# On eigenvalues and singular values of adjacency matrices of regular random graphs

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Ostrava, 2018

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# Regular graphs and adjacency matrices

$G \in \mathcal{D}_{n,d} \Leftrightarrow$ every vertex of $G$ has exactly $d$ in-neighbors and $d$ out-neighbors

$$\mathbb{P}\{G \in \Gamma\} = \frac{|\Gamma|}{|\mathcal{D}_{n,d}|}, \quad \Gamma \subset \mathcal{D}_{n,d}.$$

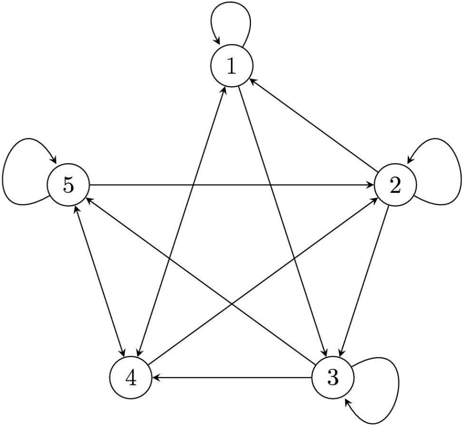
$M \in \mathcal{M}_{n,d} \Leftrightarrow$

$$M_{ij} = \begin{cases} 1, & \text{if there is an edge from } i \text{ to } j; \\ 0, & \text{otherwise.} \end{cases}$$

$$\sum_{i=1}^{n} M_{ij} = \sum_{j=1}^{n} M_{ij} = d$$

A closely related model: Erdös-Renyi graphs. Each edge of an Erdös-Renyi graph is formed with probability $p$ independently of others. In our case $p = d/n$.

$n = 5, d = 3$




$G \in \mathcal{D}_{n,d}$

| | | | | |
|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 | 1 |

$M \in \mathcal{M}_{n,d}$

# Questions we are interested in:

1. Invertibility of adjacency matrices of regular graphs.
2. Singular values.
   In particular, quantitative estimates for the smallest singular value.
3. Delocalization properties of the eigenvectors.
4. Limiting distributions of eigenvalues of $M \in \mathcal{M}_{n,d}$ as $n \to \infty$:
   - Circular Law if $d = d(n) \to \infty$,
   - Complex Kesten–McKay distribution if $d$ is fixed.

# Invertibility of adjacency matrices of regular graphs

Conjecture

**For every $3 \leq d \leq n-3$, the probability that the adjacency matrix corresponding to an undirected $d$-regular graph is singular goes to zero as $n \to \infty$.**

Theorem [ Nicholas A. Cook, 2014]

*For* $d \gg \ln^2 n$, $\quad \mathbb{P}\{M \in \mathcal{M}_{n,d} \quad \textit{is singular}\} \leq 1/d^c$.

Theorem [LLTTY,2015]

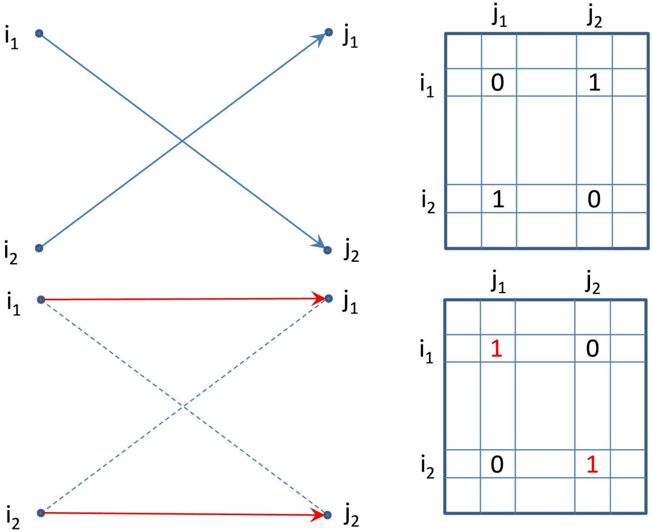
*For* $d \geq C$, $\quad \mathbb{P}\{M \in \mathcal{M}_{n,d} \quad \textit{is singular}\} \leq C \ln^3 d/\sqrt{d}$.

Theorem [Jiaoyang Huang, 2018]

*For* $d \geq 3$, $(d < \ln \ln n)$, $\quad \mathbb{P}\{M \in \mathcal{M}_{n,d} \quad \textit{is singular}\} = o(1), \quad n \to \infty$.

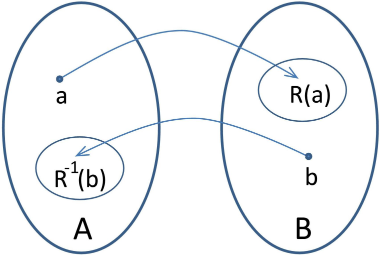
Symmetric case: András Mészáros, 2018, Hoi H. Nguyen, Melanie Matchett Wood 2018.

# Switching and multimaps



McKay'81 used the simple *switching* procedure to estimate cardinalities of subsets of graphs:

Let $A, B \subset \mathcal{D}_{n,d}$. To compare cardinalities $|A|$ and $|B|$, one uses switching to construct a multimap $R$ between $A$ and $B$, then estimate the cardinalities of images and preimages of this multimap and apply the following simple statement:




**Claim.** *Let $R : A \to B$ be a multimap,*

$$\forall a \in A \quad |R(a)| \geq s \geq 1,$$
$$\forall b \in B \quad |R^{-1}(b)| \leq t.$$

*Then*

$$\frac{|A|}{|B|} \leq \frac{t}{s}$$

# Expansion properties of $d$-regular graphs

With high probability:

- There are no large $I \times J$ zero minors in adjacency matrices ($|I|, |J| \geq cn/d$).
- Supports of any two rows (columns) almost do not intersect.
- For any set $J$ of vertices ($|J| \geq cn/d$), the union of the supports of its vertices is concentrated near its maximum $d|J|$.

# Komlós' strategy. Bernoulli matrices

## Theorem (Komlós,1977)

*Let $B$ be a random sign matrix with the iid uniform $\pm 1$ entries. Then*

$$\mathbb{P}\left\{B \text{ is singular }\right\} = O(n^{-1/2}).$$

Kahn-Komlós-Szemerédi'95, Tao-Vu'06, Bourgain-Vu-Wood'09.
**Conjecture:** $\big(1/2 + o(1)\big)^n$

Key ingredient: **anti-concentration** Littlewood-Offord type inequalities:
Let $\xi_1, \ldots, \xi_n$ be iid Bernoulli $\pm 1$ and let $|x_i| \geq 1$, $i \leq m$. Then

$$\sup_{a \in \mathbb{C}} \mathbb{P}\left( \Big| \sum_{i=1}^{m} \xi_i x_i - a \Big| < t \right) \leq \frac{Ct}{\sqrt{m}}, \quad \forall t \geq 1.$$

*In particular,* $\forall v \in \mathbb{R}^n$, $\mathbb{P}\left\{Row(B) \cdot v = 0\right\} = O\big(|supp\, v|^{-1/2}\big)$.

## The strategy of Komlós:

**Step 1. Eliminate sparse null vectors** $Sparse_\eta := \{x : |\operatorname{supp} x| < \eta n\}$.

$$G := \{B : Sparse_\eta \cap (\mathrm{Ker} B \cup \mathrm{Ker} B^T) = \{0\}\} \Rightarrow \mathbb{P}\{G\} > 1 - e^{-cn}$$

**Step 2. Treating non-sparse null vectors.** Let $\mathcal{E}_{bad} := \{B : \det B = 0\} \cap G$.

$$V_i := \operatorname{span}\{R_j\}_{j \neq i}, \quad \text{fix} \quad v^{(i)} \perp V_i \quad i \leq n.$$

$$\text{Let } B \in \mathcal{E}_{bad}. \text{ Then } \exists x : \sum_{i \in \operatorname{supp} x} x_i R_i = 0, \quad |\operatorname{supp} x| \geq \eta n.$$

$$\forall i \in \operatorname{supp} x \quad R_i \in V_i \quad \Rightarrow \quad \eta n \mathbb{P}\{\mathcal{E}_{bad}\} \leq \sum_{i=1}^{n} \mathbb{P}\{R_i \in V_i\} = n\mathbb{P}\{R_1 \in V_1\}.$$

$$\Rightarrow \mathbb{P}\{\mathcal{E}_{bad}\} \leq \eta^{-1}\mathbb{P}\{R_1 \in V_1 \mid R_2, \cdots, R_n\}$$
$$\leq \eta^{-1}\mathbb{P}\{R_1 \cdot v^{(1)} = 0 \mid R_2, \cdots, R_n\} = O(n^{-1/2}).$$

# Quantitative estimates

## Theorem (Cook, 2017)

*Let $d > C \ln^{11} n$. Then the smallest singular number of $M$ satisfies*

$$\mathbb{P}\left(s_n > 1/n^{C(\ln n)/\ln d}\right) > 1 - C \ln^{5.5} n/\sqrt{d}.$$

## Theorem (LLTTP 2017)

*Let $C < d < n/\ln^2 n$. Then*

$$\mathbb{P}\left(s_n > 1/n^6\right) > 1 - C \ln^2 d/\sqrt{d}.$$

Conjecture: $s_n \approx \sqrt{d}/n$.

# Circular law

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of a random matrix $M_n$.

The *empirical spectral distribution* (ESD) of $M_n$:

$$\mu_{M_n}(A) := \frac{1}{n}|\{i : \lambda_i \in A\}|, \quad \forall A \in \mathcal{B}(\mathbb{C}).$$

It was conjectured in 1950s, that if entries of $M_n$ are iid satisfying some mild conditions then $\mu_{M_n}$ converges to the uniform probability measure on the unit disk $D$ of the complex plane, that is,

$$\mu_{M_n/\sqrt{n}} \to \mu_\circ = \pi^{-1}\mathbf{1}_D\, dxdy, \qquad \text{where} \quad D = \{|z| \leq 1\}.$$

# Circular law: results

Let $M_{ij}$ be iid copies of a centered r.v. $\xi$ with variance 1.

Mehta **(1967)**: $\xi$ is a standard complex Gaussian variable (using the joint density function of the eigenvalues, discovered by Ginibre **(1965)**)

Girko **(1984)**: $\mathbf{E}\,|\xi|^{2+\varepsilon} < \infty$ (but the proof has gaps)

Edelman **(1997)**: $\xi$ is a standard real Gaussian variable

Bai **(1997)**: $\xi$ has bounded density and bounded 6th moment (later improved to $(2+\varepsilon)$-moment in his book with Silverstein **(2010)**)

Girko **(2004)**: $\mathbf{E}\,|\xi|^{4+\varepsilon} < \infty$ (no density conditions!)

Pan, Zhou **(2010)**: $\mathbf{E}\,|\xi|^{4} < \infty$

Tao, Vu **(2008)**: $\mathbf{E}\,|\xi|^{2+\varepsilon} < \infty$

Götze, Tikhomirov **(2010)**: $\mathbf{E}\,|\xi|^{2}(\ln|\xi|)^{20} < \infty$

Tao, Vu **(2010)**: Universality: No additional conditions!

Many recent works on matrices with non iid entries. In particular, for sparse matrices: Götze–Tikhomirov, Tao–Vu, Basak–Rudelson.

# Circular law in our setting

In our setting $M \in \mathcal{M}_{n,d}$ is uniformly distributed in the set of $n \times n$ matrices with $0/1$ entries, such that sums in rows and in columns are equal to $d$.

## Theorem (Cook, 2017)

*The circular law holds for $d^{-1/2}M$ provided that $d > \ln^{96} n$.*

## Theorem (LLTTY, 2018)

*The circular law holds for $d^{-1/2}M$ provided that $d = d(n) \to \infty$ as $n \to \infty$.*

Consider an $n \times n$ **real symmetric random matrix**

$$M_n = (M_{jk})_{j,k=1}^{n} = \begin{pmatrix} M_{11} & \dots & M_{1n} \\ \vdots & & \vdots \\ M_{n1} & \dots & M_{nn} \end{pmatrix},$$

$M_{jk}$ are random variables. Denote the eigenvalues of $M_n$ by

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n.$$

**The Normalized Counting Measure (NCM)** of eigenvalues: $\forall \Delta \subset \mathbb{R}$

$$N_n(\Delta) = \frac{|\{k : \lambda_k \in \Delta\}|}{n} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(\lambda_k \in \Delta).$$

For many classes of properly normalized random matrices, their eigenvalues possess a self-averaging property: their NCMs of eigenvalues converge to a non-random limit as $n \to \infty$.

## Wigner real symmetric matrices

$$M_n = n^{-1/2} W_n$$

- $W_n = \{W_{jk}\}_{j,k=1}^{n}$, $W_{jk} = W_{kj} \in \mathbb{R}$,
- $W_{jk}$, $1 \leq j \leq k \leq n$, are independent,
- $\mathbf{E} W_{jk} = 0$,
- $\mathbf{E} W_{jk}^2 = 1$.

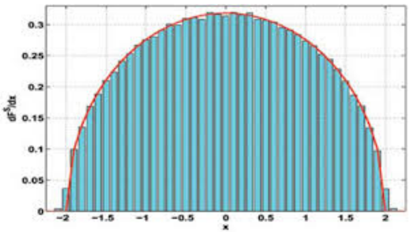
**Eugene Paul Wigner**

## Wigner's Semicircle Law

For any bounded continuous function $\varphi$, with probability 1,

$$\lim_{n\to\infty}\int_{\mathbb{R}}\varphi(\lambda)dN_n(\lambda)=\int_{-2}^{2}\varphi(\lambda)\rho_{sc}(\lambda)d\lambda,$$

$$\rho_{sc}(\lambda)=\frac{1}{2\pi}\sqrt{(4-\lambda^2)_+}.$$



## Sample Covariance Matrices

Consider $m$ independent random vectors in $\mathbb{R}^n$ with zero mean

$$\mathbf{X}_1 = \begin{pmatrix} X_{11} \\ \vdots \\ X_{n1} \end{pmatrix}, \quad \ldots, \mathbf{X}_m = \begin{pmatrix} X_{1m} \\ \vdots \\ X_{nm} \end{pmatrix}$$

Put

$$M_n = n^{-1} \sum_{\alpha=1}^{m} \mathbf{X}_\alpha \mathbf{X}_\alpha^T = n^{-1} B_n B_n^T,$$

where

$$B_n = \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \ldots & \mathbf{X}_m \end{bmatrix}.$$

We suppose that $m \to \infty, \; m/n \to c \in (0, \infty) \quad \text{as} \quad n \to \infty.$

# Marchenko-Pastur distribution

Let $M_n = n^{-1} B_n B_n^T, \quad B_n = (X_{j\alpha})_{j,\alpha=1}^{n,m},$

$\{X_{j\alpha}\}_{j,\alpha}$ are independent,

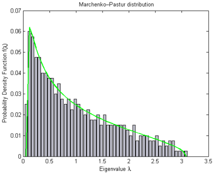
$\mathbf{E} X_{j\alpha} = 0, \quad \mathbf{E} X_{j\alpha}^2 = a^2,$

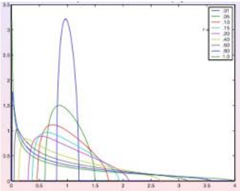
$m, n \to \infty, \ m/n \to c \geq 1.$

Then $N_n(d\lambda) \to \rho_{MP}(\lambda) d\lambda$ a.s.,

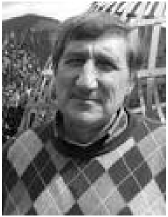
$$\rho_{MP}(\lambda) = \frac{\sqrt{((\lambda - a_-)(a_+ - \lambda))_+}}{2\pi a^2 \lambda},$$

$$a_\pm = a^2(\sqrt{c} \pm 1)^2.$$



**Vladimir Marchenko**



**Leonid Pastur**

## Stieltjes transform of a non-negative finite measure $m$:

$$s(z) = \int_{\mathbb{R}} \frac{m(d\lambda)}{\lambda - z}, \quad \Im z \neq 0$$

- the *Stieltjes - Perron inversion formula*:

$$m(\Delta) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{\Delta} \Im s(\lambda + i\varepsilon) d\lambda;$$

- There is a one-to-one correspondence between finite non-negative measures and their Stieltjes transforms. This correspondence is continuous if we use the uniform convergence of analytic functions on compact subsets of $\mathbb{C} \setminus \mathbb{R}$ for Stieltjes transforms and the weak convergence of measures.

- $$s_n(z) := \int_{\mathbb{R}} \frac{N_n(d\lambda)}{\lambda - z} = \frac{1}{n} \mathrm{Tr}(M_n - z)^{-1}, \quad \Im z \neq 0.$$

## Main steps of the proof.

$$M_n = \sum_{\beta=1}^{m} \mathbf{X}_\beta \mathbf{X}_\alpha^T, \quad G = (M_n - z)^{-1}, \quad s_n = \frac{1}{n} \operatorname{Tr} G, \quad M_n^\alpha = M_n - \mathbf{X}_\alpha \mathbf{X}_\alpha^T.$$

Let $\mathbb{E}(A\mathbf{X}_\alpha, \mathbf{X}_\alpha) = \operatorname{Tr} A + O(n^{-1})$ and $\mathbf{Var}(A\mathbf{X}_\alpha, \mathbf{X}_\alpha) = o(1)$ for every $A$. Since

$$zG = -1 + GM \quad \text{and} \quad G - G^\alpha = -\frac{G^\alpha \mathbf{X}_\alpha \mathbf{X}_\alpha^T G^\alpha}{1 + (G^\alpha \mathbf{X}_\alpha, \mathbf{X}_\alpha)},$$

we have

$$\begin{aligned} z\mathbb{E}s_n &= -1 + \frac{m}{n} - \frac{1}{n} \sum_{\alpha=1}^{m} \mathbb{E} \frac{1}{1 + (G^\alpha \mathbf{y}_\alpha, \mathbf{y}_\alpha)} \\ &= -1 + \frac{m}{n} - \frac{1}{n} \sum_{\alpha=1}^{m} \frac{1}{1 + \mathbb{E}s_n} + o(1). \end{aligned}$$

Hence, $zs(z) = -1 + c - c(1 + s(z))^{-1}$.

# Logarithmic potential

Let

$$\mu_{M_n}(A) := \frac{1}{n}|\{i : \lambda_i \in A\}|, \quad (\lambda_i)_i \text{ are eigenvalues of } M_n.$$

$M_n = M_n^*$. The Stieltjes transform of $\mu_{M_n}$:

$$g_n(z) := \int_{\mathbb{R}} \frac{d\mu_{M_n}(\lambda)}{\lambda - z} = n^{-1}\,\mathrm{Tr}(M_n - z)^{-1}.$$

$M_n \neq M_n^*$. The logarithmic potential of $\mu_{M_n}$:

$$U_{\mu_{M_n}}(z) = -\int_0^\infty \ln|\lambda - z| d\mu_{M_n}(\lambda).$$

# Hermitization

Let $s_1 \geq \ldots \geq s_n$ be the singular values of an $n \times n$ matrix $B$.
The *singular values distribution* of $B$:

$$\nu_B := \frac{1}{n}|\{s_i \in A\}|, \quad A \in \mathcal{B}(\mathbb{R}).$$

Hermitization:

$$U_{\mu_{M_n}}(z) = -\int_0^\infty \ln|\lambda - z| d\mu_{M_n}(\lambda)$$

$$= -\frac{1}{n}\sum_j \ln|\lambda_j - z| = -\frac{1}{n}\ln\Big|\prod_j \lambda_j - z\Big|$$

$$= -\frac{1}{n}\ln\Big|\det(M_n - z)\Big| = -\frac{1}{n}\ln\sqrt{\Big|\det(M_n - z)(M_n^* - \bar{z})\Big|}$$

$$= -\frac{1}{n}\sum_j \ln(s_j(M_n - z)) = -\int_0^\infty \ln(t)\, d\nu_{M_n - zI}(t).$$

# Open questions:

1. Invertibility of adjacency matrices of regular graphs:
   - directed case,
   - undirected case (symmetric matrices), $d = d(n) \to \infty$.

2. Singular values.
   Quantitative estimates for the smallest singular value.
   To get optimal bound.

3. Delocalization properties of the eigenvectors.

4. Limiting distributions of eigenvalues of $M \in \mathcal{M}_{n,d}$:
   - $n \to \infty$, $d = d(n) \to \infty$ (circular law),
   - $n \to \infty$, $d$ is fixed (complex Kesten–McKay distribution). Conjecture: as $n \to \infty$ the normalized counting measures of eigenvalues of $M \in \mathcal{M}_{n,d}$ converge to the probability measure

$$\frac{1}{\pi} \frac{d^2(d-1)}{(d^2 - |z|^2)^2} \chi_{\{|z|<\sqrt{d}\}} dxdy.$$

Thank you!