Bousfield's localization of groups

Sergei O. Ivanov (joint with R. Mikhailov)

Modern algebra and applications, St. Petersburg University

Modern Algebra and Applications

- **Category theory.** Localization of an object with respect to a class of morphisms.

- **Category theory.** Localization of an object with respect to a class of morphisms.
- **Homotopy theory.** Sullivan localization of simply connected spaces and Bousfield localization of non-simply connected spaces.

- **Category theory.** Localization of an object with respect to a class of morphisms.
- **Homotopy theory.** Sullivan localization of simply connected spaces and Bousfield localization of non-simply connected spaces.
- **Group theory.** HR-localization of groups. The second homology group of completions.

Category theory

Localizations in categories

- Let $\mathcal{C}$ be a category and $\mathcal{W} \subseteq \mathsf{Mor}(\mathcal{C})$.

Localizations in categories

- Let $\mathcal{C}$ be a category and $\mathcal{W} \subseteq \mathsf{Mor}(\mathcal{C})$.
- An object $L \in \mathcal{C}$ is **local** (with respect to $\mathcal{W}$) if for any morphism $w : X \to Y$ in $\mathcal{W}$ the induced map

$$w^* : \mathcal{C}(Y, L) \xrightarrow{\cong} \mathcal{C}(X, L)$$

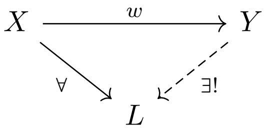
is a bijection.

Localizations in categories

- Let $\mathcal{C}$ be a category and $\mathcal{W} \subseteq \mathsf{Mor}(\mathcal{C})$.
- An object $L \in \mathcal{C}$ is **local** (with respect to $\mathcal{W}$) if for any morphism $w : X \to Y$ in $\mathcal{W}$ the induced map

$$w^* : \mathcal{C}(Y, L) \xrightarrow{\cong} \mathcal{C}(X, L)$$

  is a bijection.
- In other words

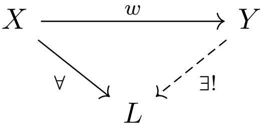


Localizations in categories

- Let $\mathcal{C}$ be a category and $\mathcal{W} \subseteq \mathsf{Mor}(\mathcal{C})$.
- An object $L \in \mathcal{C}$ is **local** (with respect to $\mathcal{W}$) if for any morphism $w : X \to Y$ in $\mathcal{W}$ the induced map

$$w^* : \mathcal{C}(Y, L) \xrightarrow{\cong} \mathcal{C}(X, L)$$

  is a bijection.
- In other words



- L is local if it "deals" with morphisms from $\mathcal{W}$ as with isomorphisms.

Localizations in categories

- A **localization** of an **object** $X \in \mathcal{C}$ is a morphism
$$w : X \to L,$$
where $w \in \mathcal{W}$ and L is local.

Localizations in categories

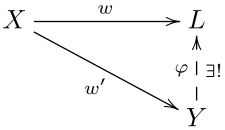
- A **localization** of an **object** $X \in \mathcal{C}$ is a morphism
$$w : X \to L,$$
where $w \in \mathcal{W}$ and L is local.
- If a localization exists, then it is unique up to isomorphism.

Localizations in categories

- A **localization** of an **object** $X \in \mathcal{C}$ is a morphism
$$w : X \to L,$$
where $w \in \mathcal{W}$ and L is local.
- If a localization exists, then it is unique up to isomorphism.
- Localization $X \to L$ satisfies two universal properties:

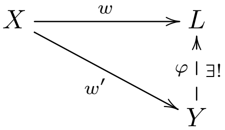
Localizations in categories

- A **localization** of an **object** $X \in \mathcal{C}$ is a morphism
$$w : X \to L,$$
where $w \in \mathcal{W}$ and L is local.
- If a localization exists, then it is unique up to isomorphism.
- Localization $X \to L$ satisfies two universal properties:
  1. For any $w' : X \to Y$ from $\mathcal{W}$ there exists a unique $\varphi : Y \to L$ such that

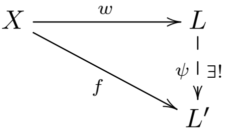


Localizations in categories

- A **localization** of an **object** $X \in \mathcal{C}$ is a morphism
$$w : X \to L,$$
where $w \in \mathcal{W}$ and L is local.
- If a localization exists, then it is unique up to isomorphism.
- Localization $X \to L$ satisfies two universal properties:
  1. For any $w' : X \to Y$ from $\mathcal{W}$ there exists a unique $\varphi : Y \to L$ such that



  2. For any $f : X \to L'$, where L' is local, there exists a unique $\psi : L \to L'$ such that



Localizations in categories

- Denote by $\mathsf{Loc}(\mathcal{W})$ the full subcategory of $\mathcal{C}$ consisted of $\mathcal{W}$-local objects.

Localizations in categories

- Denote by $\mathsf{Loc}(\mathcal{W})$ the full subcategory of $\mathcal{C}$ consisted of $\mathcal{W}$-local objects.
- **Assumption:** for any $X \in \mathcal{C}$ there exists a localization $X \to L$.

Localizations in categories

- Denote by $\mathsf{Loc}(\mathcal{W})$ the full subcategory of $\mathcal{C}$ consisted of $\mathcal{W}$-local objects.
- **Assumption:** for any $X \in \mathcal{C}$ there exists a localization $X \to L$.
- There exists functor

$$\mathcal{L} : \mathcal{C} \longrightarrow \mathsf{Loc}(\mathcal{W})$$

that that

$$X \to \mathcal{L}(X),$$

is the localization of X.

Localizations in categories

- Denote by $\mathsf{Loc}(\mathcal{W})$ the full subcategory of $\mathcal{C}$ consisted of $\mathcal{W}$-local objects.
- **Assumption:** for any $X \in \mathcal{C}$ there exists a localization $X \to L$.
- There exists functor

$$\mathcal{L} : \mathcal{C} \longrightarrow \mathsf{Loc}(\mathcal{W})$$

  that that

$$X \to \mathcal{L}(X),$$

  is the localization of X.
- We call this functor $\mathcal{L}$ the reflective $\mathcal{W}$-localization.

Localizations in categories

- Denote by $\mathsf{Loc}(\mathcal{W})$ the full subcategory of $\mathcal{C}$ consisted of $\mathcal{W}$-local objects.
- **Assumption:** for any $X \in \mathcal{C}$ there exists a localization $X \to L$.
- There exists functor

$$\mathcal{L} : \mathcal{C} \longrightarrow \mathsf{Loc}(\mathcal{W})$$

  that that

$$X \to \mathcal{L}(X),$$

  is the localization of X.
- We call this functor $\mathcal{L}$ the reflective $\mathcal{W}$-localization.
- The functor $\mathcal{L} : \mathcal{C} \longrightarrow \mathsf{Loc}(\mathcal{W})$ is left adjoint to the embedding $U : \mathsf{Loc}(\mathcal{W}) \hookrightarrow \mathcal{C}$ and $\eta : \mathsf{Id} \to U\mathcal{L}$ is the unit.

Localizations in categories

- Denote by $\mathsf{Loc}(\mathcal{W})$ the full subcategory of $\mathcal{C}$ consisted of $\mathcal{W}$-local objects.
- **Assumption:** for any $X \in \mathcal{C}$ there exists a localization $X \to L$.
- There exists functor
$$\mathcal{L} : \mathcal{C} \longrightarrow \mathsf{Loc}(\mathcal{W})$$
that that
$$X \to \mathcal{L}(X),$$
is the localization of X.
- We call this functor $\mathcal{L}$ the reflective $\mathcal{W}$-localization.
- The functor $\mathcal{L} : \mathcal{C} \longrightarrow \mathsf{Loc}(\mathcal{W})$ is left adjoint to the embedding $U : \mathsf{Loc}(\mathcal{W}) \hookrightarrow \mathcal{C}$ and $\eta : \mathsf{Id} \to U\mathcal{L}$ is the unit.
- In this case $\mathsf{Loc}(\mathcal{W}) \simeq \mathcal{C}[\mathcal{W}^{-1}]$, where $\mathcal{C}[\mathcal{W}^{-1}]$ is the usual localization of a category $\mathcal{C}$ by $\mathcal{W}$.

Example: $- \otimes \mathbb{Q} : \mathsf{Ab} \to \mathsf{Vect}(\mathbb{Q})$

- Let Ab be the category of abelian groups and $\mathcal{W} \subseteq \mathsf{Mor}(\mathsf{Ab})$ consists of homomorphisms $w : A \to B$ that induce an isomorphism

$$w \otimes \mathbb{Q} : A \otimes \mathbb{Q} \xrightarrow{\cong} B \otimes \mathbb{Q}.$$

Example: $- \otimes \mathbb{Q} : \mathsf{Ab} \to \mathsf{Vect}(\mathbb{Q})$

- Let Ab be the category of abelian groups and $\mathcal{W} \subseteq \mathsf{Mor}(\mathsf{Ab})$ consists of homomorphisms $w : A \to B$ that induce an isomorphism

$$w \otimes \mathbb{Q} : A \otimes \mathbb{Q} \xrightarrow{\cong} B \otimes \mathbb{Q}.$$

- An abelian group is $\mathcal{W}$-local iff it is a $\mathbb{Q}$-vector space

$$\mathsf{Loc}_{\mathcal{W}}(\mathsf{Ab}) \cong \mathsf{Vect}(\mathbb{Q}).$$

Example: $- \otimes \mathbb{Q} : \mathsf{Ab} \to \mathsf{Vect}(\mathbb{Q})$

- Let Ab be the category of abelian groups and $\mathcal{W} \subseteq \mathsf{Mor}(\mathsf{Ab})$ consists of homomorphisms $w : A \to B$ that induce an isomorphism

$$w \otimes \mathbb{Q} : A \otimes \mathbb{Q} \xrightarrow{\cong} B \otimes \mathbb{Q}.$$

- An abelian group is $\mathcal{W}$-local iff it is a $\mathbb{Q}$-vector space

$$\mathsf{Loc}_{\mathcal{W}}(\mathsf{Ab}) \cong \mathsf{Vect}(\mathbb{Q}).$$

- The map $A \to A \otimes \mathbb{Q}$ is the $\mathcal{W}$-localization of A.

Example: $- \otimes \mathbb{Q} : \mathsf{Ab} \to \mathsf{Vect}(\mathbb{Q})$

- Let Ab be the category of abelian groups and $\mathcal{W} \subseteq \mathsf{Mor}(\mathsf{Ab})$ consists of homomorphisms $w : A \to B$ that induce an isomorphism

$$w \otimes \mathbb{Q} : A \otimes \mathbb{Q} \xrightarrow{\cong} B \otimes \mathbb{Q}.$$

- An abelian group is $\mathcal{W}$-local iff it is a $\mathbb{Q}$-vector space

$$\mathsf{Loc}_{\mathcal{W}}(\mathsf{Ab}) \cong \mathsf{Vect}(\mathbb{Q}).$$

- The map $A \to A \otimes \mathbb{Q}$ is the $\mathcal{W}$-localization of A.
- The functor $\mathcal{L}(A) = A \otimes \mathbb{Q}$ is the reflective localization.

Example: $- \otimes \mathbb{Q} : \mathsf{Ab} \to \mathsf{Vect}(\mathbb{Q})$

- Let Ab be the category of abelian groups and $\mathcal{W} \subseteq \mathsf{Mor}(\mathsf{Ab})$ consists of homomorphisms $w : A \to B$ that induce an isomorphism

$$w \otimes \mathbb{Q} : A \otimes \mathbb{Q} \xrightarrow{\cong} B \otimes \mathbb{Q}.$$

- An abelian group is $\mathcal{W}$-local iff it is a $\mathbb{Q}$-vector space

$$\mathsf{Loc}_{\mathcal{W}}(\mathsf{Ab}) \cong \mathsf{Vect}(\mathbb{Q}).$$

- The map $A \to A \otimes \mathbb{Q}$ is the $\mathcal{W}$-localization of A.
- The functor $\mathcal{L}(A) = A \otimes \mathbb{Q}$ is the reflective localization.
- This can be generalized to the case of arbitrary commutative ring R and a submonoid (multiplicative system) $S \subseteq R$.

Example: $-\otimes\mathbb{Q} : \mathsf{Ab} \to \mathsf{Vect}(\mathbb{Q})$

- Let Ab be the category of abelian groups and $\mathcal{W} \subseteq \mathsf{Mor}(\mathsf{Ab})$ consists of homomorphisms $w : A \to B$ that induce an isomorphism

$$w \otimes \mathbb{Q} : A \otimes \mathbb{Q} \xrightarrow{\cong} B \otimes \mathbb{Q}.$$

- An abelian group is $\mathcal{W}$-local iff it is a $\mathbb{Q}$-vector space

$$\mathsf{Loc}_{\mathcal{W}}(\mathsf{Ab}) \cong \mathsf{Vect}(\mathbb{Q}).$$

- The map $A \to A \otimes \mathbb{Q}$ is the $\mathcal{W}$-localization of A.
- The functor $\mathcal{L}(A) = A \otimes \mathbb{Q}$ is the reflective localization.
- This can be generalized to the case of arbitrary commutative ring R and a submonoid (multiplicative system) $S \subseteq R$.
- Localization theory from commutative algebra gives examples for this general categorial notion.

Example: $\mathsf{ab} : \mathsf{Gr} \to \mathsf{Ab}$

- Let $\mathcal{C} = \mathsf{Gr}$.

Example: $\mathsf{ab} : \mathsf{Gr} \to \mathsf{Ab}$

- Let $\mathcal{C} = \mathsf{Gr}$.
- $\mathcal{W}$ consists of homomorphisms $f : G \to H$ such that $f_{ab} : G_{ab} \to H_{ab}$ is an isomorphism.

Example: $\mathsf{ab} : \mathsf{Gr} \to \mathsf{Ab}$

- Let $\mathcal{C} = \mathsf{Gr}$.
- $\mathcal{W}$ consists of homomorphisms $f : G \to H$ such that $f_{ab} : G_{ab} \to H_{ab}$ is an isomorphism.
- Then a group is local iff it is abelian.

Example: $\mathsf{ab} : \mathsf{Gr} \to \mathsf{Ab}$

- Let $\mathcal{C} = \mathsf{Gr}$.
- $\mathcal{W}$ consists of homomorphisms $f : G \to H$ such that $f_{ab} : G_{ab} \to H_{ab}$ is an isomorphism.
- Then a group is local iff it is abelian.
- The reflective localization is the abelianization

$$\mathsf{ab} : \mathsf{Gr} \longrightarrow \mathsf{Ab},$$

$$\mathsf{ab}(G) = G/[G,G].$$

Reflective subcategories

- A full subcategory $\mathcal{D} \subseteq \mathcal{C}$ is called **reflective** if the functor of embedding $\mathcal{D} \hookrightarrow \mathcal{C}$ has an adjoint functor

$$\mathcal{L} : \mathcal{C} \to \mathcal{D}$$

which is called **reflection**.

Reflective subcategories

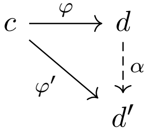
- A full subcategory $\mathcal{D} \subseteq \mathcal{C}$ is called **reflective** if the functor of embedding $\mathcal{D} \hookrightarrow \mathcal{C}$ has an adjoint functor

$$\mathcal{L} : \mathcal{C} \to \mathcal{D}$$

  which is called **reflection**.
- Equivalently $\mathcal{D}$ is reflective if for any $c \in \mathcal{C}$ there exists a "universal" map

$$\varphi : c \to d$$

  to an object of $\mathcal{D}$ such that for any $\varphi' : c \to d'$ there exists a unique $\alpha : d \to d'$



Reflective subcategories

- Denote by $\mathcal{W}(\mathcal{L})$ the class of morphisms $w : X \to Y$ in $\mathcal{C}$ such that $\mathcal{L}(w)$ is an isomorphism.

Reflective subcategories

- Denote by $\mathcal{W}(\mathcal{L})$ the class of morphisms $w : X \to Y$ in $\mathcal{C}$ such that $\mathcal{L}(w)$ is an isomorphism.
- For a reflective subcategory $\mathcal{D}$ the map $X \to \mathcal{L}(X)$ is the localization of X with respect to $\mathcal{W}(\mathcal{L})$.

Reflective subcategories

- Denote by $\mathcal{W}(\mathcal{L})$ the class of morphisms $w : X \to Y$ in $\mathcal{C}$ such that $\mathcal{L}(w)$ is an isomorphism.
- For a reflective subcategory $\mathcal{D}$ the map $X \to \mathcal{L}(X)$ is the localization of X with respect to $\mathcal{W}(\mathcal{L})$.
- For any class of morphisms $\mathcal{W} \subseteq \mathsf{Mor}(\mathcal{C})$, if the localization of any object exists, then $\mathsf{Loc}(\mathcal{W})$ is a reflective subcategory.

Reflective subcategories

- Denote by $\mathcal{W}(\mathcal{L})$ the class of morphisms $w : X \to Y$ in $\mathcal{C}$ such that $\mathcal{L}(w)$ is an isomorphism.
- For a reflective subcategory $\mathcal{D}$ the map $X \to \mathcal{L}(X)$ is the localization of X with respect to $\mathcal{W}(\mathcal{L})$.
- For any class of morphisms $\mathcal{W} \subseteq \mathsf{Mor}(\mathcal{C})$, if the localization of any object exists, then $\mathsf{Loc}(\mathcal{W})$ is a reflective subcategory.
- The language of localizations with respect to a class of morphisms $\mathcal{W}$ and the language of reflective subcategories are more or less equivalent.

Homotopy theory

Sullivan's R-localization of simply connected spaces

- Let X be a simply connected space.

Sullivan's R-localization of simply connected spaces

- Let X be a simply connected space.
- $R = \mathbb{Z}[P^{-1}]$, where P is a set of primes.

Sullivan's R-localization of simply connected spaces

- Let X be a simply connected space.
- $R = \mathbb{Z}[P^{-1}]$, where P is a set of primes.
- $H_*(X, \mathbb{Z})$ – complicated; $H_*(X, R)$ – simpler.

Sullivan's R-localization of simply connected spaces

- Let X be a simply connected space.
- $R = \mathbb{Z}[P^{-1}]$, where P is a set of primes.
- $H_*(X, \mathbb{Z})$ – complicated; $H_*(X, R)$ – simpler.
- We want to replace X by a "simpler" space X_R such that

$$H_*(X_R, \mathbb{Z}) = H_*(X, R).$$

Sullivan's R-localization of simply connected spaces

- Let X be a simply connected space.
- $R = \mathbb{Z}[P^{-1}]$, where P is a set of primes.
- $H_*(X, \mathbb{Z})$ – complicated; $H_*(X, R)$ – simpler.
- We want to replace X by a "simpler" space X_R such that

$$H_*(X_R, \mathbb{Z}) = H_*(X, R).$$

- A simply connected space L is *R-local* if $H_*(L, \mathbb{Z})$ is an R-module.

Sullivan's R-localization of simply connected spaces

- Let X be a simply connected space.
- $R = \mathbb{Z}[P^{-1}]$, where P is a set of primes.
- $H_*(X, \mathbb{Z})$ – complicated; $H_*(X, R)$ – simpler.
- We want to replace X by a "simpler" space X_R such that

$$H_*(X_R, \mathbb{Z}) = H_*(X, R).$$

- A simply connected space L is R-*local* if $H_*(L, \mathbb{Z})$ is an R-module.
- R-localization of X is the universal map to an R-local space

$$X \to X_R.$$

Sullivan's R-localization of simply connected spaces

- Let X be a simply connected space.
- $R = \mathbb{Z}[P^{-1}]$, where P is a set of primes.
- $H_*(X, \mathbb{Z})$ – complicated; $H_*(X, R)$ – simpler.
- We want to replace X by a "simpler" space X_R such that

$$H_*(X_R, \mathbb{Z}) = H_*(X, R).$$

- A simply connected space L is R-*local* if $H_*(L, \mathbb{Z})$ is an R-module.
- R-localization of X is the universal map to an R-local space

$$X \to X_R.$$

- $\pi_*(X_R) = \pi_*(X) \otimes R$

Sullivan's R-localization of simply connected spaces

- A morphism in the homotopy category $w : X \to Y$ is called R-homological equivalence, if it induces an isomorphism

$$H_*(X, R) \cong H_*(Y, R).$$

Sullivan's R-localization of simply connected spaces

- A morphism in the homotopy category $w : X \to Y$ is called R-homological equivalence, if it induces an isomorphism

$$H_*(X, R) \cong H_*(Y, R).$$

- Sullivan's R-localization is the localization with respect to the class of R-homological equivalences in the homotopy category of simply connected spaces.

Bousfield's R-localization of non-simply connected spaces

- The theory is much more complicated for non-simply connected spaces.

Bousfield's R-localization of non-simply connected spaces

- The theory is much more complicated for non-simply connected spaces.
- By definition R-localization

$$X \to X_R$$

of a space X is the localization with respect to R-homological equivalences.

Bousfield's R-localization of non-simply connected spaces

- The theory is much more complicated for non-simply connected spaces.
- By definition R-localization

$$X \to X_R$$

  of a space X is the localization with respect to R-homological equivalences.
- **Theorem** (Bousfield'75). For any space there exists the R-localization $X \to X_R$.

Bousfield's R-localization of non-simply connected spaces

- The theory is much more complicated for non-simply connected spaces.
- By definition R-localization

$$X \to X_R$$

  of a space X is the localization with respect to R-homological equivalences.
- **Theorem** (Bousfield'75). For any space there exists the R-localization $X \to X_R$.
- $H_*(X, R) \cong H_*(X_R, R)$

Bousfield's R-localization of non-simply connected spaces

- The theory is much more complicated for non-simply connected spaces.
- By definition R-localization

$$X \to X_R$$

  of a space X is the localization with respect to R-homological equivalences.
- **Theorem** (Bousfield'75). For any space there exists the R-localization $X \to X_R$.
- $H_*(X, R) \cong H_*(X_R, R)$
- $\pi_*(X_R) = ??$ (complicated and interesting even for $R = \mathbb{Z}$).

Bousfield's R-localization of non-simply connected spaces

- The theory is much more complicated for non-simply connected spaces.
- By definition R-localization

$$X \to X_R$$

of a space X is the localization with respect to R-homological equivalences.

- **Theorem** (Bousfield'75). For any space there exists the R-localization $X \to X_R$.
- $H_*(X, R) \cong H_*(X_R, R)$
- $\pi_*(X_R) =$?? (complicated and interesting even for $R = \mathbb{Z}$).
- $BGL(A)^+ = BGL(A)_{\mathbb{Z}}$ and $K_*(A) = \pi_*(BGL(A)_{\mathbb{Z}})$.

Bousfield's R-localization of non-simply connected spaces

- The theory is much more complicated for non-simply connected spaces.
- By definition R-localization

$$X \to X_R$$

  of a space X is the localization with respect to R-homological equivalences.
- **Theorem** (Bousfield'75). For any space there exists the R-localization $X \to X_R$.
- $H_*(X, R) \cong H_*(X_R, R)$
- $\pi_*(X_R) =??$ (complicated and interesting even for $R = \mathbb{Z}$).
- $BGL(A)^+ = BGL(A)_{\mathbb{Z}}$ and $K_*(A) = \pi_*(BGL(A)_{\mathbb{Z}})$.
- $R = \mathbb{Z}[P^{-1}]$ and $R = \mathbb{Z}/p$.

Bousfield's R-localization of non-simply connected spaces

- $\pi_1(X_R) =?$

Bousfield's R-localization of non-simply connected spaces

- $\pi_1(X_R)$ =?
- There is a functor $L_{HR} : \mathsf{Gr} \to \mathsf{Gr}$ such that

$$\pi_1(X_R) \cong L_{HR}(\pi_1(X)).$$

Bousfield's R-localization of non-simply connected spaces

- $\pi_1(X_R) =?$
- There is a functor $L_{HR} : \mathsf{Gr} \to \mathsf{Gr}$ such that

$$\pi_1(X_R) \cong L_{HR}(\pi_1(X)).$$

- So, $\pi_1(X_R)$ depends only on the group $\pi_1(X)$.

Bousfield's R-localization of non-simply connected spaces

- $\pi_1(X_R) =?$
- There is a functor $L_{HR} : \mathsf{Gr} \to \mathsf{Gr}$ such that

$$\pi_1(X_R) \cong L_{HR}(\pi_1(X)).$$

- So, $\pi_1(X_R)$ depends only on the group $\pi_1(X)$.
- We are interested in the functor

$$L_{HR} : \mathsf{Gr} \to \mathsf{Gr},$$

which is called HR-localization of a group.

Group theory

Bousfield's HR-localization of groups

- A homomorphism $f : G \to G'$ is called R-2-connected, if $H_1(G,R) \to H_1(G',R)$ is iso and $H_2(G,R) \to H_2(G',R)$ is epi.

Bousfield's HR-localization of groups

- A homomorphism $f : G \to G'$ is called R-2-connected, if $H_1(G,R) \to H_1(G',R)$ is iso and $H_2(G,R) \to H_2(G',R)$ is epi.
- Let $\mathcal{C} = \mathsf{Gr}$ and $\mathcal{W}_R$ is the class of R-2-connected homomorphisms.

Bousfield's HR-localization of groups

- A homomorphism $f : G \to G'$ is called R-2-connected, if $H_1(G,R) \to H_1(G',R)$ is iso and $H_2(G,R) \to H_2(G',R)$ is epi.
- Let $\mathcal{C} = \mathsf{Gr}$ and $\mathcal{W}_R$ is the class of R-2-connected homomorphisms.
- HR-localization of a group is the localization with respect to the class of R-2-connected homomorphisms.

Bousfield's HR-localization of groups

- We will discuss in details $H\mathbb{Z}$-localization of groups.

Bousfield's HR-localization of groups

- We will discuss in details $H\mathbb{Z}$-localization of groups.
- There are 4 different equivalent definitions of $H\mathbb{Z}$-localization

Bousfield's HR-localization of groups

- We will discuss in details $H\mathbb{Z}$-localization of groups.
- There are 4 different equivalent definitions of $H\mathbb{Z}$-localization
    1. Via description of the class of $H\mathbb{Z}$-local groups.

Bousfield's HR-localization of groups

- We will discuss in details $H\mathbb{Z}$-localization of groups.
- There are 4 different equivalent definitions of $H\mathbb{Z}$-localization
  1. Via description of the class of $H\mathbb{Z}$-local groups.
  2. Via 2-connected homomorphisms (historically the first)

Bousfield's HR-localization of groups

- We will discuss in details $H\mathbb{Z}$-localization of groups.
- There are 4 different equivalent definitions of $H\mathbb{Z}$-localization
  1. Via description of the class of $H\mathbb{Z}$-local groups.
  2. Via 2-connected homomorphisms (historically the first)
  3. As the algebraic closure with respect to "Γ-systems of equations".

Bousfield's HR-localization of groups

- We will discuss in details $H\mathbb{Z}$-localization of groups.
- There are 4 different equivalent definitions of $H\mathbb{Z}$-localization
  1. Via description of the class of $H\mathbb{Z}$-local groups.
  2. Via 2-connected homomorphisms (historically the first)
  3. As the algebraic closure with respect to "Γ-systems of equations".
  4. Via explicit transfinite construction by relative central extensions.

Bousfield's HR-localization of groups

- We will discuss in details $H\mathbb{Z}$-localization of groups.
- There are 4 different equivalent definitions of $H\mathbb{Z}$-localization
  1. Via description of the class of $H\mathbb{Z}$-local groups.
  2. Via 2-connected homomorphisms (historically the first)
  3. As the algebraic closure with respect to "Γ-systems of equations".
  4. Via explicit transfinite construction by relative central extensions.
- (1)$\Leftrightarrow$ (2) proved by Bousfield'77. (2)$\Leftrightarrow$ (4) proved by Bousfield'77 but it was formulated in different terms and reformulated by -, R.Mikhailov'16. (3)$\Leftrightarrow$ (1) proved by Farjoun, Orr, Shelah'89.

Bousfield's $H\mathbb{Z}$-localization of groups

- **Theorem** (Bousfield'77). *The class of $H\mathbb{Z}$-local groups is the minimal nonempty class closed with respect to central extensions and small limits.*

Bousfield's $H\mathbb{Z}$-localization of groups

- **Theorem** (Bousfield'77). *The class of $H\mathbb{Z}$-local groups is the minimal nonempty class closed with respect to central extensions and small limits.*
- $H\mathbb{Z}$-localization is the universal map to an $H\mathbb{Z}$-local group

$$G \to LG.$$

Bousfield's $H\mathbb{Z}$-localization of groups

- **Theorem** (Bousfield'77). *The class of $H\mathbb{Z}$-local groups is the minimal nonempty class closed with respect to central extensions and small limits.*
- $H\mathbb{Z}$-localization is the universal map to an $H\mathbb{Z}$-local group

$$G \to LG.$$

- Nilpotent groups are $H\mathbb{Z}$-local.

Bousfield's $H\mathbb{Z}$-localization of groups

- **Theorem** (Bousfield'77). *The class of $H\mathbb{Z}$-local groups is the minimal nonempty class closed with respect to central extensions and small limits.*
- $H\mathbb{Z}$-localization is the universal map to an $H\mathbb{Z}$-local group

$$G \to LG.$$

- Nilpotent groups are $H\mathbb{Z}$-local.
- A finitely generated group is $H\mathbb{Z}$-local iff it is nilpotent.

Bousfield's $H\mathbb{Z}$-localization of groups

- **Theorem** (Bousfield'77). *The class of $H\mathbb{Z}$-local groups is the minimal nonempty class closed with respect to central extensions and small limits.*
- $H\mathbb{Z}$-localization is the universal map to an $H\mathbb{Z}$-local group

$$G \to LG.$$

- Nilpotent groups are $H\mathbb{Z}$-local.
- A finitely generated group is $H\mathbb{Z}$-local iff it is nilpotent.
- We denote by $\gamma_n = \gamma_n(G)$ the lower central series and by γ_α the transfinite lower central series.

Bousfield's $H\mathbb{Z}$-localization of groups

- **Theorem** (Bousfield'77). *The class of $H\mathbb{Z}$-local groups is the minimal nonempty class closed with respect to central extensions and small limits.*
- $H\mathbb{Z}$-localization is the universal map to an $H\mathbb{Z}$-local group

$$G \to LG.$$

- Nilpotent groups are $H\mathbb{Z}$-local.
- A finitely generated group is $H\mathbb{Z}$-local iff it is nilpotent.
- We denote by $\gamma_n = \gamma_n(G)$ the lower central series and by γ_α the transfinite lower central series.
- Pronilpotent completion $\hat{G} = \varprojlim G/\gamma_n$ is $H\mathbb{Z}$-local.

Bousfield's $H\mathbb{Z}$-localization of groups

- **Theorem** (Bousfield'77). *The class of $H\mathbb{Z}$-local groups is the minimal nonempty class closed with respect to central extensions and small limits.*
- $H\mathbb{Z}$-localization is the universal map to an $H\mathbb{Z}$-local group

$$G \to LG.$$

- Nilpotent groups are $H\mathbb{Z}$-local.
- A finitely generated group is $H\mathbb{Z}$-local iff it is nilpotent.
- We denote by $\gamma_n = \gamma_n(G)$ the lower central series and by γ_α the transfinite lower central series.
- Pronilpotent completion $\hat{G} = \varprojlim G/\gamma_n$ is $H\mathbb{Z}$-local.
- Usually $G \to \hat{G}$ is not $H\mathbb{Z}$-localization (but is it difficult to prove it for concrete examples).

Bousfield's $H\mathbb{Z}$-localization of groups

- **Theorem** (Bousfield'77). *The class of $H\mathbb{Z}$-local groups is the minimal nonempty class closed with respect to central extensions and small limits.*
- $H\mathbb{Z}$-localization is the universal map to an $H\mathbb{Z}$-local group

$$G \to LG.$$

- Nilpotent groups are $H\mathbb{Z}$-local.
- A finitely generated group is $H\mathbb{Z}$-local iff it is nilpotent.
- We denote by $\gamma_n = \gamma_n(G)$ the lower central series and by γ_α the transfinite lower central series.
- Pronilpotent completion $\hat{G} = \varprojlim G/\gamma_n$ is $H\mathbb{Z}$-local.
- Usually $G \to \hat{G}$ is not $H\mathbb{Z}$-localization
(but is it difficult to prove it for concrete examples).
- If G is finitely generated, then

$$LG/\gamma_\omega = \hat{G}.$$

Bousfield's $H\mathbb{Z}$-localization of groups

- If G is finitely generated, then

$$LG/\gamma_\omega = \hat{G}.$$

Bousfield's $H\mathbb{Z}$-localization of groups

- If G is finitely generated, then

$$LG/\gamma_\omega = \hat{G}.$$

- Any $H\mathbb{Z}$-local group is transfinitely nilpotent.

Bousfield's $H\mathbb{Z}$-localization of groups

- If G is finitely generated, then

$$LG/\gamma_\omega = \hat{G}.$$

- Any $H\mathbb{Z}$-local group is transfinitely nilpotent.
- We try to understand LG/γ_α for $\alpha = \omega + 1, \omega + 2, \dots$.

Bousfield's $H\mathbb{Z}$-localization of groups

- If G is finitely generated, then

$$LG/\gamma_\omega = \hat{G}.$$

- Any $H\mathbb{Z}$-local group is transfinitely nilpotent.
- We try to understand LG/γ_α for $\alpha = \omega + 1, \omega + 2, \dots$.
- $$H_2(G, \mathbb{Z}) \to H_2(\hat{G}, \mathbb{Z}) \to LG/\gamma_{\omega+1} \to \hat{G} \to 1$$

Bousfield's $H\mathbb{Z}$-localization of groups

- If G is finitely generated, then
$$LG/\gamma_\omega = \hat{G}.$$
- Any $H\mathbb{Z}$-local group is transfinitely nilpotent.
- We try to understand LG/γ_α for $\alpha = \omega + 1, \omega + 2, \dots$.
-
$$H_2(G, \mathbb{Z}) \to H_2(\hat{G}, \mathbb{Z}) \to LG/\gamma_{\omega+1} \to \hat{G} \to 1$$
- $F = F_n$ is the free group $2 \leq n < \infty$.

Bousfield's $H\mathbb{Z}$-localization of groups

- If G is finitely generated, then

$$LG/\gamma_\omega = \hat{G}.$$

- Any $H\mathbb{Z}$-local group is transfinitely nilpotent.
- We try to understand LG/γ_α for $\alpha = \omega + 1, \omega + 2, \dots$.
-

$$H_2(G,\mathbb{Z}) \to H_2(\hat{G},\mathbb{Z}) \to LG/\gamma_{\omega+1} \to \hat{G} \to 1$$

- $F = F_n$ is the free group $2 \le n < \infty$.
- **Theorem** (Bousfield'77). $H_2(\hat{F},\mathbb{Z})$ is uncountable.

Bousfield's $H\mathbb{Z}$-localization of groups

- If G is finitely generated, then
$$LG/\gamma_\omega = \hat{G}.$$
- Any $H\mathbb{Z}$-local group is transfinitely nilpotent.
- We try to understand LG/γ_α for $\alpha = \omega+1, \omega+2, \dots$.
- $$H_2(G,\mathbb{Z}) \to H_2(\hat{G},\mathbb{Z}) \to LG/\gamma_{\omega+1} \to \hat{G} \to 1$$
- $F = F_n$ is the free group $2 \le n < \infty$.
- **Theorem** (Bousfield'77). $H_2(\hat{F},\mathbb{Z})$ is uncountable.
- Hence $LF \ncong \hat{F}$ and $\gamma_\omega(LF) \neq \gamma_{\omega+1}(LF)$.

Bousfield's $H\mathbb{Z}$-localization of groups

- If G is finitely generated, then
$$LG/\gamma_\omega = \hat{G}.$$
- Any $H\mathbb{Z}$-local group is transfinitely nilpotent.
- We try to understand LG/γ_α for $\alpha = \omega + 1, \omega + 2, \dots$.
-
$$H_2(G,\mathbb{Z}) \to H_2(\hat{G},\mathbb{Z}) \to LG/\gamma_{\omega+1} \to \hat{G} \to 1$$
- $F = F_n$ is the free group $2 \le n < \infty$.
- **Theorem** (Bousfield'77). $H_2(\hat{F},\mathbb{Z})$ is uncountable.
- Hence $LF \ncong \hat{F}$ and $\gamma_\omega(LF) \neq \gamma_{\omega+1}(LF)$.
- **Theorem** (-, R.Mikhailov'16). $\gamma_{\omega+1}(LF) \neq \gamma_{\omega+2}(LF)$.

Bousfield's $H\mathbb{Z}$-localization of groups

- If G is finitely generated, then

$$LG/\gamma_\omega = \hat{G}.$$

- Any $H\mathbb{Z}$-local group is transfinitely nilpotent.
- We try to understand LG/γ_α for $\alpha = \omega+1, \omega+2, \dots$.
- $$H_2(G,\mathbb{Z}) \to H_2(\hat{G},\mathbb{Z}) \to LG/\gamma_{\omega+1} \to \hat{G} \to 1$$
- $F = F_n$ is the free group $2 \le n < \infty$.
- **Theorem** (Bousfield'77). $H_2(\hat{F},\mathbb{Z})$ is uncountable.
- Hence $LF \ncong \hat{F}$ and $\gamma_\omega(LF) \ne \gamma_{\omega+1}(LF)$.
- **Theorem** (-, R.Mikhailov'16). $\gamma_{\omega+1}(LF) \ne \gamma_{\omega+2}(LF)$.
- **Conjecture.** $\gamma_\alpha(LF) \ne \gamma_{\alpha+1}(LF)$ for $\alpha < \omega^2$.

Bousfield's $H\mathbb{Z}/p$-localization and profinite groups

- Pro-p-group is an inverse limit of finite p-groups.

Bousfield's $H\mathbb{Z}/p$-localization and profinite groups

- Pro-p-group is an inverse limit of finite p-groups.
- If G is finitely generated,

$$L_pG/\gamma_\omega^p = \hat{G}_{\text{pro}-p}$$

Bousfield's $H\mathbb{Z}/p$-localization and profinite groups

- Pro-p-group is an inverse limit of finite p-groups.
- If G is finitely generated,

$$L_pG/\gamma_\omega^p = \hat{G}_{\mathrm{pro}-p}$$

- Probably, over fields things go better than over $\mathbb{Z}$?

Bousfield's $H\mathbb{Z}/p$-localization and profinite groups

- Pro-p-group is an inverse limit of finite p-groups.
- If G is finitely generated,

$$L_pG/\gamma_\omega^p = \hat{G}_{\text{pro}-p}$$

- Probably, over fields things go better than over $\mathbb{Z}$?
- **Bousfield's problem**: it this true that $L_pG = \hat{G}_p$ for f.p. G?

Bousfield's $H\mathbb{Z}/p$-localization and profinite groups

- Pro-p-group is an inverse limit of finite p-groups.
- If G is finitely generated,

$$L_pG/\gamma_\omega^p = \hat{G}_{\text{pro}-p}$$

- Probably, over fields things go better than over $\mathbb{Z}$?
- **Bousfield's problem**: it this true that $L_pG = \hat{G}_p$ for f.p. G?
- **Theorem**(-,R.Mikhailov'17). $H_2(\hat{F}_{\text{pro}-p}, \mathbb{Z}/p)$ is uncountable.

Bousfield's $H\mathbb{Z}/p$-localization and profinite groups

- Pro-p-group is an inverse limit of finite p-groups.
- If G is finitely generated,

$$L_pG/\gamma_\omega^p = \hat{G}_{\text{pro}-p}$$

- Probably, over fields things go better than over $\mathbb{Z}$?
- **Bousfield's problem**: it this true that $L_pG = \hat{G}_p$ for f.p. G?
- **Theorem**(-,R.Mikhailov'17). $H_2(\hat{F}_{\text{pro}-p}, \mathbb{Z}/p)$ is uncountable.
- Hence $L_pF \neq \hat{F}_{\text{pro}-p}$

Bousfield's $H\mathbb{Z}/p$-localization and profinite groups

- Pro-p-group is an inverse limit of finite p-groups.
- If G is finitely generated,

$$L_pG/\gamma_\omega^p = \hat{G}_{\text{pro}-p}$$

- Probably, over fields things go better than over $\mathbb{Z}$?
- **Bousfield's problem**: it this true that $L_pG = \hat{G}_p$ for f.p. G?
- **Theorem**(-,R.Mikhailov'17). $H_2(\hat{F}_{\text{pro}-p}, \mathbb{Z}/p)$ is uncountable.
- Hence $L_pF \neq \hat{F}_{\text{pro}-p}$
- **Theorem**(Bousfield'92). $H_i(\hat{F}_{\text{pro}-p}, \mathbb{Z}/p) \neq 0$ for some $i \in \{2, 3\}$.

Bousfield's $H\mathbb{Z}/p$-localization and profinite groups

- Pro-p-group is an inverse limit of finite p-groups.
- If G is finitely generated,

$$L_pG/\gamma_\omega^p = \hat{G}_{\text{pro}-p}$$

- Probably, over fields things go better than over $\mathbb{Z}$?
- **Bousfield's problem**: it this true that $L_pG = \hat{G}_p$ for f.p. G?
- **Theorem**(-,R.Mikhailov'17). $H_2(\hat{F}_{\text{pro}-p}, \mathbb{Z}/p)$ is uncountable.
- Hence $L_pF \neq \hat{F}_{\text{pro}-p}$
- **Theorem**(Bousfield'92). $H_i(\hat{F}_{\text{pro}-p}, \mathbb{Z}/p) \neq 0$ for some $i \in \{2,3\}$.
- Negative answer on the following conjecture.
  Conjecture (Fernandez-Alcober, Kazatchkov, Remeslennikov, Symonds): For a finitely presented pro-p group $\mathcal{G}$

$$H_2^{\mathsf{disc}}(\mathcal{G}, \mathbb{Z}/p) \cong H_2^{\mathsf{cont}}(\mathcal{G}, \mathbb{Z}/p).$$

Bousfield's $H\mathbb{Z}/p$-localization and profinite groups

- Pro-p-group is an inverse limit of finite p-groups.
- If G is finitely generated,

$$L_p G/\gamma_\omega^p = \hat{G}_{\mathrm{pro}-p}$$

- Probably, over fields things go better than over $\mathbb{Z}$?
- **Bousfield's problem**: it this true that $L_p G = \hat{G}_p$ for f.p. G?
- **Theorem**(-,R.Mikhailov'17). $H_2(\hat{F}_{\mathrm{pro}-p}, \mathbb{Z}/p)$ is uncountable.
- Hence $L_p F \neq \hat{F}_{\mathrm{pro}-p}$
- **Theorem**(Bousfield'92). $H_i(\hat{F}_{\mathrm{pro}-p}, \mathbb{Z}/p) \neq 0$ for some $i \in \{2, 3\}$.
- Negative answer on the following conjecture.
  Conjecture (Fernandez-Alcober, Kazatchkov, Remeslennikov, Symonds): For a finitely presented pro-p group $\mathcal{G}$

$$H_2^{\mathsf{disc}}(\mathcal{G}, \mathbb{Z}/p) \cong H_2^{\mathsf{cont}}(\mathcal{G}, \mathbb{Z}/p).$$

- We use results of Nikolov and Segal about profinite groups.

Bousfield's $H\mathbb{Q}$-localization and rational homotopy theory

- Malcev's completion is the extension of the functor $-\otimes\mathbb{Q}$ from the category of nilpotent groups to itself, which sends short exact sequences to short exact sequences.

Bousfield's $H\mathbb{Q}$-localization and rational homotopy theory

- Malcev's completion is the extension of the functor $- \otimes \mathbb{Q}$ from the category of nilpotent groups to itself, which sends short exact sequences to short exact sequences.
- $\hat{G}_{\mathbb{Q}} = \varprojlim\, G/\gamma_n \otimes \mathbb{Q}$.

Bousfield's $H\mathbb{Q}$-localization and rational homotopy theory

- Malcev's completion is the extension of the functor $-\otimes\mathbb{Q}$ from the category of nilpotent groups to itself, which sends short exact sequences to short exact sequences.
- $\hat{G}_{\mathbb{Q}} = \varprojlim G/\gamma_n \otimes \mathbb{Q}$.
- If G is finitely generated

$$L_{\mathbb{Q}} G/\gamma_\omega^{\mathbb{Q}} = \hat{G}_{\mathbb{Q}}.$$

Bousfield's $H\mathbb{Q}$-localization and rational homotopy theory

- Malcev's completion is the extension of the functor $-\otimes\mathbb{Q}$ from the category of nilpotent groups to itself, which sends short exact sequences to short exact sequences.
- $\hat{G}_{\mathbb{Q}} = \varprojlim G/\gamma_n \otimes \mathbb{Q}$.
- If G is finitely generated

$$L_{\mathbb{Q}} G/\gamma_\omega^{\mathbb{Q}} = \hat{G}_{\mathbb{Q}}.$$

- Probably over $\mathbb{Q}$ things go better then over $\mathbb{Z}$ and $\mathbb{Z}/p$?

Bousfield's $H\mathbb{Q}$-localization and rational homotopy theory

- Malcev's completion is the extension of the functor $-\otimes\mathbb{Q}$ from the category of nilpotent groups to itself, which sends short exact sequences to short exact sequences.
- $\hat{G}_{\mathbb{Q}} = \varprojlim G/\gamma_n \otimes \mathbb{Q}$.
- If G is finitely generated

$$L_{\mathbb{Q}} G/\gamma_{\omega}^{\mathbb{Q}} = \hat{G}_{\mathbb{Q}}.$$

- Probably over $\mathbb{Q}$ things go better then over $\mathbb{Z}$ and $\mathbb{Z}/p$?
- **Bousfield's problem:** Is it true that $L_{\mathbb{Q}} G = \hat{G}_{\mathbb{Q}}$ for f.p. G?

Bousfield's $H\mathbb{Q}$-localization and rational homotopy theory

- Malcev's completion is the extension of the functor $-\otimes\mathbb{Q}$ from the category of nilpotent groups to itself, which sends short exact sequences to short exact sequences.
- $\hat{G}_{\mathbb{Q}} = \varprojlim G/\gamma_n \otimes \mathbb{Q}$.
- If G is finitely generated

$$L_{\mathbb{Q}}G/\gamma_\omega^{\mathbb{Q}} = \hat{G}_{\mathbb{Q}}.$$

- Probably over $\mathbb{Q}$ things go better then over $\mathbb{Z}$ and $\mathbb{Z}/p$?
- **Bousfield's problem:** Is it true that $L_{\mathbb{Q}}G = \hat{G}_{\mathbb{Q}}$ for f.p. G?
- **Theorem**(-, R. Mikhailov'17). $H_2(\hat{F}_{\mathbb{Q}}, \mathbb{Q})$ is uncountable.

Bousfield's $H\mathbb{Q}$-localization and rational homotopy theory

- Malcev's completion is the extension of the functor $- \otimes \mathbb{Q}$ from the category of nilpotent groups to itself, which sends short exact sequences to short exact sequences.
- $\hat{G}_{\mathbb{Q}} = \varprojlim G/\gamma_n \otimes \mathbb{Q}$.
- If G is finitely generated

$$L_{\mathbb{Q}} G/\gamma_{\omega}^{\mathbb{Q}} = \hat{G}_{\mathbb{Q}}.$$

- Probably over $\mathbb{Q}$ things go better then over $\mathbb{Z}$ and $\mathbb{Z}/p$?
- **Bousfield's problem:** Is it true that $L_{\mathbb{Q}} G = \hat{G}_{\mathbb{Q}}$ for f.p. G?
- **Theorem**(-, R. Mikhailov'17). $H_2(\hat{F}_{\mathbb{Q}}, \mathbb{Q})$ is uncountable.
- Hence $L_{\mathbb{Q}} F \ncong \hat{F}_{\mathbb{Q}}$.

Bousfield's $H\mathbb{Q}$-localization and rational homotopy theory

- Malcev's completion is the extension of the functor $- \otimes \mathbb{Q}$ from the category of nilpotent groups to itself, which sends short exact sequences to short exact sequences.
- $\hat{G}_{\mathbb{Q}} = \varprojlim G/\gamma_n \otimes \mathbb{Q}$.
- If G is finitely generated

$$L_{\mathbb{Q}} G/\gamma_{\omega}^{\mathbb{Q}} = \hat{G}_{\mathbb{Q}}.$$

- Probably over $\mathbb{Q}$ things go better then over $\mathbb{Z}$ and $\mathbb{Z}/p$?
- **Bousfield's problem:** Is it true that $L_{\mathbb{Q}} G = \hat{G}_{\mathbb{Q}}$ for f.p. G?
- **Theorem**(-, R. Mikhailov'17). $H_2(\hat{F}_{\mathbb{Q}}, \mathbb{Q})$ is uncountable.
- Hence $L_{\mathbb{Q}} F \ncong \hat{F}_{\mathbb{Q}}$.
- Rational homotopy theory works not only for simply connected spaces but for $\mathbb{Q}$-good spaces.

Bousfield's $H\mathbb{Q}$-localization and rational homotopy theory

- Malcev's completion is the extension of the functor $- \otimes \mathbb{Q}$ from the category of nilpotent groups to itself, which sends short exact sequences to short exact sequences.
- $\hat{G}_{\mathbb{Q}} = \varprojlim G/\gamma_n \otimes \mathbb{Q}$.
- If G is finitely generated

$$L_{\mathbb{Q}} G/\gamma_\omega^{\mathbb{Q}} = \hat{G}_{\mathbb{Q}}.$$

- Probably over $\mathbb{Q}$ things go better then over $\mathbb{Z}$ and $\mathbb{Z}/p$?
- **Bousfield's problem:** Is it true that $L_{\mathbb{Q}} G = \hat{G}_{\mathbb{Q}}$ for f.p. G?
- **Theorem**(-, R. Mikhailov'17). $H_2(\hat{F}_{\mathbb{Q}}, \mathbb{Q})$ is uncountable.
- Hence $L_{\mathbb{Q}} F \ncong \hat{F}_{\mathbb{Q}}$.
- Rational homotopy theory works not only for simply connected spaces but for $\mathbb{Q}$-good spaces.
- There was a conjecture that any finite CW-complex is $\mathbb{Q}$-good.

Bousfield's $H\mathbb{Q}$-localization and rational homotopy theory

- Malcev's completion is the extension of the functor $-\otimes\mathbb{Q}$ from the category of nilpotent groups to itself, which sends short exact sequences to short exact sequences.
- $\hat{G}_{\mathbb{Q}} = \varprojlim G/\gamma_n \otimes \mathbb{Q}$.
- If G is finitely generated

$$L_{\mathbb{Q}}G/\gamma_{\omega}^{\mathbb{Q}} = \hat{G}_{\mathbb{Q}}.$$

- Probably over $\mathbb{Q}$ things go better then over $\mathbb{Z}$ and $\mathbb{Z}/p$?
- **Bousfield's problem:** Is it true that $L_{\mathbb{Q}}G = \hat{G}_{\mathbb{Q}}$ for f.p. G?
- **Theorem**(-, R. Mikhailov'17). $H_2(\hat{F}_{\mathbb{Q}}, \mathbb{Q})$ is uncountable.
- Hence $L_{\mathbb{Q}}F \ncong \hat{F}_{\mathbb{Q}}$.
- Rational homotopy theory works not only for simply connected spaces but for $\mathbb{Q}$-good spaces.
- There was a conjecture that any finite CW-complex is $\mathbb{Q}$-good.
- **Corollary**(-, R.Mikhailov'17). $S^1 \vee S^1$ is $\mathbb{Q}$-bad.

R-good and R-bad spaces

Theorem (Bousfield'77+Bousfield'92+ Mikhailov and I'17)

$S^1 \vee S^1$ is R-bad for any $R \in \{\mathbb{Z}, \mathbb{Z}/p, \mathbb{Q}\}$.

In all 3 cases it was the first known example of a finite R-bad space.

Bousfield's problem for metabelian groups

- **Theorem** (-, R.Mikhailov'14). If G is metabelian and finitely presented, then

$$L_p G = \hat{G}_{\text{pro}-p}, \qquad L_{\mathbb{Q}} G = \hat{G}_{\mathbb{Q}}.$$

Bousfield's problem for metabelian groups

- **Theorem** (-, R.Mikhailov'14). If G is metabelian and finitely presented, then

$$L_pG = \hat{G}_{\text{pro}-p}, \qquad L_{\mathbb{Q}}G = \hat{G}_{\mathbb{Q}}.$$

- For finitely generated groups this is not true.

Bousfield's problem for metabelian groups

- **Theorem** (-, R.Mikhailov'14). If G is metabelian and finitely presented, then

$$L_pG = \hat{G}_{\text{pro}-p}, \qquad L_{\mathbb{Q}}G = \hat{G}_{\mathbb{Q}}.$$

- For finitely generated groups this is not true.
- $G := C \ltimes \mathbb{Z}[C]$, where C is the infinite cyclic group.

$$G = \langle x, y \mid [y, y^{x^i}] = 1, i \in \mathbb{Z} \rangle.$$

Then $L_pG \ncong \hat{G}_{\text{pro}-p}$ and $L_{\mathbb{Q}}G \ncong \hat{G}_{\mathbb{Q}}$.

Sketch of the proof for $R = \mathbb{Z}/p$.

- Hopf like formulas:

Sketch of the proof for $R = \mathbb{Z}/p$.

- Hopf like formulas:
- (Hopf's formula) If $H \triangleleft F$, then

$$H_2(F/H) = \frac{H \cap [F,F]}{[H,F]}$$

Sketch of the proof for $R = \mathbb{Z}/p$.

- Hopf like formulas:
- (Hopf's formula) If $H \lhd F$, then

$$H_2(F/H) = \frac{H \cap [F,F]}{[H,F]}$$

- If G is a group and $H \lhd G$, then

$$H_2(G) \longrightarrow H_2(G/H) \longrightarrow \frac{H \cap [G,G]}{[H,G]} \longrightarrow 0$$

$$H_2(G,\mathbb{Z}/p) \longrightarrow H_2(G/H,\mathbb{Z}/p) \longrightarrow \frac{H \cap [G,G]G^p}{[H,G]H^p} \longrightarrow 0$$

Sketch of the proof for $R = \mathbb{Z}/p$.

- Hopf like formulas:
- (Hopf's formula) If $H \lhd F$, then

$$H_2(F/H) = \frac{H \cap [F,F]}{[H,F]}$$

- If G is a group and $H \lhd G$, then

$$H_2(G) \longrightarrow H_2(G/H) \longrightarrow \frac{H \cap [G,G]}{[H,G]} \longrightarrow 0$$

$$H_2(G,\mathbb{Z}/p) \longrightarrow H_2(G/H,\mathbb{Z}/p) \longrightarrow \frac{H \cap [G,G]G^p}{[H,G]H^p} \longrightarrow 0$$

- If $\mathcal{G}$ is a profinite group and $\mathcal{H}$ is a normal closed subgroup, then

$$H_2^{\mathsf{cont}}(\mathcal{G},\mathbb{Z}/p) \longrightarrow H_2^{\mathsf{cont}}(\mathcal{G}/\mathcal{H},\mathbb{Z}/p) \longrightarrow \frac{\mathcal{H} \cap \overline{[\mathcal{G},\mathcal{G}]\mathcal{G}^p}}{\overline{[\mathcal{H},\mathcal{G}]\mathcal{H}^p}} \longrightarrow 0$$

Sketch of the proof for $R = \mathbb{Z}/p = \mathbb{F}_p$.

- **Theorem**(Nikolov, Segal, 2007, Ann. of Math.) Let $\mathcal{G}$ be a finitely generated profinite group and $\mathcal{H}$ be a normal closed subgroup. Then $[\mathcal{H},\mathcal{G}]$ and $[\mathcal{H},\mathcal{G}]\mathcal{H}^p$ are closed.

Sketch of the proof for $R = \mathbb{Z}/p = \mathbb{F}_p$.

- **Theorem**(Nikolov, Segal, 2007, Ann. of Math.) Let $\mathcal{G}$ be a finitely generated profinite group and $\mathcal{H}$ be a normal closed subgroup. Then $[\mathcal{H},\mathcal{G}]$ and $[\mathcal{H},\mathcal{G}]\mathcal{H}^p$ are closed.
- Then the following cokernels coincide

$$\begin{array}{ccccccc}
H_2^{\mathsf{disc}}(\mathcal{G},\mathbb{Z}/p) & \longrightarrow & H_2^{\mathsf{disc}}(\mathcal{G}/\mathcal{H},\mathbb{Z}/p) & \longrightarrow & Q^{\mathsf{disc}} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \cong} & & \\
H_2^{\mathsf{cont}}(\mathcal{G},\mathbb{Z}/p) & \longrightarrow & H_2^{\mathsf{cont}}(\mathcal{G}/\mathcal{H},\mathbb{Z}/p) & \longrightarrow & Q^{\mathsf{cont}} & \longrightarrow & 0
\end{array}$$

Sketch of the proof for $R = \mathbb{Z}/p = \mathbb{F}_p$.

- **Theorem**(Nikolov, Segal, 2007, Ann. of Math.) Let $\mathcal{G}$ be a finitely generated profinite group and $\mathcal{H}$ be a normal closed subgroup. Then $[\mathcal{H},\mathcal{G}]$ and $[\mathcal{H},\mathcal{G}]\mathcal{H}^p$ are closed.
- Then the following cokernels coincide

$$\begin{array}{ccccccc}
H_2^{\mathsf{disc}}(\mathcal{G},\mathbb{Z}/p) & \longrightarrow & H_2^{\mathsf{disc}}(\mathcal{G}/\mathcal{H},\mathbb{Z}/p) & \longrightarrow & Q^{\mathsf{disc}} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \cong} & & \\
H_2^{\mathsf{cont}}(\mathcal{G},\mathbb{Z}/p) & \longrightarrow & H_2^{\mathsf{cont}}(\mathcal{G}/\mathcal{H},\mathbb{Z}/p) & \longrightarrow & Q^{\mathsf{cont}} & \longrightarrow & 0
\end{array}$$

- If $\mathcal{G}$ is a finitely generated pro-p group and $\mathcal{G} = \mathcal{F}/\mathcal{R}$ is its pro-p-presentation, then

$$H_2^{\mathsf{disc}}(\mathcal{F},\mathbb{Z}/p) \longrightarrow H_2^{\mathsf{disc}}(\mathcal{G},\mathbb{Z}/p) \longrightarrow H_2^{\mathsf{cont}}(\mathcal{G},\mathbb{Z}/p) \longrightarrow 0.$$

Sketch of the proof for $R = \mathbb{Z}/p = \mathbb{F}_p$.

- **Theorem**(Nikolov, Segal, 2007, Ann. of Math.) Let $\mathcal{G}$ be a finitely generated profinite group and $\mathcal{H}$ be a normal closed subgroup. Then $[\mathcal{H},\mathcal{G}]$ and $[\mathcal{H},\mathcal{G}]\mathcal{H}^p$ are closed.
- Then the following cokernels coincide

$$\begin{array}{ccccccc} H_2^{\mathsf{disc}}(\mathcal{G},\mathbb{Z}/p) & \longrightarrow & H_2^{\mathsf{disc}}(\mathcal{G}/\mathcal{H},\mathbb{Z}/p) & \longrightarrow & Q^{\mathsf{disc}} & \longrightarrow & 0 \\ \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \cong} & & \\ H_2^{\mathsf{cont}}(\mathcal{G},\mathbb{Z}/p) & \longrightarrow & H_2^{\mathsf{cont}}(\mathcal{G}/\mathcal{H},\mathbb{Z}/p) & \longrightarrow & Q^{\mathsf{cont}} & \longrightarrow & 0 \end{array}$$

- If $\mathcal{G}$ is a finitely generated pro-p group and $\mathcal{G} = \mathcal{F}/\mathcal{R}$ is its pro-p-presentation, then

$$H_2^{\mathsf{disc}}(\mathcal{F},\mathbb{Z}/p) \longrightarrow H_2^{\mathsf{disc}}(\mathcal{G},\mathbb{Z}/p) \longrightarrow H_2^{\mathsf{cont}}(\mathcal{G},\mathbb{Z}/p) \longrightarrow 0.$$

- We need to find a group $\mathcal{G}$ such that the kernel of

$$H_2^{\mathsf{disc}}(\mathcal{G},\mathbb{Z}/p) \xrightarrow{\varphi_2} H_2^{\mathsf{cont}}(\mathcal{G},\mathbb{Z}/p)$$

is uncountable.

Sketch of the proof for $R = \mathbb{Z}/p = \mathbb{F}_p$.

- The following map is well defined

$$\mathbb{Z}_p \longrightarrow \mathbb{F}_p[[x]], \qquad \alpha \mapsto (1+x)^\alpha,$$

where $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^i$ is the group of p-adic integers.

Sketch of the proof for $R = \mathbb{Z}/p = \mathbb{F}_p$.

- The following map is well defined

$$\mathbb{Z}_p \longrightarrow \mathbb{F}_p[[x]], \qquad \alpha \mapsto (1+x)^\alpha,$$

where $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^i$ is the group of p-adic integers.

- We take the pro-p-completion of the double version of p-lamplighter group

$$\mathcal{G} = \mathbb{F}_p[[x]]^2 \rtimes \mathbb{Z}_p.$$

Sketch of the proof for $R = \mathbb{Z}/p = \mathbb{F}_p$.

- The following map is well defined
$$\mathbb{Z}_p \longrightarrow \mathbb{F}_p[[x]], \qquad \alpha \mapsto (1+x)^\alpha,$$
where $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^i$ is the group of p-adic integers.
- We take the pro-p-completion of the double version of p-lamplighter group
$$\mathcal{G} = \mathbb{F}_p[[x]]^2 \rtimes \mathbb{Z}_p.$$
- Using the spectral sequence of the extension we obtain

$$\begin{array}{ccc} \mathbb{F}_p[[x]] \otimes_{\mathbb{F}_p[\mathbb{Z}_p]} \mathbb{F}_p[[x]] & \longrightarrow & \mathbb{F}_p[[x]] \\ \downarrow & & \downarrow \\ H_2^{\mathsf{disc}}(\mathcal{G}, \mathbb{Z}/p) & \longrightarrow & H_2^{\mathsf{cont}}(\mathcal{G}, \mathbb{Z}/p) \end{array}$$

Sketch of the proof for $R = \mathbb{Z}/p = \mathbb{F}_p$.

- The following map is well defined
$$\mathbb{Z}_p \longrightarrow \mathbb{F}_p[[x]], \qquad \alpha \mapsto (1+x)^\alpha,$$
where $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^i$ is the group of p-adic integers.
- We take the pro-p-completion of the double version of p-lamplighter group
$$\mathcal{G} = \mathbb{F}_p[[x]]^2 \rtimes \mathbb{Z}_p.$$
- Using the spectral sequence of the extension we obtain
$$\begin{array}{ccc} \mathbb{F}_p[[x]] \otimes_{\mathbb{F}_p[\mathbb{Z}_p]} \mathbb{F}_p[[x]] & \longrightarrow & \mathbb{F}_p[[x]] \\ \downarrow & & \downarrow \\ H_2^{\mathsf{disc}}(\mathcal{G}, \mathbb{Z}/p) & \longrightarrow & H_2^{\mathsf{cont}}(\mathcal{G}, \mathbb{Z}/p) \end{array}$$
- It is enough to prove that the kernel of the map
$$\mathbb{F}_p[[x]] \otimes_{\mathbb{F}_p[\mathbb{Z}_p]} \mathbb{F}_p[[x]] \longrightarrow \mathbb{F}_p[[x]]$$
is uncountable.

Sketch of the proof for $R = \mathbb{Z}/p$.

- In order to proof that the kernel of

$$\mathbb{F}_p[[x]] \otimes_{\mathbb{F}_p[\mathbb{Z}_p]} \mathbb{F}_p[[x]] \longrightarrow \mathbb{F}_p[[x]]$$

  is uncountable, we need the following lemma.

Sketch of the proof for $R = \mathbb{Z}/p$.

- In order to proof that the kernel of

$$\mathbb{F}_p[[x]] \otimes_{\mathbb{F}_p[\mathbb{Z}_p]} \mathbb{F}_p[[x]] \longrightarrow \mathbb{F}_p[[x]]$$

  is uncountable, we need the following lemma.

- **Lemma**. Let $\mathbb{F}_p((x))$ be the field of Laurent power series and K be the subfield generated by the image of $\mathbb{Z}_p$. Then $[\mathbb{F}_p((x)) : K]$ is uncountable.

Sketch of the proof for $R = \mathbb{Z}/p$.

- In order to proof that the kernel of

$$\mathbb{F}_p[[x]] \otimes_{\mathbb{F}_p[\mathbb{Z}_p]} \mathbb{F}_p[[x]] \longrightarrow \mathbb{F}_p[[x]]$$

  is uncountable, we need the following lemma.
- **Lemma**. Let $\mathbb{F}_p((x))$ be the field of Laurent power series and K be the subfield generated by the image of $\mathbb{Z}_p$. Then $[\mathbb{F}_p((x)) : K]$ is uncountable.
- In order to prove this lemma we consider $\mathbb{F}_p[[x]]$ as a complete metric space and use the **Baire theorem** about countable unions of nowhere dense subsets.

Sketch of the proof for $R = \mathbb{Z}/p$.

- In order to proof that the kernel of

$$\mathbb{F}_p[[x]] \otimes_{\mathbb{F}_p[\mathbb{Z}_p]} \mathbb{F}_p[[x]] \longrightarrow \mathbb{F}_p[[x]]$$

  is uncountable, we need the following lemma.

- **Lemma**. Let $\mathbb{F}_p((x))$ be the field of Laurent power series and K be the subfield generated by the image of $\mathbb{Z}_p$. Then $[\mathbb{F}_p((x)) : K]$ is uncountable.

- In order to prove this lemma we consider $\mathbb{F}_p[[x]]$ as a complete metric space and use the **Baire theorem** about countable unions of nowhere dense subsets.

- We use the theory of profinite groups, field extensions and metric spaces.