On the asymptotic eigenvalue distribution of Toeplitz matrices and generalizations

Frantisek Štampach

&

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Contents

Basic definitions and facts

- (Semi-infinite) Toeplitz matrix:

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- Finite Toeplitz matrix: $T_n(a)$ stands for the upper-left $n \times n$ section of $T(a)$.
- Recall that if $\sum |a_n| < \infty$, then $T(a)$ determines a well-defined bounded operator on $\ell^2(\mathbb{N})$ and one has [Toeplitz, Wiener]

$$\operatorname{spec} T(a) = a(\mathbb{T}) \cup \{z \in \mathbb{C} \setminus a(\mathbb{T}) \mid \text{wind}(a - z) \neq 0\}.$$

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$$\operatorname{spec} T(a) = a(\mathbb{T}) \cup \{z \in \mathbb{C} \setminus a(\mathbb{T}) \mid \text{wind}(a - z) \neq 0\}.$$

- Note that a is real-valued on $\mathbb{T}$, if and only if $T(a) = T(a)^*$.

The limiting set and measure

- The limiting set:

$$\Lambda(a) = \{\lambda \in \mathbb{C} \mid \liminf_{n\to\infty} \operatorname{dist}(\lambda, \operatorname{spec}(T_n(a)) = 0\},$$

equivalently

$$\lambda \in \Lambda(a) \quad \Leftrightarrow \quad \exists n_k \quad \exists \lambda_k \in \operatorname{spec}\left(T_{n_k}(a)\right) \quad \text{s.t.} \quad \lim_{k\to\infty} \lambda_k = \lambda.$$

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- The eigenvalue-counting measure:

$$\mu_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{\lambda_k^{(n)}},$$

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where $\lambda_1^{(n)}, \dots, \lambda_n^{(n)}$ are eigenvalues of $T_n(a)$.
- If the weak limit, say μ, of μ_n for $n \to \infty$ exists, i.e.,
$$\lim_{n\to\infty} \int_{\mathbb{C}} f(z)\mathrm{d}\mu_n(z) \equiv \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\lambda_k^{(n)}\right) = \int_{\mathbb{C}} f(z)\mathrm{d}\mu(z), \quad \forall f \in C_0(\mathbb{C}),$$
then μ is called a.e.d./limiting measure/density of states.

Three sets

- Naturally, there are 3 sets to compare:

$$\operatorname{spec} T(a) \quad \text{vs.} \quad \Lambda(a) \quad \text{vs.} \quad \operatorname{supp} \mu,$$

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- At this point it is essential to distinguish:

$$\begin{matrix} \text{self-adjoint case} & & \text{non-self-adjoint case} \\ & \text{vs.} & \\ a_n = \overline{a_{-n}} & & a_n \neq \overline{a_{-n}} \end{matrix}$$

The self-adjoint case

- Here we assume $\sum |a_n| < \infty$ and $a_n = \overline{a_{-n}}$ for all $n \in \mathbb{Z}$.

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- The Weierstrass approximation theorem implies

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i.e., $\mu_n \xrightarrow{w} \mu$, where

$$\mu((\alpha,\beta]) = \frac{1}{2\pi} \left|\{t \in (-\pi,\pi] \mid \alpha < a(e^{\mathrm{i}t}) \le \beta\}\right|.$$

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- Clearly, $\operatorname{supp}\mu = [\min_{|z|=1} a(z), \max_{|z|=1} a(z)]$, and hence

$$\boxed{\operatorname{supp}\mu = \operatorname{spec} T(a).}$$

The self-adjoint case

- A consequence of Szegő's result:

$$\lim_{n\to\infty}\frac{N_n(\alpha,\beta)}{n}=\frac{1}{2\pi}\left|\{t\in(-\pi,\pi]\mid\alpha<a(e^{\mathrm{i}t})<\beta\}\right|$$

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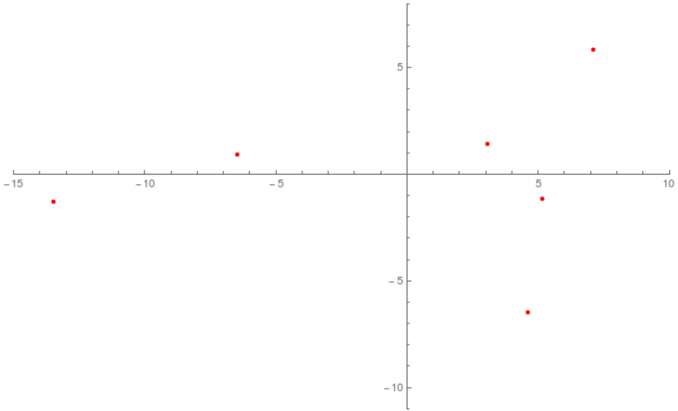
Moreover, μ is determined by

$$\mu((\alpha,\beta])=\frac{1}{2\pi}\left|\{t\in(-\pi,\pi]\mid \alpha<a(e^{it})\leq\beta\}\right|.$$

The non-self-adjoint case

Q: What happen if the assumption of self-adjointness of $T(a)$ is relaxed?



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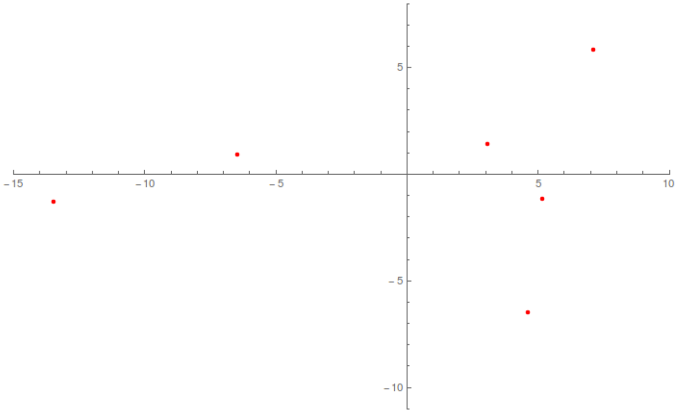
Q: What happen if the assumption of self-adjointness of $T(a)$ is relaxed?

... a numerical experiment for

$$a(z) = 2z^{-2} + 4iz^{-1} + 1 - 2iz + 5z^2 + 7iz^3 - z^4 + 19z^5 + (i+2)z^6 + 28z^7$$



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- To get some results, we restrict ourself to banded Toeplitz matrices. So the symbol is the Laurent polynomial:

$$b(z) = \sum_{j=-r}^{s} a_j z^j, \quad r, s \geq 1, \quad a_{-r} \neq 0, \; a_s \neq 0,$$

(we also exclude lower/upper triangular matrices).

The non-self-adjoint case - the result of Schmidt & Spitzer

- Denote $z_1(\lambda), \dots, z_{r+s}(\lambda)$ the roots of the polynomial $z \mapsto z^r\,(b(z) - \lambda)$ labeled such that

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- An elegant description of $\Lambda(b)$ for banded Toeplitz matrices is due to Schmidt & Spitzer:
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Theorem (Schmidt, Spitzer, Ullman - 60's):

$\Lambda(b)$ is a connected set that equals the union of a finite number of pairwise disjoint open analytic arcs and a finite number of the so called exceptional points (basically: branching points and end-points).

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- Open problem: It is not know for what b the set $\mathbb{C} \setminus \Lambda(b)$ is connected.

The non-self-adjoint case - the result of Hirschman Jr.

- Also the problem of a.e.d. has been solved for banded Toeplitz matrices. The limiting measure μ exists and one has

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Theorem (Hirschman Jr. - 1967)

On each arc Γ of $\Lambda(b)$, the limiting measure μ is a.c. and its density can be expressed as follows:

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(\lambda) = \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{r} \left(\frac{z_j'(\lambda+)}{z_j(\lambda+)} - \frac{z_j'(\lambda-)}{z_j(\lambda-)} \right).$$

Here $\mathrm{d}\lambda$ is the complex line element on Γ taken with respect to a chosen orientation on Γ and $z_j(\lambda\pm)$ are one-side limits of $z_j(\lambda')$, as λ' approaches $\lambda \in \Gamma$ from the left/right side of Γ determined by the chosen orientation.

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- For more general symbols, no similar results are known.

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4. Kuijlaars and Van Assche (1999) studied the asymptotic distribution of zeros of OG polynomials with variable coefficients - a special (tridiagonal) case of KMS matrices.

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$$T_n(a) = \left[a_{k-l}\left(\frac{k+l}{2n+2}\right)\right]_{k,l=0}^{n-1}$$

and called them *Generalized Toeplitz matrices* (if $a_k(t) = a_k$, $T_n(a)$ is a Toeplitz matrix).

An interesting history:

1. Introduced by Kac, Murdock, and Szegő in 1953.
2. After 1958 almost forgotten (no citation in 1958-1999 according to MathSciNet).
3. Tilli rediscovered these matrices in 1998 and called them *locally Toeplitz matrices*.
4. Kuijlaars and Van Assche (1999) studied the asymptotic distribution of zeros of OG polynomials with variable coefficients - a special (tridiagonal) case of KMS matrices.
5. After 2000, a renewed interest...

The result of Kac, Murdock, and Szegő

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Theorem (Kac, Murdock, Szegő - 1953)

With the assumptions above, one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[\lambda_k^{(n)}\right]^m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^1 \left[a(e^{\mathrm{i}t}, x)\right]^m \mathrm{d}x\mathrm{d}t, \quad \forall m \in \mathbb{N}_0,$$

where $\lambda_1^{(n)}, \dots, \lambda_n^{(n)}$ are eigenvalues of $T_n(a)$.

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where $\lambda_1^{(n)},\dots,\lambda_n^{(n)}$ are eigenvalues of $T_n(a)$.

- By applying the Weierstrass approximation theorem (and the fact that the eigenvalues remain in a compact interval for all n), we prove that the a.e.d. of $T_n(a)$, as $n\to\infty$, exists and is given by

$$\mu((\alpha,\beta]) = \frac{1}{2\pi}\left|\{(t,x)\in(-\pi,\pi]\times[0,1] \mid \alpha < a(e^{\mathrm{i}t},x) \leq \beta\}\right|.$$

A special case - orthogonal polynomials with variable coefficients

- From the special case with the trinomial symbol

$$a(z,x) = a_{-1}(x)z^{-1} + a_0(x) + a_1(x)z,$$

one can deduce the result of Kuijlaars & Van Assche that can be formulated as follows.

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Let $a : [0,1] \to \mathbb{R}_+$ and $b : [0,1] \to \mathbb{R}$ be continuous and $p_k^{(n)}$ be a family of polynomials generated by the recurrence

$$p_{k+1}^{(n)}(z) = \left(z - b\left(\frac{k}{n}\right)\right) p_k^{(n)}(z) - \left(a\left(\frac{k-1}{n}\right)\right)^2 p_{k-1}^{(n)}(z)$$

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$$\frac{\mathrm{d}\omega_{[\alpha,\beta]}(x)}{\mathrm{d}x} = \frac{1}{\pi\sqrt{(\beta-x)(x-\alpha)}}, \quad \text{for} \quad \alpha < x < \beta.$$

An alternative formulation - sampling Jacobi matrix

- Alternatively, the previous statement says that the a.e.d. of a self-adjoint sampling Jacobi matrix

$$J_n(a,b) = \begin{pmatrix} b\left(\frac{1}{n}\right) & a\left(\frac{1}{n}\right) & & & & \\ a\left(\frac{1}{n}\right) & b\left(\frac{2}{n}\right) & a\left(\frac{2}{n}\right) & & & \\ & a\left(\frac{2}{n}\right) & b\left(\frac{3}{n}\right) & a\left(\frac{3}{n}\right) & & \\ & & \ddots & \ddots & \ddots & \\ & & & a\left(\frac{n-2}{n}\right) & b\left(\frac{n-1}{n}\right) & a\left(\frac{n-1}{n}\right) \\ & & & & a\left(\frac{n-1}{n}\right) & b(1) \end{pmatrix},$$

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- The last formula fails to hold, if the assumption on self-adjointness is relaxed and no generalization is known.

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- Our inability to solve this problem in its generality motivates us to investigate some special cases - collaboration with O. Turek, work very much in progress.
- Typically, the special choices of a and b correspond to well-known families of polynomials where more properties are available.

The strategy for the derivation of the limiting measure

Definition:

The *Cauchy transform* of a Borel measure μ is a function defined by

$$C_\mu(z) := \int_{\mathbb{C}} \frac{\mathrm{d}\mu(x)}{z - x}, \quad z \in \mathbb{C} \setminus \operatorname{supp} \mu.$$

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Theorem

Let μ_n is a sequence of probability measures supported uniformly in a compact set $K \subset \mathbb{C}$. Assume that

$$\lim_{n\to\infty} C_{\mu_n}(z) = C(z), \quad \text{a.e. } z \in \mathbb{C}.$$

Then C is the Cauchy transform of a probability measure μ which is a weak limit of μ_n for $n \to \infty$. Moreover, one has

$$\mu = \frac{1}{\pi}\partial_{\overline{z}} C \quad \text{in the generalized sense.}$$

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- Although the generalized formula $\mu = \frac{1}{\pi}\partial_{\overline{z}}C_{\mu}$ is elegant, it can be difficult to deduce μ from it in cocrete cases.

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Plemelj–Sokhotski's formula

Let γ be an oriented analytic curve, C_μ analytic on $\mathbb{C} \setminus \gamma$ and can be continuously extended onto γ from the left(+)/right(-) side. Then one has

$$\boxed{\frac{\mathrm{d}\mu}{\mathrm{d}z}(z) = -\frac{1}{2\pi\mathrm{i}}\left(C_\mu(z+) - C_\mu(z-)\right)}$$

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- The main difficultly of the strategy: $\boxed{p_n(z) \sim ?}$ for $n \to \infty$.
- There are many powerful methods for the asymptotic analysis (Saddle point method, Riemann–Hilbert problem,...) but it usually requires a more detailed knowledge about p_n (generating functions, integral representations,...).

An appetizer - one example

$$a(x) = \sqrt{ax}, \quad (a > 0),$$

$$b(x) = \mathrm{i}x,$$

$$J_n = \begin{pmatrix} b\left(\frac{1}{n}\right) & a\left(\frac{1}{n}\right) & & \\ a\left(\frac{1}{n}\right) & b\left(\frac{2}{n}\right) & a\left(\frac{2}{n}\right) & \\ & \ddots & \ddots & \ddots \\ & & a\left(\frac{n-1}{n}\right) & b\left(1\right) \end{pmatrix},$$

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- Simple estimates on the quadratic form of J_n show that

$$\mathrm{spec}(J_n) \subset (-2\sqrt{a}, 2\sqrt{a}) + \mathrm{i}(0,1], \quad \forall n \in \mathbb{N}.$$

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- Moreover, $\operatorname{spec}(J_n)$ is the set of zeros of the polynomial

$$p_n(z) := {}_2F_0\left(-n, -an - \mathrm{i}nz - 1; -; a^{-1}n^{-1}\right),$$

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- Namely,

$$p_n(z) = C_n^{(-an)}\left(-an - \mathrm{i}zn - 1\right),$$

where $C_n^{(\alpha)}(x)$ are the Charlier polynomials.

An appetizer - asymptotic analysis

- From the hypergeometric representation, it follows that $\overline{p_n(z)} = p_n(-\overline{z})$. Hence, $\mathrm{spec}(J_n)$ is symmetric w.r.t. the imaginary line and we may restrict ourself to the half-plane $\Re z > 0$.

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- Certain nice properties of the Charlier polynomials yields the integral representation

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- This is a suitable form for the application of the Saddle point method:

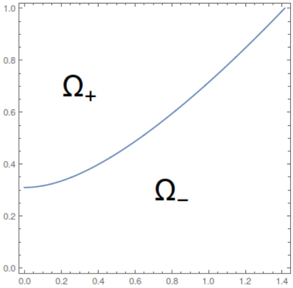
$$\boxed{p_n(z) \sim A_n(z) e^{-np(\xi_\pm,z)},} \qquad \text{if } \Re\, p(\xi_+, z) \lessgtr \Re\, p(\xi_-, z).$$

where $\xi_\pm = \xi_\pm(z, a)$ are two stationary points of $p(\cdot, z)$, i.e., the solutions of

$$a\xi^2 - (1 + \mathrm{i}z)\xi - 1 = 0.$$

An appetizer - the Cauchy transform

$$\Omega_{\pm} := \left\{ z \in (0, 2\sqrt{a}) + \mathrm{i}(0,1) \;\middle|\; \Re\, p(\xi_+, z) \lessgtr \Re\, p(\xi_-, z) \right\}$$



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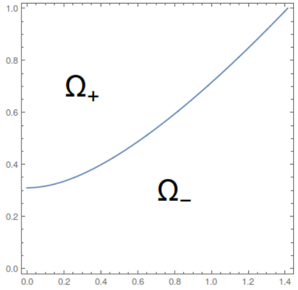

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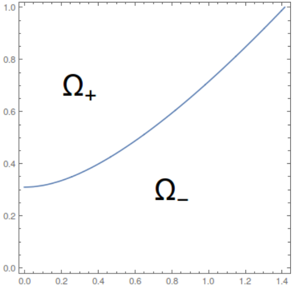

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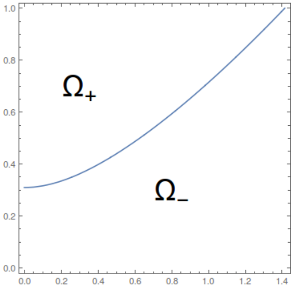
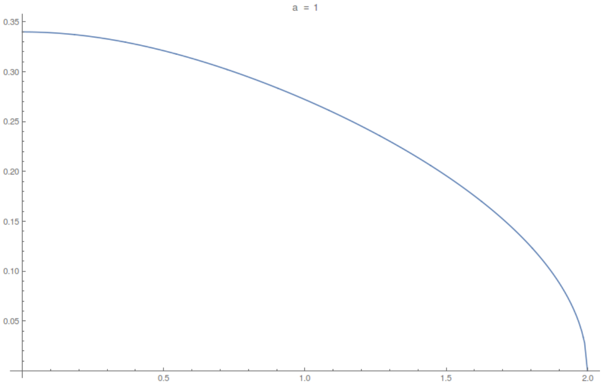


- If the curve is parametrized by the real part of the variable:
$$\gamma(x) := x + \mathrm{i}y(x), \quad x \in (0, 2\sqrt{a}),$$
then one can show that
$$y'(x) = -\frac{\Im \log\left((1+\xi_+)/(1+\xi_-)\right)}{\Re \log\left((1+\xi_+)/(1+\xi_-)\right)}.$$

An appetizer - the limiting measure on Arc 1

- The application of Plemelj–Sokhotski's formula yields

$$\boxed{\frac{\mathrm{d}\mu}{\mathrm{d}x}(x) = \frac{1}{2\pi}\frac{\left|\log\left((1+\xi_+)/(1+\xi_-)\right)\right|^2}{\Re\log\left((1+\xi_+)/(1+\xi_-)\right)},} \quad x \in (0, 2\sqrt{a}).$$



An appetizer - threshold

- Since $\overline{p_n(z)} = p_n(-\overline{z})$, one has $\boxed{\overline{C_\mu(z)} = -C_\mu(-\overline{z})}$

  which allows us to extend the Cauchy transform to the left half-plane $\Re z < 0$.

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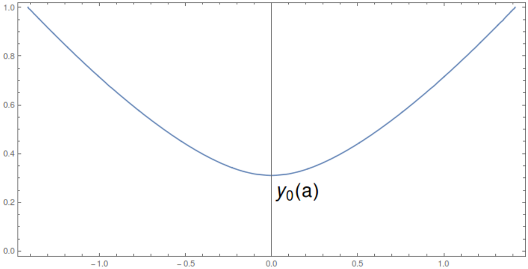
Denote by $y_0(a)$ the imaginary part of the point where the curve γ intersects the imaginary line.



If $\boxed{a > y_0(a),}$ C_μ is analytic everywhere but on the curve γ.

An appetizer - threshold

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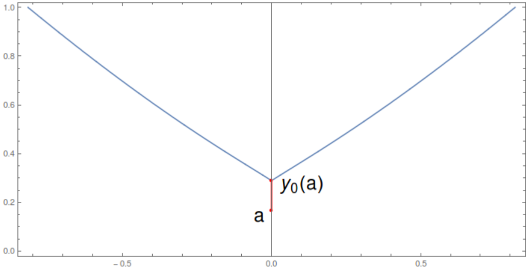

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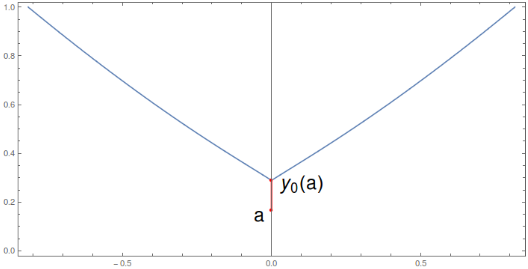

If $\boxed{a < y_0(a),}$ C_μ has an additional branch cut on the line segment $\mathrm{i}(a, y_0(a))$. Plemelj–Sokhotski implies

$$\boxed{\frac{\mathrm{d}\mu}{\mathrm{d}y}(y) = 1,} \quad y \in (a, y_0(a)).$$

An appetizer - summary

There are two regimes according to the value of a:

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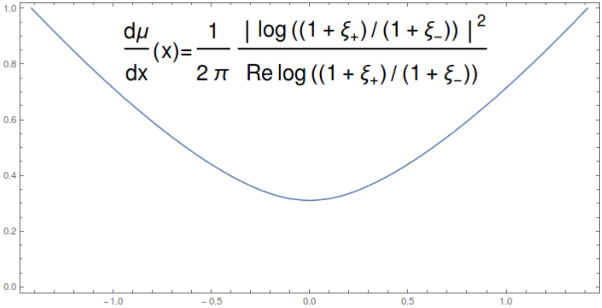

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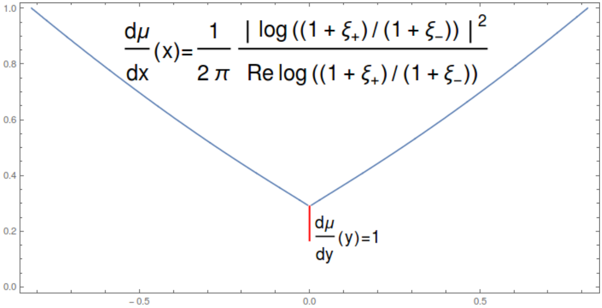

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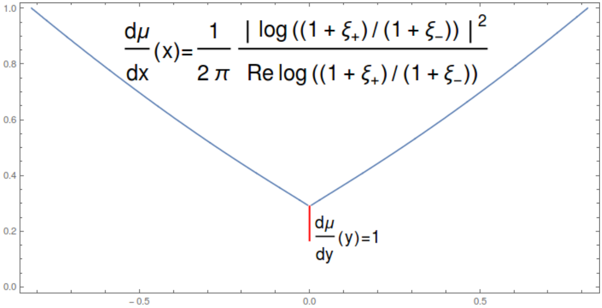
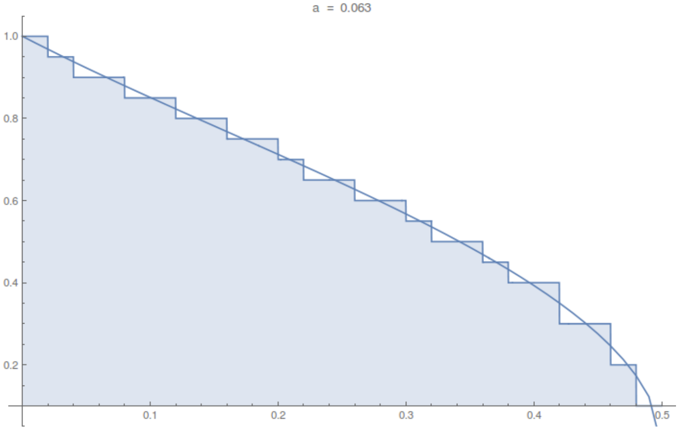


- The threshold $a = y_0(a)$ occurs for $a > 0$ the unique solution of the equation

$$\boxed{ae^{1+a} = 1}$$

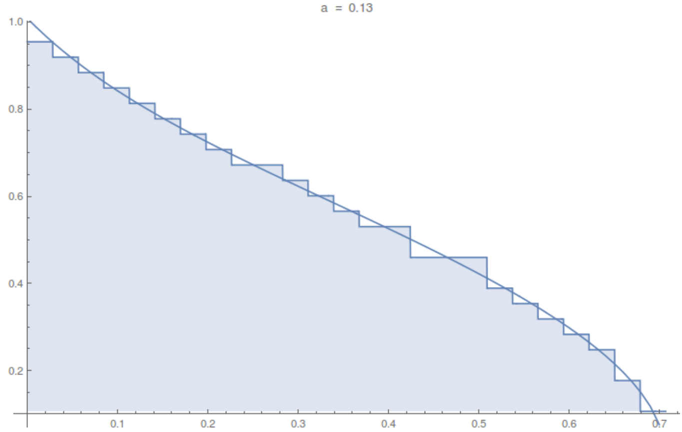
i.e, $a = 0.278465....$

An appetizer - numerical demonstrations



The histogram of eigenvalues of J_{1000} compared with the limiting density in $\Re z > 0$.

An appetizer - numerical demonstrations



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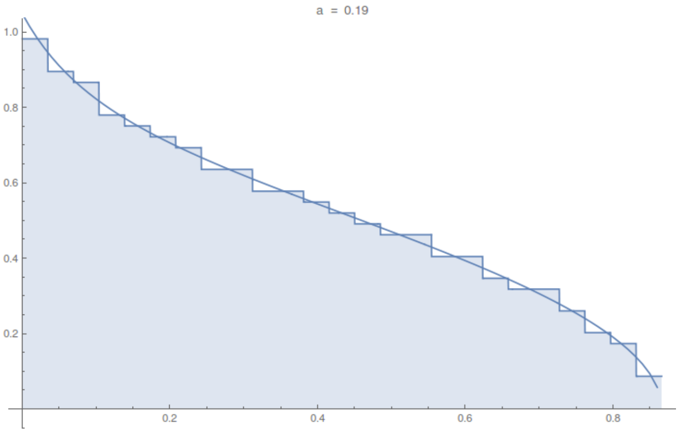

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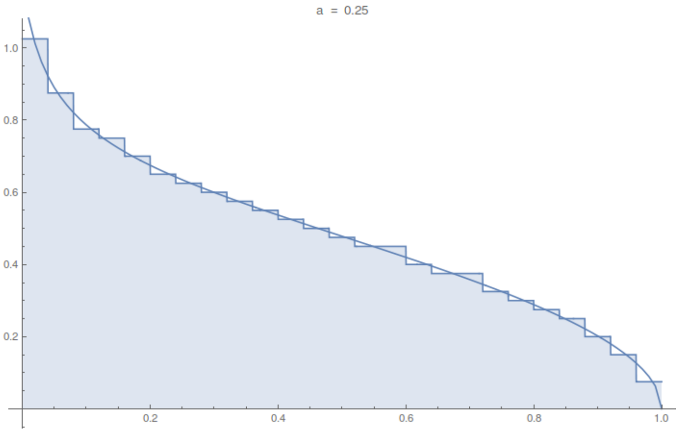

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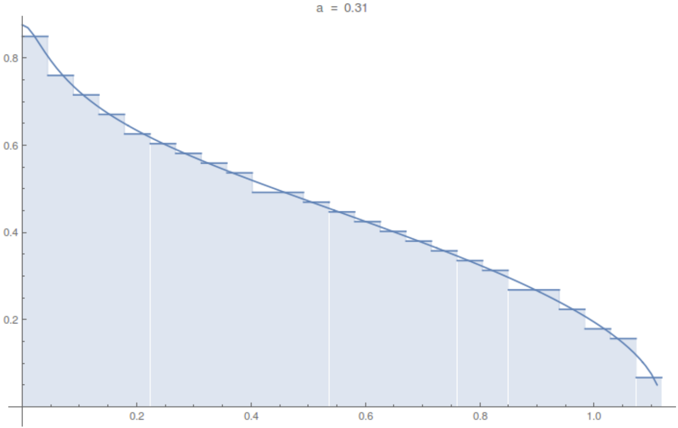

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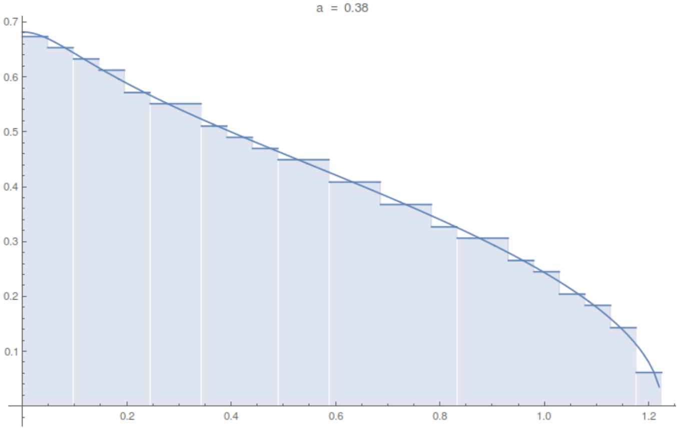

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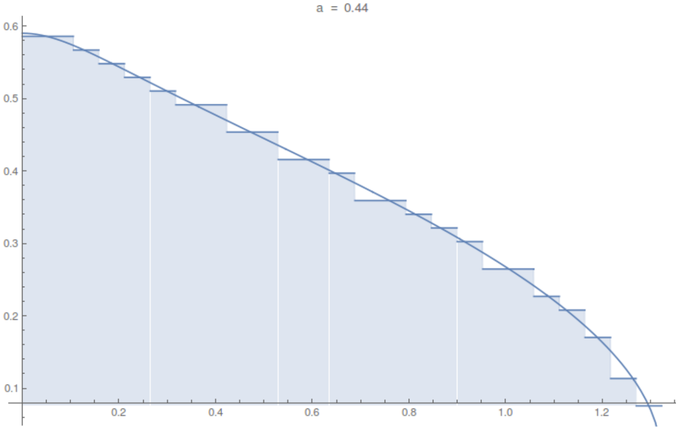

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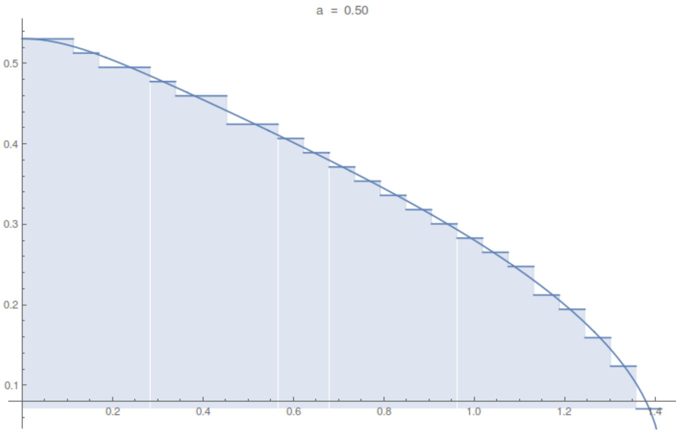

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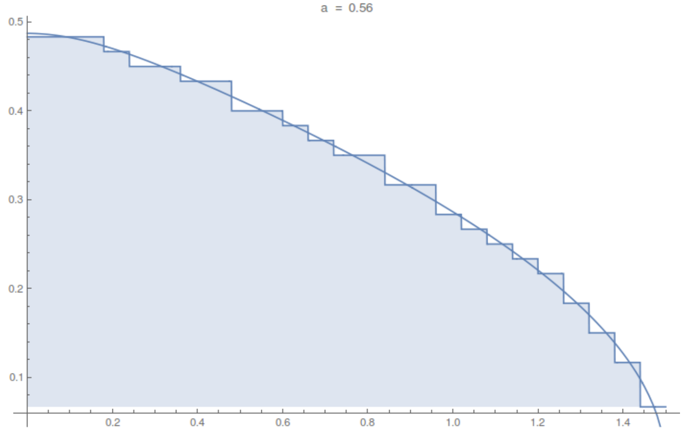

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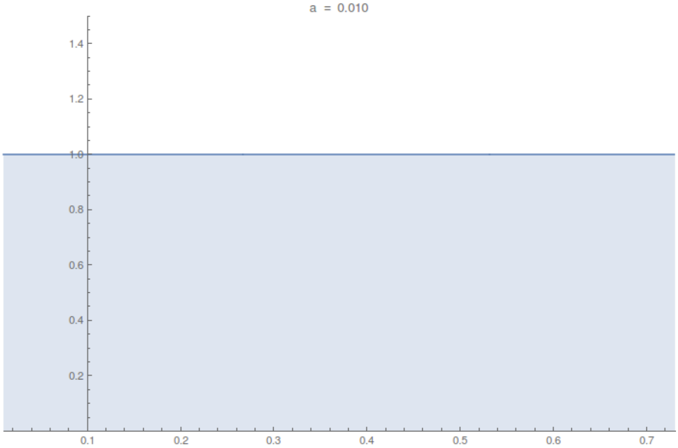
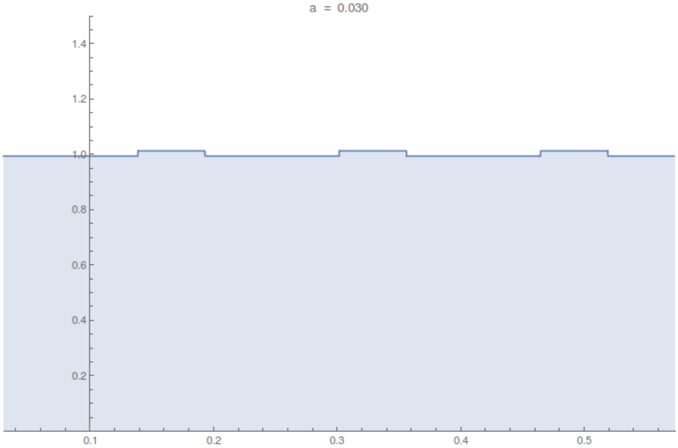


The histogram of eigenvalues of J_{1000} on $\Re z = 0$ (when present).

An appetizer - numerical demonstrations



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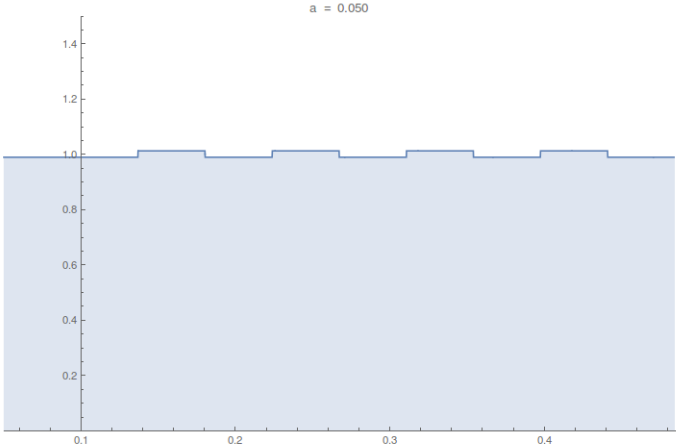

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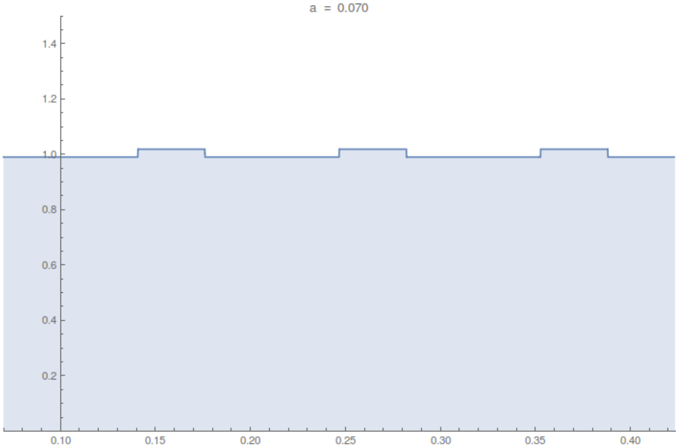

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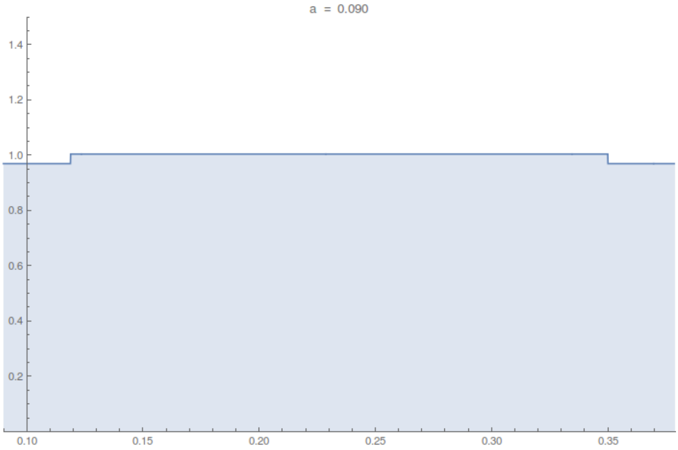

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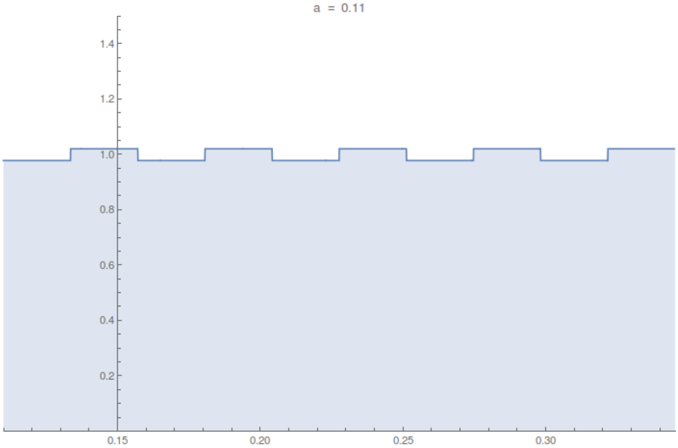

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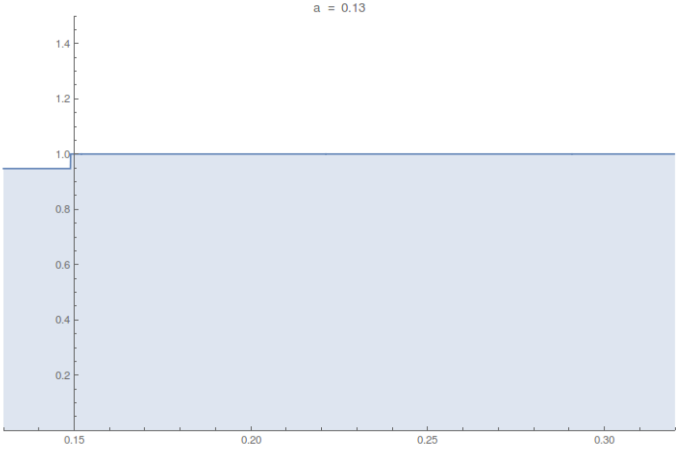

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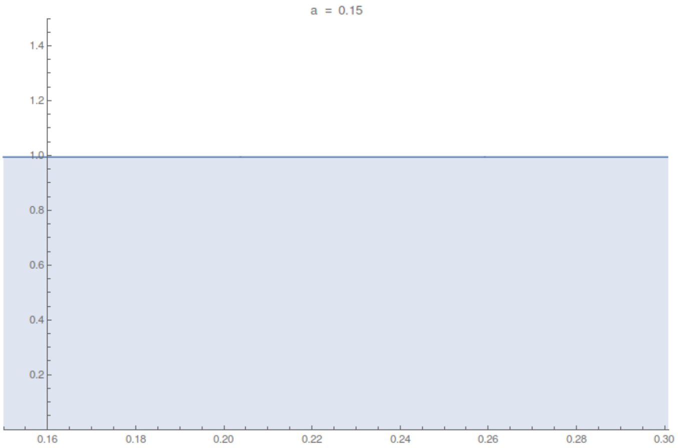

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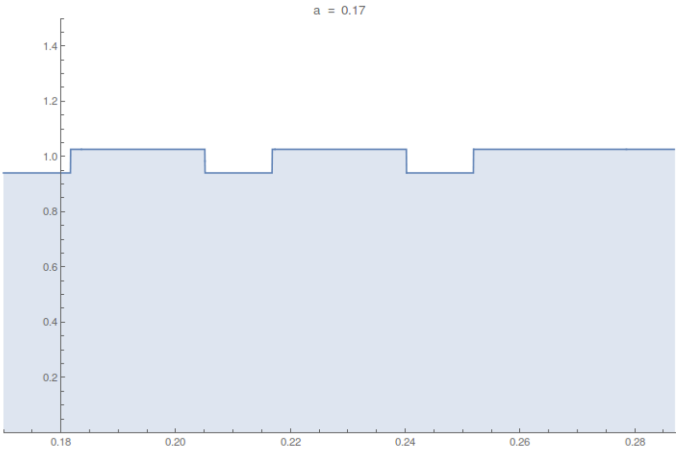

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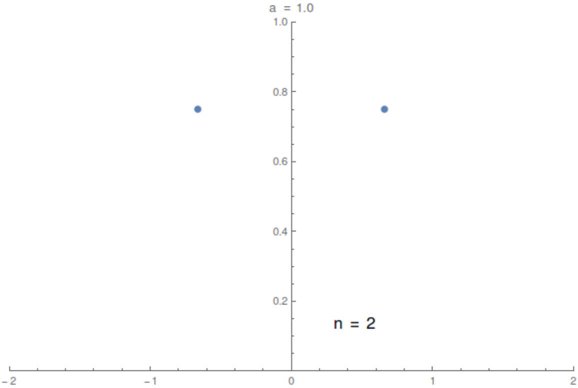
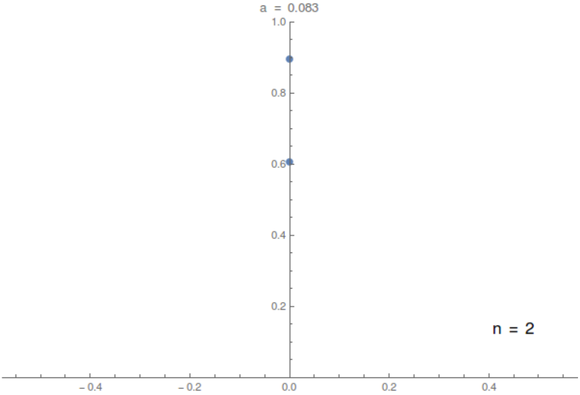


The distribution of eigenvalues in Regime 1: $a = 1 > y_0(a) = 0.32$.

An appetizer - numerical demonstrations



The distribution of eigenvalues in Regime 2: $a = 0.08 < y_0(a) = 0.4$.

Thank you!