vartheta + f(i, j) + f(i + j, k), i + j + k),\end{aligned}$$

and

$$\begin{aligned}(\zeta, i) \cdot [(\xi, j) \cdot (\vartheta, k)] &= (\zeta, i) \cdot (\xi + \vartheta + f(j, k), j + k) \\ &= (\zeta + \xi + \vartheta + f(i, j + k) + f(j, k), i + j + k).\end{aligned}$$

It follows that the product in G is associative if and only if f satisfies the identity

$$f(i, j) + f(i + j, k) = f(i, j + k) + f(j, k) \tag{3}$$

for all $i, j, k \in \mathbb{Z}_n$.

Construction all groups G such that $G/\mathbb{Z} \cong \mathbb{Z}_n$

Thus the set of elements (ζ, i), $\zeta \in \mathbb{Z}$, $i \in \mathbb{Z}_n$, with the product rule

$$(\zeta, i) \cdot (\xi, k) := (\zeta + \xi + f(i, k), i + k),$$

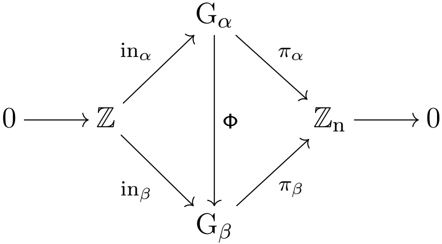
where $f : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}$ satisfies (2) and (3), defines a group G with normal subgroup $\mathbb{Z}$ and $G/\mathbb{Z} \cong \mathbb{Z}_n$.
Next, set

$$f(i, j) = \begin{cases} 0 & \text{if } i + j \leq n - 1, \\ \alpha \in \mathbb{Z} & \text{if } i + j \geq n. \end{cases}$$

With these definition we easily verify that (2) and (3) are satisfied, and so all groups G such that $G/\mathbb{Z} \cong \mathbb{Z}_n$ are defined.

Construction all groups G such that $G/\mathbb{Z} \cong \mathbb{Z}_n$

Thus for every $\alpha \in \mathbb{Z}$ we've obtained the extension G_α of $\mathbb{Z}$ by $\mathbb{Z}_n$. Let $\alpha, \beta \in \mathbb{Z}$ and consider the corresponding extensions G_α, G_β. We want to know whether they are equivalent, i.e., whether there is an homomorphism $\Phi : G_\alpha \to G_\beta$.
We thus get the following commutative diagram



Set $\Phi(\zeta, i) = (\Phi(\zeta), \Phi(i))$. By $\Phi \circ \mathrm{in}_\alpha = \mathrm{in}_\beta$, $\Phi(\zeta, 0) = (\zeta, 0)$, and by $\pi_\beta \circ \Phi = \pi_\alpha$, $\Phi(i) = i$.

Construction all groups G such that $G/\mathbb{Z} \cong \mathbb{Z}_n$

Thus, we can put $\Phi(\zeta, i) := (\zeta + \varphi(i), i)$, where $\varphi : \mathbb{Z}_n \to \mathbb{Z}$ is a function with $\varphi(0) = 0$. Since Φ is a homomorphism then one can easy obtain that α and β must satisfy the following identity

$$\alpha - \beta = \varphi(i + j) - \varphi(i) - \varphi(j). \tag{4}$$

From these identities it follows that there exist only n non equivalent extensions of $\mathbb{Z}$ by $\mathbb{Z}_n$. Indeed, let us consider the case $n = 3$. We then get

$$f(0,0) = f(1,0) = f(0,1) = f(1,1) = f(2,0) = f(0,2) = 0,$$

and

$$f(1,2) = f(2,1) = f(2,2) = \alpha \in \mathbb{Z},$$

Construction all groups G such that $G/\mathbb{Z} \cong \mathbb{Z}_n$

Thus the multiplication tableau in G_α has the following form

	$(\zeta, 0)$	$(\xi, 1)$	$(\vartheta, 2)$
$(\zeta', 0)$	$(\zeta' + \zeta, 0)$	$(\zeta' + \xi, 1)$	$(\zeta' + \vartheta, 2)$
$(\xi', 1)$	$(\xi' + \zeta, 1)$	$(\xi' + \xi, 2)$	$(\xi' + \vartheta + \alpha, 0)$
$(\vartheta', 2)$	$(\vartheta' + \zeta, 2)$	$(\vartheta' + \xi + \alpha, 0)$	$(\vartheta' + \vartheta + \alpha, 1)$

Further, by (4), any two extensions G_α, G_β are equivalent if and only if there exist a function $\varphi : \mathbb{Z}_3 \to \mathbb{Z}$, $\varphi(0) = 0$, such that

$$\alpha - \beta = \varphi(i + j) - \varphi(i) - \varphi(j),$$

for every $i, j \in \mathbb{Z}_3$.

Construction all groups G such that $G/\mathbb{Z} \cong \mathbb{Z}_n$

We get

$$\begin{cases} \alpha - \beta = 2\varphi(2) - \varphi(1), \\ \alpha - \beta = \varphi(1) + \varphi(2), \end{cases}$$

hence $\varphi(2) = 2\varphi(1)$, therefore $\alpha - \beta = 3\varphi(1)$. This implies that G_α and G_β are equivalent if and only if $\alpha \equiv \beta \mod (3)$. Thus we get only 3 non equivalent extensions of $\mathbb{Z}$ by $\mathbb{Z}_3$.

We've got the following complex $(\mathscr{G}_\bullet, \partial_n)$,

$$\mathscr{G}_n := \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G]^{\otimes(n+1)}, M) \cong \mathrm{Hom}_{\mathbb{Z}}(G^{\times n}, M),$$

with

$$\begin{aligned}(\partial^n f)(g_1, \dots, g_{n+1}) &= g_1 f(g_2, \dots, g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \\ &+ (-1)^{n+1} f(g_1, \dots, g_n).\end{aligned}$$

In particular

$$(\partial^2 f)(g, h, w) = g f(h, w) - f(gh, w) + f(g, hw) - f(g, h).$$

And now this identity can be rewritten in a form which should look familiar:

$$\mu_g(f(h, w)) - f(gh, w) + f(g, hw) - f(g, h) = 0.$$

Thus f can be regarded as a 2-cochain of the complex $\mathscr{G}$ for computing $H^*(G, A)$ and the identity says precisely that f is a cocycle and we therefore can say that

$$\begin{pmatrix} \text{extensions with} \\ \text{a normalised section} \end{pmatrix} \rightleftarrows \begin{pmatrix} \text{normalized 2-cocycles of} \\ \text{G with coefficients in A} \end{pmatrix}$$

Finally, one can prove the following

Theorem

Let A be a G-module and let $\mathscr{E}(G, A)$ be the set of equivalence classes of extensions of G by A giving rise to the given action of G on A. Then there is a bijection

$$\mathscr{E}(G, A) \longleftrightarrow H^2(G, A).$$

Composition–Diamond Lemma

Here we present the concepts of Composition–Diamond lemma and Gröbner–Shirshov basis. In the classical version of Composition–Diamond lemma, it assumed that considered algebras is over a field, here we consider the general case.

Let K be an arbitrary commutative ring with unit, $\mathrm{K}\langle \mathrm{X}\rangle$ the free associative algebra over K generated by X, and let X^* be the free monoid generated by X, where empty word is the identity, denoted by 1_{X^*}. Assume that X^* is a well-ordered set. Take $\mathrm{f} \in \mathrm{K}\langle \mathrm{X}\rangle$ with the leading word (term) $\bar{\mathrm{f}}$ and $\mathrm{f} = \kappa\bar{\mathrm{f}} + \mathrm{r_f}$, where $0 \neq \kappa \in \mathrm{K}$ and $\overline{\mathrm{r_f}} < \bar{\mathrm{f}}$. We call f is monic if $\kappa = 1$. We denote by deg(f) the degree of $\bar{\mathrm{f}}$.

Zero Devisors in Group Rings

The problem is

let G be a free torsion group, does $\mathbb{Z}[G]$ contain zero devisors?

Given a group G is presented as follows

$$G = \langle x_1, \dots, x_n \,|\, r_1, \dots, r_m \rangle.$$

We then get the following exact sequence of $\mathbb{Z}[G]$-modules

$$0 \to \pi_2(K) \xrightarrow{\mathfrak{p}} \bigoplus_{i=1}^{m} \mathbb{Z}[G] \xrightarrow{d_1} \bigoplus_{i=1}^{n} \mathbb{Z}[G] \xrightarrow{d_0} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where K is the standard 2-complex associated with G. The homomorphisms d_1 is given by

$$(\alpha_1, \dots, \alpha_n)^T \mapsto \sum_{i=1}^{n} \alpha_i (x_i - 1),$$

the homomorphism d_2 is given by the matrix (="Jacobian") $\left(\overline{\dfrac{\partial r_j}{\partial x_i}}\right)_{\substack{1 \le i \le n, \\ 1 \le j \le m}}$

Further, take a $\beta \in \pi_2(K)$, we then get $\mathfrak{p}(\beta) = (\beta_1, \dots, \beta_m)^T$, and hence

$$\sum_{j=1}^{m} \beta_j \overline{\frac{\partial r_j}{\partial x_i}} = 0,$$

for every $1 \le i \le n$.

Assume now that for a fixed i, $\overline{\dfrac{\partial r_j}{\partial x_i}} = S_j f$, $1 \le j \le m$, then by the previous equality we then have

$$\sum_{j=1}^{m} (\beta_j S_j) f = 0,$$

it follows that $\mathbb{Z}[G]$ has zero divisors.

Example

$G = \mathrm{sgr}\langle x, y, x^{-1}, y^{-1} \mid r_{11} = r_{12}, r_{21} = r_{22} \rangle$

$$r_1 = \{r_{11} = r_{12}\} = \{y^p x^n y^m x y^t = y^k x^n y^m x y^l\},$$
$$r_2 = \{r_{21} = r_{22}\} = \{y^1 x^n y^m x y^s = y^r x^n y^m x y^h\},$$

and the Fox derivatives $\dfrac{\partial r_1}{\partial x}, \dfrac{\partial r_2}{\partial x}$ have common divisor $f = x^n y^m + x^{n-1} + \cdots + x + 1.$

Composition–Diamond Lemma

A well ordering $\leqslant$ on X^* is called monomial if for $u, v \in X^*$, we have:

$$u \leqslant v \Longrightarrow w\big|_u \leqslant w\big|_v, \qquad \forall w \in X^*,$$

where $w\big|_u := w\big|_{x \to u}$ and x's are the same individuality of the letter $x \in X$ in w.
A standard example of monomial ordering on X^* is the deg-lex ordering (i.e., degree and lexicographical), in which two words are compared first by the degree and then lexicographically, where X is a well-ordering set.

Composition–Diamond Lemma

Fix a monomial ordering $\leqslant$ on X^*, and let φ and ψ be two monic polynomials in $\mathrm{K}\langle\mathrm{X}\rangle$. There are two kinds of compositions:

(i) If w is a word (i.e, it lies in X^*) such that $\mathrm{w} = \overline{\varphi}\mathrm{b} = \mathrm{a}\overline{\psi}$ for some $\mathrm{a}, \mathrm{b} \in \mathrm{X}^*$ with $\deg(\overline{\varphi}) + \deg(\overline{\psi}) > \deg(\mathrm{w})$, then the polynomial $(\varphi, \psi)_{\mathrm{w}} := \varphi\mathrm{b} - \mathrm{a}\psi$ is called the intersection composition of φ and ψ with respect to w.

(ii) If $\mathrm{w} = \overline{\varphi} = \mathrm{a}\overline{\psi}\mathrm{b}$ for some $\mathrm{a}, \mathrm{b} \in \mathrm{X}^*$, then the polynomial $(\varphi, \psi)_{\mathrm{w}} := \varphi - \mathrm{a}\psi\mathrm{b}$ is called the inclusion composition of φ and ψ with respect to w.

We then note that $\overline{(\varphi, \psi)}_{\mathrm{w}} \leq \mathrm{w}$ and $(\varphi, \psi)_{\mathrm{w}}$ lies in the ideal (φ, ψ) of $\mathrm{K}\langle\mathrm{X}\rangle$ generated by φ and ψ.

Composition–Diamond Lemma

Let $S \subseteq K\langle X\rangle$ be a monic set (i.e., it is a set of monic polynomials). Take $f \in K\langle X\rangle$ and $w \in X^*$. We call f is trivial modulo (S, w), denoted by

$$f \equiv 0 \bmod (S, w),$$

if $f = \sum_{s\in S} \kappa a s b$, where $\kappa \in K$, $a, b \in X^*$, and $a\bar{s}b \leqslant w$.
A monic set $S \subseteq K\langle X\rangle$ is called a Gröbner–Shirshov basis in $K\langle X\rangle$ with respect to the monomial ordering $\leq$ if every composition of polynomials in S is trivial modulo S and the corresponding w.

Theorem (Composition Diamond Lemma)

Let K be an arbitrary commutative ring with unit, $\leqslant$ a monomial ordering on X^* and let $I(S)$ be the ideal of $K\langle X\rangle$ generated by the monic set $S \subseteq K\langle X\rangle$. Then the following statements are equivalent:

(1) S is a Gröbner–Shirshov basis in $K\langle X\rangle$.

(2) if $f \in I(S)$ then $\bar{f} = a\bar{s}b$ for some $s \in S$ and $a, b \in X^*$.

(3) the set of irreducible words

$$\mathrm{Irr}(S) := \{u \in X^* : u \neq a\bar{s}b,\ s \in S,\ a, b \in X^*\}$$

is a linear basis of the algebra $K\langle X|S\rangle := K\langle X\rangle / I(S)$.

Example

Let K be an arbitrary commutative ring and consider the following algebra $\Lambda = K\langle x, y\rangle/(x^2 - y^2)$. Let us consider the polynomials $\varphi = x^2 - y^2$, $\psi = xy^2 - y^2x$, and let $y \leqslant x$. It is not hard to see that the set $S = \{\varphi, \psi\}$ is a Gröbner–Shirshov basis of Λ. Indeed,

$$\begin{aligned}(\varphi, \varphi)_w &= \varphi x - x\varphi \\ &= x^3 - y^2x - (x^3 - xy^2) = \psi,\end{aligned}$$

for $w = x^3$, and

$$\begin{aligned}(\varphi, \psi)_w &= \varphi y^2 - x\psi \\ &= x^2y^2 - y^2y^2 - (x^2y^2 - xy^2x) \\ &= \psi x + y^2\varphi,\end{aligned}$$

for $w = x^2y^2$.

Example

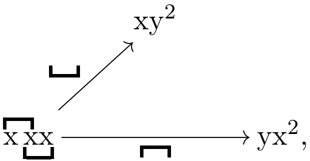
Since the set S is monic, then the set

$$\mathrm{Irr}(\mathrm{S}) = \bigcup_{n>0} \left\{ 1, x, xy, y^n, y^n x, (xy)^n, (yx)^n, (yxy)^n \right\}$$

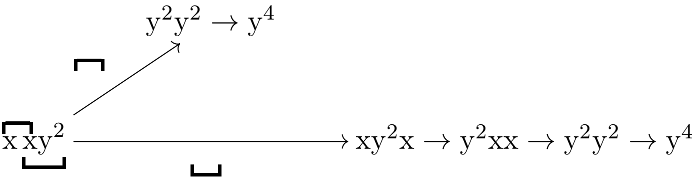
is the K-basis for Λ, by Composition–Diamond lemma.

In practice, to calculate a Gröbner–Shirshov basis, it is better to use the following way;

(1) for the given $\varphi = \mathrm{x}^3 - \mathrm{y}^3$, $\mathrm{x}^3 = \overline{\varphi}$, we have



(2) we thus have to add a polynomial $\psi = \mathrm{xy}^2 - \mathrm{yx}^2$. We have $\overline{\psi} = \mathrm{xy}^2$ and thus



The Anick Resolution

Let F be a field and let Λ be an associative F-algebra with unit. Let X be a set of generators for Λ. Suppose that $\leq$ is a well-ordering on the free monoid generated by X.

Definition

Let $\mathrm{GSB}_\Lambda = \{f_i\}$ be a Gröbner–Shirshov basis for Λ and let $\overline{f}_1, \ldots, \overline{f}_\ell$ be leading terms of GSB_Λ, such that for $j = 1, \ldots, \ell - 1$ we have $\overline{f}_j = a_j b_j$ with $a_{j+1} = b_j$, $|a_j|, |b_j| \geq 1$. Then we call the word $\overline{f}_1 \cdots \overline{f}_\ell$ the ℓth Anick's chain.

We say that elements from X are 0-chains, further, we say that leading terms are 1-chains. Denote by $\Lambda^{(\ell)}$ the the set of all ℓth Anick's chains.

The Anick Resolution

Example

Let $\Lambda = \mathrm{F}\langle \mathrm{x}, \mathrm{y} | \mathrm{x}^2 = \mathrm{y}^2 \rangle$. Consider the polynomials $\varphi = \mathrm{x}^2 - \mathrm{y}^2$, $\psi = \mathrm{xy}^2 - \mathrm{y}^2\mathrm{x}$, and let $\mathrm{y} \leqslant \mathrm{x}$. We have seen that the set $\{\mathrm{x}^2 - \mathrm{y}^2, \mathrm{xy}^2 - \mathrm{y}^2\mathrm{x}\}$ is a Gröbner–Shirshov basis of Λ. We thus get

$$\Lambda^{(0)} = \{\mathrm{x}, \mathrm{y}\},$$
$$\Lambda^{(1)} = \left\{\mathrm{x}^2, \mathrm{xy}^2\right\},$$
$$\Lambda^{(2)} = \left\{\mathrm{xxx}, \mathrm{xxy}^2\right\},$$
$$\Lambda^{(3)} = \left\{\mathrm{xxxx}, \mathrm{xxxy}^2\right\}, \text{etc.}$$

$\square$

Theorem (Anick's Resolution)

Let Λ be an associative augmented F-algebra, generated as a F-algebra by the set X and let $\leq$ be an well-ordering on the free monoid generated by X. Let GSB_Λ be a Gröbner–Shirshov basis for Λ let $\Lambda^{(1)}$ be a set of leading terms (1-chains) of GSB_Λ and let $\Lambda^{(n)}$ be a set of n-chains. Then there is a free Λ-resolution of F,

$$0 \leftarrow F \xleftarrow{\varepsilon} \Lambda \xleftarrow{d_0} F\Lambda^{(0)} \otimes_F \Lambda \xleftarrow{d_1} F\Lambda^{(1)} \otimes_F \Lambda \xleftarrow{d_2} F\Lambda^{(2)} \otimes_F \Lambda \xleftarrow{d_3} \dots$$

in which $d_0(x \otimes 1) = x - \varepsilon(x)$, for $x \in \Lambda^{(0)}$ and for $n \geq 1$, $d_n(a_n \otimes 1) = a_{n-1} \otimes b + \omega$, where $a_n = a_{n-1}b$, $a_n \in \Lambda^{(n)}$, $a_{n-1} \in \Lambda^{(n-1)}$, $HT(\omega) < a_n$ if $\omega \neq 0$.

□

Algebraic Discrete Morse Theory

The discrete Morse theory developed by Forman provides a way to reduce the number of cells in a CW-complex without changing the homotopy type.

There are a few different ways to express discrete Morse theory, the way that works best for the algebraic setting is in terms of acyclic matchings in the Hasse diagram of the face poset of the complex. Let $\Gamma = (V, E)$ be a directed graph. A subset $M \subseteq E$ is a matching if every vertex is in at most one of the edges in M. A matching is acyclic if the graph obtained by reversing the edges in the matching contain no directed cycles. An important property of Hasse diagrams of a posets is that they contain no directed cycles. Given an acyclic matching M of Γ the elements of V that are not matched are critical.

Algebraic Discrete Morse Theory

The main theorem of discrete Morse theory can be stated as follows.

Theorem

Let X be a regular CW-complex with face poset P. If $\mathfrak{M}$ is an acyclic matching of P where the empty face is critical, then there is a CW-complex $\widetilde{X}$ homotopy equivalent to X. The critical cells are in bijection with the cells of $\widetilde{X}$, this bijection preserve dimension.

For one-dimensional complexes the theory is greatly simplified and it is always possible to find optimal matchings in the sense that the resulting complex have the minimal number of cells of any complex homotopy equivalent to the original complex.

Algebraic Discrete Morse Theory

One-dimensional complexes are essentially graphs where loops and multiple edges are allowed, the complexes obtained from discrete Morse theory are the complexes obtained by contracting non-loop edges. The matchings are pairings of a vertex with an edge containing the vertex, and the matched edge is then contracted and the new vertex is identified with the endpoint of the contracted edge not paired to the contracted edge. In particular it is possible to contract edges in a graph until there is only a single vertex in each component and there is a matching realizing this. The space of acyclic matchings for the Hasse diagram of posets of one-dimensional complexes has interesting structure and was further studied by Chari and Joswig. Batzies and Welker extended discrete Morse theory to work well with cellular resolutions.

Algebraic Discrete Morse Theory

Let X be a CW-complex with labeling map ℓ and face poset P. An acyclic matching M of the Hasse diagram of P satisfying $\sigma\tau \in \mathfrak{M} \implies \ell(\sigma) = \ell(\tau)$ is homogenous, that is the matching is homogenous if cells are only matched to cells with the same label.

The main theorem of algebraic discrete Morse theory for cellular resolutions can be stated as follows.

Theorem

Let X be a regular CW-complex with face poset P. Let ℓ be a labeling of X giving a cellular resolution of the ideal I. If $\mathfrak{M}$ is a homogenous acyclic matching of P then $\widetilde{\mathrm{X}}$ also supports a cellular resolution of I. The cell corresponding to the critical cell σ has label $\ell(\sigma)$.

Algebraic Discrete Morse Theory

Let R be a ring and $\mathscr{C}_\bullet = (\mathscr{C}_i, \partial_i)_{i\geq 0}$ be a chain complex of free R-modules $\mathscr{C}_i$. We choose a basis $X = \cup_{i\geq 0} X_i$ such that $\mathscr{C}_i \cong \bigoplus_{c\in X_i} Rc$. Write the differentials ∂_i with respect to the basis X in the following form:

$$\partial_i : \begin{cases} \mathscr{C}_i \to \mathscr{C}_{i-1} \\ c \mapsto \partial_i(c) = \sum\limits_{x' \in X_{i-1}} [c : c'] \cdot c'. \end{cases}$$

Given a complex $\mathscr{C}_\bullet$ and a basis X, we construct a directed weighted graph $\Gamma(\mathscr{C}) = (V, E)$. The set of vertices V of $\Gamma(\mathscr{C})$ is the basis $V = X$ and the set E of weighted edges is given by the rule

$$(c, c', [c : c']) \in E \quad \text{if and only if} \quad c \in X_i, c' \in X_{i-1}, \text{and } [c : c'] \neq 0.$$

Algebraic Discrete Morse Theory

Definition

A finite subset $\mathfrak{M} \subset E$ in the set of edges is called an acyclic matching if it satisfies the following three conditions:

(1) (Matching) Each vertex $v \in V$ lies in at most one edge $e \in \mathfrak{M}$.

(2) (Invertibility) For all edges $(c, c'[c : c']) \in \mathfrak{M}$ the weight $[c : c']$ lies in the center $Z(R)$ of the ring R and is a unit in R.

(3) (Acyclicity) The graph $\Gamma_{\mathfrak{M}}(V, E_{\mathfrak{M}})$ has no directed cycles, where $E_{\mathfrak{M}}$ is given by

$$E_{\mathfrak{M}} := (E \setminus \mathfrak{M}) \cup \{(c', c, [c : c']^{-1}) \text{ with } (c, c', [c : c']) \in \mathfrak{M}\}.$$

Algebraic Discrete Morse Theory

For an acyclic matching $\mathfrak{M}$ on the graph $\Gamma(\mathscr{C}_\bullet) = (V, E)$, we introduce the following notation

(1) We call a vertex $c \in V$ critical with respect to $\mathfrak{M}$ if c does not lie in an edge $e \in \mathfrak{M}$; we write

$$X_i^{\mathfrak{M}} := \{c \in X_i : c \text{ critical}\}$$

for the set of all critical vertices of homological degree i.

(2) The weight $\omega(p)$ of a path $p = c_1 \to \ldots \to c_r \in \mathrm{Path}(c_1, c_r)$ is given by

$$\omega(c_1 \to \ldots \to c_r) := \prod_{i=1}^{r-1} \omega(c_i \to c_{i+1}), \qquad \omega(c \to c') := \begin{cases} -\frac{1}{[c : c']}, \\ [c : c'], \quad c \end{cases}$$

(3) We write $\Gamma(c, c') := \sum_{p \in \mathrm{Path}(c,c')} \omega(p)$ for the sum of weights

Algebraic Discrete Morse Theory

Theorem

The chain complex $(\mathscr{C}_\bullet, \partial_\bullet)$ of free R-modules is homotopy-equivalent to the complex $(\mathscr{C}_\bullet^{\mathfrak{M}}, \partial_\bullet^{\mathfrak{M}})$ which is complex of free R-modules and

$$\mathscr{C}_i^{\mathfrak{M}} := \bigoplus_{c \in X_i^{\mathfrak{M}}} Rc, \qquad \partial_i^{\mathfrak{M}} : \begin{cases} \mathscr{C}_i^{\mathfrak{M}} \to \mathscr{C}_{i-1}^{\mathfrak{M}} \\ c \mapsto \sum_{c' \in X_{i-1}^{\mathfrak{M}}} \Gamma(c, c')c'. \end{cases}$$

Theorem (Jöllenbeck–Scöldberg–Welker)

For $\omega \in \mathrm{X}^*$, let $\Lambda_{\omega,\mathrm{i}}$ be the vertices $[\omega_1|\dots|\omega_\mathrm{n}]$ in $\Gamma_{\mathrm{B}_\bullet(\Lambda,\Bbbk)}$ such that $\omega = \omega_1 \cdots \omega_\mathrm{n}$ and i is the larger integer $\mathrm{i} \geq -1$ such that $\omega_1 \cdots \omega_{\mathrm{i}+1} \in \Lambda^{(\mathrm{i})}$ is an Anick i-chain. Let $\Lambda_\omega := \bigcup_{\mathrm{i} \geq -1} \Lambda_{\omega,\mathrm{i}}$.

Define a partial matching $\mathfrak{M}_\omega$ on $(\Gamma_{\mathscr{B}_\bullet(\Lambda,\mathrm{K})})_\omega = \Gamma_{\mathscr{B}_\bullet(\Lambda,\mathrm{K})}|_{\Lambda_\omega}$ by letting $\mathfrak{M}_\omega$ consist of all edges

$$[\omega_1|\dots|\omega'_{\mathrm{i}+2}|\omega''_{\mathrm{i}+2}|\dots|\omega_\mathrm{n}] \to [\omega_1|\dots|\omega_{\mathrm{i}+2}|\dots|\omega_\mathrm{m}]$$

when $[\omega_1|\dots|\omega_\mathrm{m}] \in \Lambda_{\omega,\mathrm{i}}$, such that $\omega'_{\mathrm{i}+2}\omega''_{\mathrm{i}+2} = \omega_{\mathrm{i}+2}$ and $[\omega_1|\dots|\omega_{\mathrm{i}+1}|\omega'_{\mathrm{i}+2}] \in \Lambda^{(\mathrm{i}+1)}$ is an Anick $(\mathrm{i}+1)$-chain.
The set of edges $\mathfrak{M} = \bigcup_\omega \mathfrak{M}_\omega$ is a Morse matching on $\Gamma_{\mathscr{B}_\bullet(\Lambda,\mathrm{K})}$, with critical cells $\mathrm{X}_\mathrm{n}^{\mathfrak{M}} = \Lambda^{(\mathrm{n}-1)}$ for all n.

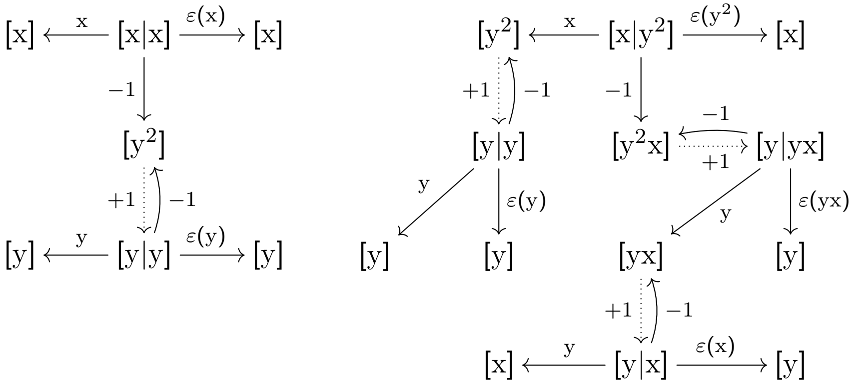
Example

Let us consider the algebra $\Lambda = \mathrm{K}\langle \mathrm{x}, \mathrm{y} | \mathrm{x}^2 - \mathrm{y}^2 = 0 \rangle$. We have

$$\Lambda^{(0)} = \{\mathrm{x}, \mathrm{y}\}, \Lambda^{(1)} = \left\{ \mathrm{x}^2, \mathrm{x}\mathrm{y}^2 \right\}, \quad \Lambda^{(2)} = \left\{ \underline{\mathrm{x}\overline{\mathrm{x}}}\overline{\mathrm{x}}, \overline{\mathrm{x}\,\underline{\mathrm{x}}}\underline{\mathrm{y}^2} \right\}, \mathrm{etc},$$

i.e., $\Lambda^{(\ell)} = \{\mathrm{x}^{\ell+1}, \mathrm{x}^{\ell}\mathrm{y}^2\}$, $\ell \geq 0$. We thus get the Anick resolution

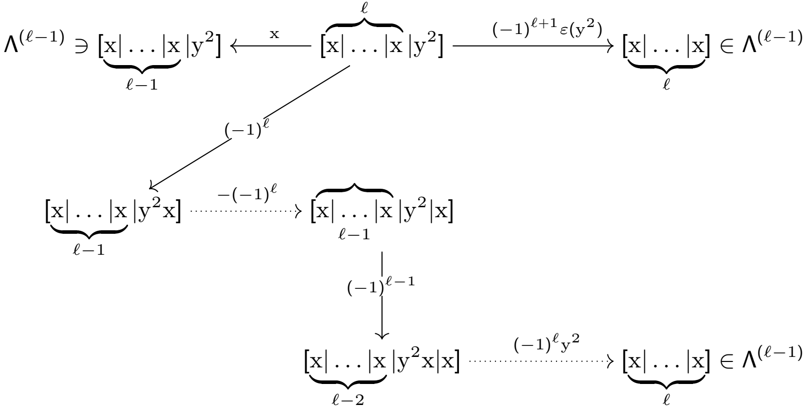
$$0 \leftarrow \mathrm{K} \xleftarrow{\varepsilon} \Lambda \xleftarrow{\mathrm{d}_0} \Lambda\mathrm{x} \oplus \Lambda\mathrm{y} \xleftarrow{\mathrm{d}_1} \Lambda\mathrm{x}^2 \oplus \Lambda\mathrm{x}\mathrm{y}^2 \xleftarrow{\mathrm{d}_2} \Lambda\mathrm{x}^3 \oplus \Lambda\mathrm{x}^2\mathrm{y}^2 \leftarrow \ldots.$$



hence

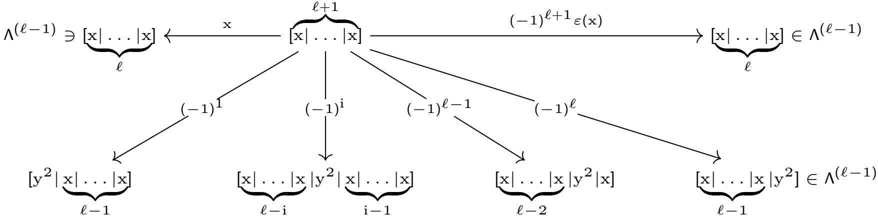
$$d_1[x|x] = x[x] + \varepsilon(x)[x] - y[y] - \varepsilon(y)[y],$$

$$d_1[x|y^2] = \varepsilon(y^2)[x] + xy[y] + \varepsilon(y)x[y] - \varepsilon(yx)[y] - y^2[x] - \varepsilon(x)y[y],$$



hence

$$d_\ell[\underbrace{x|\dots|x}_{\ell}|y^2] = x[\underbrace{x|\dots|x}_{\ell-1}|y^2] + (-1)^{\ell+1}\varepsilon(y^2)[\underbrace{x|\dots|x}_{\ell}] + (-1)^{\ell}y^2[\underbrace{x|\dots|x}_{\ell}],$$



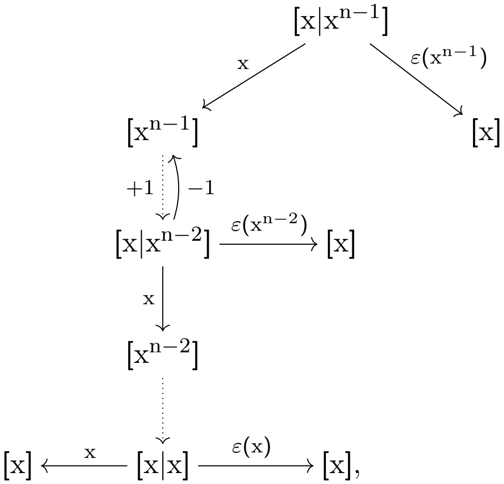
thus

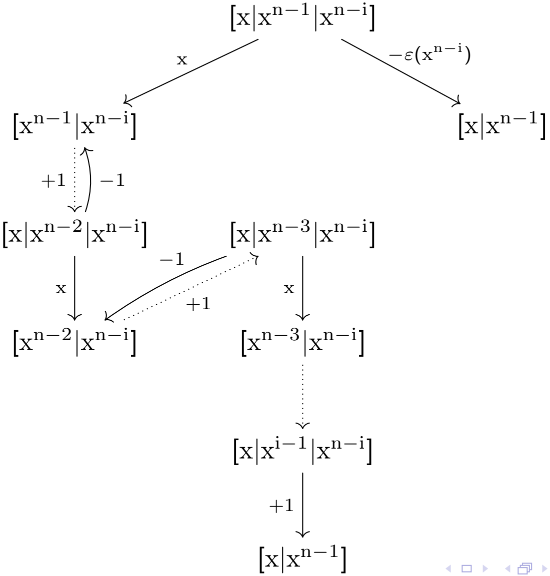
$$d_\ell[\underbrace{x|\dots|x}_{\ell+1}] = x[\underbrace{x|\dots|x}_{\ell}] + (-1)^{\ell+1}\varepsilon(x)[\underbrace{x|\dots|x}_{\ell}] + (-1)^{\ell}y^2[\underbrace{x|\dots|x}_{\ell-1}].$$

Let us consider the cyclic group $\mathbb{Z}_n = \langle \zeta | \zeta^n = 1 \rangle$ of order n. Then the powers $\zeta^0 = 1, \zeta^2, \dots, \zeta^{n-1}$ form a $\mathbb{Z}$-basis for its group ring $\Lambda = \mathbb{Z}[\mathbb{Z}_n]$. It follows that $\Lambda \cong \mathbb{Z}[x]/(x^n - 1)$. It is obviously that the set $\{x^n - 1\}$ is a Grönber–Shirshov basis of Λ. We thus get

$$\Lambda^{(0)} = \{[x]\},$$
$$\Lambda^{(1)} = \{[x|x^{n-1}]\},$$
$$\Lambda^{(2)} = \{[x|x^{n-1}|x], [x|x^{n-1}|x^2], \dots, [x|x^{n-1}|x^{n-1}]\}.$$

Calculate the differentials $d_1 : \mathbb{Z}\Lambda^{(1)} \to \mathbb{Z}\Lambda^{(0)}$, $d_2 : \mathbb{Z}\Lambda^{(2)} \to \mathbb{Z}\Lambda^{(1)}$.


[x|x^{n−1}]
x
ε(x^{n−1})
[x^{n−1}]
[x]
+1
−1
[x|x^{n−2}]
ε(x^{n−2})
[x]
x
[x^{n−2}]
[x]
x
[x|x]
ε(x)
[x],



[x|x^{n−1}|x^{n−i}]
x
−ε(x^{n−i})
[x^{n−1}|x^{n−i}]
[x|x^{n−1}]
+1
−1
[x|x^{n−2}|x^{n−i}]
[x|x^{n−3}|x^{n−i}]
−1
x
+1
x
[x^{n−2}|x^{n−i}]
[x^{n−3}|x^{n−i}]
[x|x^{i−1}|x^{n−i}]
+1
[x|x^{n−1}]

We thus obtain

$$\mathrm{d}_1[\mathrm{x}|\mathrm{x}^{\mathrm{n}-1}] = (\mathrm{x}^{\mathrm{n}-1} + \mathrm{x}^{\mathrm{n}-2}\varepsilon(\mathrm{x}) + \cdots + \varepsilon(\mathrm{x}^{\mathrm{n}-1}))[\mathrm{x}],$$
$$\mathrm{d}_2[\mathrm{x}|\mathrm{x}^{\mathrm{n}-1}|\mathrm{x}^{\mathrm{n}-\mathrm{i}}] = (\mathrm{x}^{\mathrm{n}-\mathrm{i}} - \varepsilon(\mathrm{x}^{\mathrm{n}-\mathrm{i}}))[\mathrm{x}|\mathrm{x}^{\mathrm{n}-1}], \qquad 1 \leq \mathrm{i} \leq \mathrm{n}-1.$$

$$0 \to \mathbb{Z} \xrightarrow{\delta^0} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Lambda^{(0)}, \mathbb{Z}) \xrightarrow{\delta^1} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Lambda^{(1)}, \mathbb{Z}) \xrightarrow{\delta^2} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Lambda^{(2)}, \mathbb{Z}) \to \cdot$$

Hence

$$(\delta^1\varphi)([x|x^{n-1}]) = (\underbrace{1+\cdots+1}_{n})\varphi([x]) = n\varphi([x]),$$

$$(\delta^2\psi)([x|x^{n-1}|x^{n-i}]) = (1-1)\psi([x|x^{n-1}]) = 0, \qquad 1 \le i \le n-1,$$

for every $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Lambda^{(0)}, \mathbb{Z})$ and $\psi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Lambda^{(1)}, \mathbb{Z})$. It follows that $H^2(\mathbb{Z}_n, \mathbb{Z}) \cong \mathbb{Z}_n$.

Let us calculate all extensions of $\mathbb{Z}$ and $\mathbb{Z}_2$ by the symmetric group $\mathfrak{S}_3$. It is well known that $\mathfrak{S}_3$ my be described as follows. It has generators s_1, s_2 and relations:

$$s_1^2 = s_2^2 = 1,$$
$$s_1 s_2 s_1 = s_2 s_1 s_2.$$

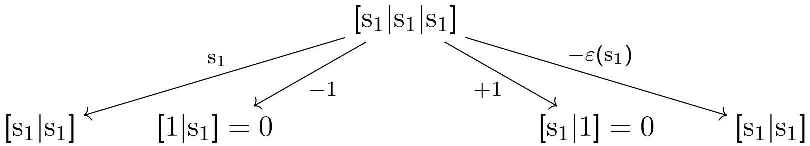
One thinks of s_i as swapping the ith and $(i+1)$th position. Next, set $s_1 > s_2$. One can easy see that the set

$$\{s_1^2 = 1, s_2^2 = 1, s_1 s_2 s_1 = s_2 s_1 s_2\}$$

is its Gröbner–Shirshov basis. We thus get

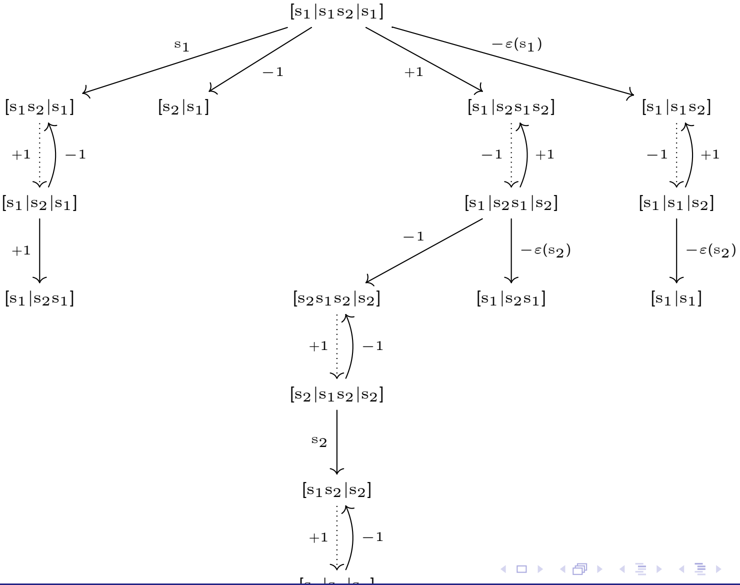
$$\mathfrak{S}_3^{(0)} = \{[s_1], [s_2]\},$$
$$\mathfrak{S}_3^{(1)} = \{[s_1|s_1], [s_2|s_2], [s_1|s_2 s_1]\},$$
$$\mathfrak{S}_3^{(2)} = \{[s_1|s_1|s_1], [s_1|s_1 s_2|s_1], [s_1|s_2 s_1|s_1], [s_2|s_2|s_2]\}.$$

To define the differentials $d_2 : \mathbb{Z}\mathfrak{S}_3^{(2)} \to \mathbb{Z}\mathfrak{S}_3^{(1)}$ and $d_1 : \mathbb{Z}\mathfrak{S}_3^{(1)} \to \mathbb{Z}\mathfrak{S}_3^{(0)}$, we have to consider the corresponding paths in the graph $\Gamma_{\mathfrak{A}}(\mathscr{B}_\bullet(\mathbb{Z}\mathfrak{S}_3))$;



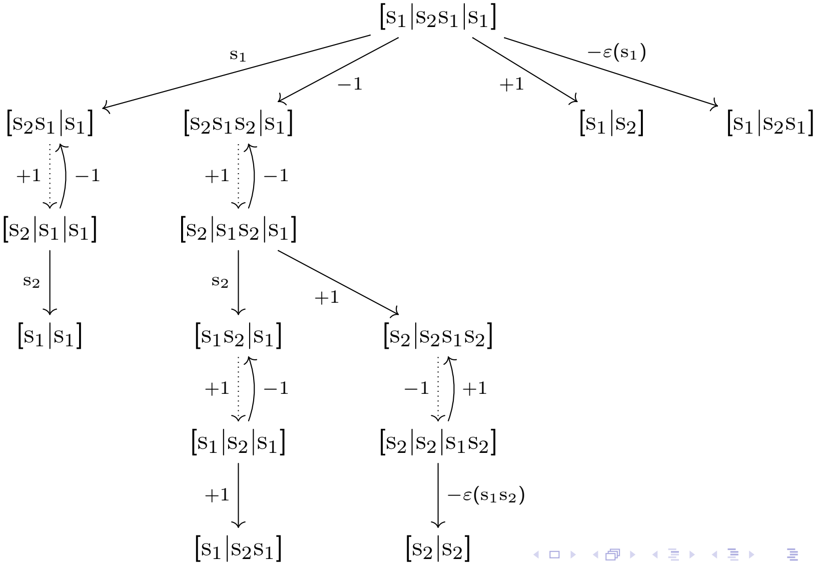
hence

$$d_2[s_1|s_1|s_1] = s_1[s_1|s_1] - \varepsilon(s_1)[s_1|s_1],$$


[s1|s1s2|s1]
s1
−1
+1
−ε(s1)
[s1s2|s1]
[s2|s1]
[s1|s2s1s2]
[s1|s1s2]
+1
−1
−1
+1
−1
+1
[s1|s2|s1]
[s1|s2s1|s2]
[s1|s1|s2]
+1
−1
−ε(s2)
−ε(s2)
[s1|s2s1]
[s2s1s2|s2]
[s1|s2s1]
[s1|s1]
+1
−1
[s2|s1s2|s2]
s2
[s1s2|s2]
+1
−1

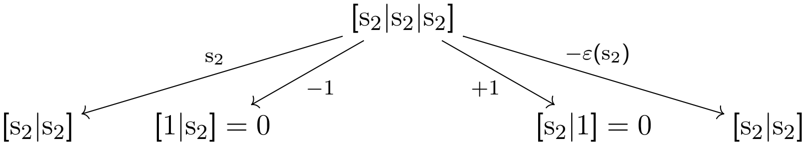
hence

$$d_2[s_1|s_1s_2|s_1] = (s_1 + \varepsilon(s_2))[s_1|s_2s_1] + s_2s_1[s_2|s_2] - \varepsilon(s_2)[s_1|s_1],$$


[s1|s2s1|s1]
s1
−1
+1
−ε(s1)
[s2s1|s1]
[s2s1s2|s1]
[s1|s2]
[s1|s2s1]
+1
−1
+1
−1
[s2|s1|s1]
[s2|s1s2|s1]
s2
s2
+1
[s1|s1]
[s1s2|s1]
[s2|s2s1s2]
+1
−1
−1
+1
[s1|s2|s1]
[s2|s2|s1s2]
+1
−ε(s1s2)
[s1|s2s1]
[s2|s2]

hence

$$d_2[s_1|s_2s_1|s_1] = s_1s_2[s_1|s_1] - (s_2 + \varepsilon(s_1))[s_1|s_2s_1] - \varepsilon(s_1s_2)[s_2|s_2],$$



hence

$$d_2[s_2|s_2|s_2] = s_2[s_2|s_2] - \varepsilon(s_2)[s_2|s_2],$$

Thus we obtain the following cochain complex

$$0 \to \mathbb{Z} \xrightarrow{d_0} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathfrak{S}_3^{(0)}, \mathbb{Z}) \xrightarrow{d_1} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathfrak{S}_3^{(1)}, \mathbb{Z}) \xrightarrow{d_2} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathfrak{S}_3^{(2)}, \mathbb{Z}) -$$

where

$$(d_2 f)([s_1|s_1|s_1]) = 0,$$
$$(d_2 f)([s_1|s_1 s_2|s_1]) = 2f([s_1|s_2 s_1]) + f([s_2|s_2]) - f([s_1|s_1]),$$
$$(d_2 f)[s_1|s_2 s_1|s_1] = f([s_1|s_1]) - 2f([s_1|s_2 s_1]) - f([s_2|s_2]),$$
$$(d_2 f)[s_2|s_2|s_2] = 0,$$
$$(d_1 \alpha)([s_1|s_1]) = 2\alpha([s_1]),$$
$$(d_1 \alpha)([s_2|s_2]) = 2\alpha([s_2]),$$
$$(d_1 \alpha)([s_1|s_2 s_1]) = \alpha([s_1]) - \alpha([s_2]),$$

$f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathfrak{S}_3^{(1)}, \mathbb{Z})$, $\alpha \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathfrak{S}_3^{(0)}, \mathbb{Z})$. Hence $f \in \mathrm{Ker}(d_2)$ if and only if f has the following values:

$$f([s_1|s_1]) = k, f([s_1|s_2 s_1]) = l, f([s_2|s_2]) = k - 2l,$$

On the other hand, by the description of d_1,

$$(d_1\alpha)([s_2|s_2]) = (d_1\alpha)([s_1|s_1]) - 2(d_1\alpha)([s_1|s_2s_1]).$$

Hence every $f \in \mathrm{Ker}(d_2)$ has a form $(d_1\alpha)$ for some $\alpha \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathfrak{S}_3^{(0)}, \mathbb{Z})$, i.e., $\mathrm{Ker}(d_2) = \mathrm{Im}(d_1)$. Thus

$$H^2(\mathfrak{S}_3, \mathbb{Z}) \cong \mathrm{Ker}(d_2)/\mathrm{Im}(d_1) = 0.$$

Next, it is easy to see that $\mathrm{Aut}(\mathbb{Z}) = \{\mathrm{id}_{\mathbb{Z}}, \iota\} \cong \mathbb{Z}_2$, where $\iota(\zeta) = -\zeta$ for every $\zeta \in \mathbb{Z}$. We then have two homomorphisms $\varphi_1, \varphi_2 : \mathfrak{S}_3 \to \mathbb{Z}_2$ are defined as follows: $\varphi_1(\mathfrak{p}) := 0$, and $\varphi_2(\mathfrak{p}) := \mathrm{sign}(\mathfrak{p})$ (= signature of the permutation $\mathfrak{p}$), for every $\mathfrak{p} \in \mathfrak{S}_3$. Thus we obtain

$$0 \to \mathbb{Z} \to \mathbb{Z} \rtimes_{\varphi_i} \mathfrak{S}_3 \to \mathfrak{S}_3 \to 1, \qquad i = 1, 2,$$

where $\mathbb{Z} \rtimes_{\varphi_1} \mathfrak{S}_3 = \mathbb{Z} \times \mathfrak{S}_3$ (=trivial action of $\mathfrak{S}_3$ over $\mathbb{Z}$), and $\mathbb{Z} \rtimes_{\varphi} \mathfrak{S}_3$ has the following group low

$$(\zeta, \mathfrak{p}) \cdot (\xi, \mathfrak{q}) := (\zeta + \mathrm{sign}(\mathfrak{p})\xi, \mathfrak{p}\mathfrak{q}).$$

Result

Thus these extensions are equivalent and we thus have only one class of extension of $\mathbb{Z}$ by $\mathfrak{S}_3$.

Further, let us find all extensions of $\mathbb{Z}_2$ by $\mathfrak{S}_3$. Using the calculations above, we get the following cochain complex

$$0 \to \mathbb{Z}_2 \xrightarrow{\mathrm{d}_0} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathfrak{S}_3^{(0)}, \mathbb{Z}_2) \xrightarrow{\mathrm{d}_1} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathfrak{S}_3^{(1)}, \mathbb{Z}_2) \xrightarrow{\mathrm{d}_2} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathfrak{S}_3^{(2)}, \mathbb{Z}_2) \to \cdot$$

where

$$(\mathrm{d}_2 \mathrm{f})([\mathrm{s}_1|\mathrm{s}_1|\mathrm{s}_1]) = 0,$$
$$(\mathrm{d}_2 \mathrm{f})([\mathrm{s}_1|\mathrm{s}_1\mathrm{s}_2|\mathrm{s}_1]) = \mathrm{f}([\mathrm{s}_2|\mathrm{s}_2]) + \mathrm{f}([\mathrm{s}_1|\mathrm{s}_1]),$$
$$(\mathrm{d}_2 \mathrm{f})[\mathrm{s}_1|\mathrm{s}_2\mathrm{s}_1|\mathrm{s}_1] = \mathrm{f}([\mathrm{s}_1|\mathrm{s}_1]) + \mathrm{f}([\mathrm{s}_2|\mathrm{s}_2]),$$
$$(\mathrm{d}_2 \mathrm{f})[\mathrm{s}_2|\mathrm{s}_2|\mathrm{s}_2] = 0,$$
$$(\mathrm{d}_1 \alpha)([\mathrm{s}_1|\mathrm{s}_1]) = 0,$$
$$(\mathrm{d}_1 \alpha)([\mathrm{s}_2|\mathrm{s}_2]) = 0,$$
$$(\mathrm{d}_1 \alpha)([\mathrm{s}_1|\mathrm{s}_2\mathrm{s}_1]) = \alpha([\mathrm{s}_1]) + \alpha([\mathrm{s}_2]),$$

Result

There exists only two non equivalent extensions of $\mathbb{Z}_2$ by $\mathfrak{S}_3$, namely;

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathfrak{S}_3 \to \mathfrak{S}_3 \to 1,$$
$$0 \to \mathbb{Z}_2 \to \mathrm{E}_\mathrm{f} \to \mathfrak{S}_3 \to 1,$$

where E_f is generated by the following set of pairs $\{(\mathrm{i}, \mathfrak{p}) : \mathrm{i} \in \mathbb{Z}_2, \mathfrak{p} \in \mathfrak{S}_3\}$, and the group low is given by

$$(\mathrm{i}, \mathfrak{p}) \cdot (\mathrm{j}, \mathfrak{q}) := (\mathrm{i} + \mathrm{j} + \mathrm{f}(\mathfrak{p}, \mathfrak{q}), \mathfrak{p}\mathfrak{q}),$$

here $\mathrm{f}(\mathrm{s}_1, \mathrm{s}_1) = \mathrm{f}(\mathrm{s}_2, \mathrm{s}_2)$ and $\mathrm{f}(\mathfrak{p}, \mathfrak{q}) = \alpha \in \mathbb{Z}_2$ otherwise.