About some generalizations of a theorem of Moens

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History

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Lie algebras

An algebra L is called *Lie* if it satisfies the following identities:

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$d \in \mathrm{End}_{\mathbb{F}}(A)$ is a *derivation* of A, if

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The space of all derivations of an algebra A gives the structure of a Lie algebra under new multiplication $[D_1, D_2] = D_1 D_2 - D_2 D_1$.

Main Question

What information about the structure of an algebra is contained in its Lie algebra of derivations?

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Nilpotency

An algebra A is called *nilpotent* if there exists $n \in \mathbb{N}$ such that $A^n = 0$, where

$$A^1 = A, A^2 = AA, \ldots, A^i = A^{i-1}A^1 + A^{i-2}A^2 + \ldots + A^2A^{i-2} + A^1A^{i-1}.$$

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Theorem (Jacobson, 1955)

A finite dimensional Lie algebra over a field $\mathbb{F}$ of characteristic 0 is nilpotent if it admits an invertible derivation.

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What about generalizations?

Prederivations (Muller, 1989)

$d \in \mathrm{End}_{\mathbb{F}}(A)$ is a *prederivation* of A, if

$$d((xy)z) = (d(x)y)z + (xd(y))z + (xy)d(z); \ \ x, y, z \in A.$$

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Prederivations of Lie algebra $\mathfrak{g}$ are derivations of the Lie triple system induced by $\mathfrak{g}$.

Theorem (Muller, 1989)

Any prederivation of a semisimple finite-dimensional Lie algebra over a field of characteristic 0 is a derivation.

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Theorem (Bajo, 1997)

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Bajo, 1997, Burde, 1999 — examples of nilpotent Lie algebras possessing only nilpotent prederivations.

History

Leibniz-derivations (Moens, 2012)

A linear map $d \in \mathrm{End}_{\mathbb{F}}(A)$ is a *Leibniz-derivation of order n* of A, if

$$d(x_1(x_2(x_3 \ldots (x_{n-1}x_n) \ldots)) = \sum_{i=1}^{n}(x_1(x_2(\ldots d(x_i) \ldots (x_{n-1}x_n) \ldots).$$

Problems for Leibniz-derivations

1) Are the radicals invariant under Leibniz-derivations?
2) Do well-studied classes of algebras (e.g. semisimple algebras) admit nontrivial Leibniz-derivations?
3) How possessing an invertible Leibniz-derivation relates with nilpotency?

Theorem (Moens, 2013)

Let $\mathfrak{g}$ be a Lie algebra, Rad be the solvable radical of $\mathfrak{g}$, d a Leibniz-derivation of $\mathfrak{g}$. Then $d(\mathrm{Rad}) \subseteq \mathrm{Rad}$.

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Theorem (Moens, 2013)

A Lie algebra is nilpotent if and only if it admits an invertible Leibniz-derivation.

Leibniz algebras

An algebra L is called *right Leibniz* if it satisfies the following identity:

$$[[x, y], z] = [x, [y, z]] + [[x, z], y].$$

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Example (Fialowski, Khudoyberdiev, Omirov, 2013)

An example of a right Leibniz algebra admitting an invertible Leibniz derivation, but not nilpotent.

f-Leibniz derivations

Let f be an arrangement of brackets on product of n elements. A linear map $d \in \mathrm{End}_{\mathbb{F}}(A)$ is a *f-Leibniz-derivation of order n* of A, if

$$d((x_1, x_2, \ldots, x_n)_f) = \sum_{i=1}^{n} (x_1, x_2, \ldots, d(x_i), \ldots, x_n)_f.$$

If $(x_1, x_2, \ldots, x_n)_f = ((\ldots(x_1x_2)x_3\ldots)x_n)$, then d is called a *left Leibniz derivation*.
If $(x_1, x_2, \ldots, x_n)_f = (x_1(x_2(x_3\ldots(x_{n-1}x_n)\ldots)))$, then d is called a *right Leibniz derivation*.
If d is a f-Leibniz derivation for all f, then d is called a *Leibniz derivation*.

History

Theorem (Fialowski, Khudoyberdiev, Omirov, 2013)

Solvable and nilpotent radicals of a right Leibniz algebra are invariant under its left Leibniz-derivations.

Theorem (Fialowski, Khudoyberdiev, Omirov, 2013)

A right Leibniz algebra is nilpotent if and only if it admits an invertible left Leibniz-derivation.

Alternative algebras

An algebra A is called *alternative*, if it satisfies the following identities:

$$(xy)y = xy^2, \; x(xy) = x^2y$$

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Theorem (Kaygorodov, Popov, 2014)

The radical (solvable, nilpotent,...) of an alternative algebra is invariant under its left Leibniz-derivations.

Theorem (Kaygorodov, Popov, 2014)

An alternative algebra is nilpotent if and only if it admits an invertible left Leibniz-derivation.

Main results

Jordan algebras

An algebra J is called *Jordan* if it satisfies the following identities:

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$(-1,1)$-algebras

An algebra A is called a $(-1,1)$-*algebra* if it satisfies the following identities:

$$(yx)x = yx^2, \ (x,y,z) + (z,x,y) + (y,z,x) = 0.$$

Malcev algebras

An algebra M is called *Malcev* if it satisfies the following identities:

$$[x, y] = -[y, x], \ J(x, y, [x, z]) = [J(x, y, z), x],$$

where

$$J(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x], y]$$

is the jacobian of elements x, y, z.

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Let $\mathfrak{M}$ be a variety of Jordan, $(-1,1)$ or Malcev algebras.

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Theorem (Kaygorodov & Popov)

Let $A \in \mathfrak{M}$, Rad be its (solvable, nilpotent) radical, d be a left Leibniz-derivation of A. Then $d(\mathrm{Rad}) \subseteq \mathrm{Rad}$.

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Theorem (Kaygorodov & Popov)

An algebra $A \in \mathfrak{M}$ is nilpotent if and only if it admits an invertible left Leibniz-derivation.

Corollaries

Noncommutative Jordan algebras

An algebra A is called a *noncommutative Jordan algebra*, if in it the following operator identities hold:

$$[R_x, L_y] = [L_x, R_y],$$

$$[R_{x \circ y}, L_z] + [R_{y \circ z}, L_x] + [R_{z \circ x}, L_y] = 0.$$

A is noncommutative Jordan algebra if and only if it is flexible (satisfies the identity $(x, y, x) = 0$) and $A^{(+)}$ is a Jordan algebra.

Corollaries

Malcev-admissible algebras

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Corollary

A noncommutative Jordan Malcev-admissible algebra is nilpotent if and only if it admits an invertible Leibniz-derivation.

Corollaries

Mutations

Let $(A, \cdot)$ be an algebra over the field F, $\lambda \in \mathbb{F}$. Then $(A, \cdot_\lambda)$, $x \cdot_\lambda y = \lambda x \cdot y + (1 - \lambda) y \cdot x$ is called a λ-*mutation* of A and denoted by $A^{(\lambda)}$.

Quasiassociative and quasialternative algebras

The algebra A is said to be *quasiassociative* (*quasialternative*), if the algebra $A^{(\lambda)}$ is associative (alternative).

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Quasiassociative and quasialternative algebras

The algebra A is said to be *quasiassociative* (*quasialternative*), if the algebra $A^{(\lambda)}$ is associative (alternative).

Corollary

A quasiassociative (quasialternative) algebra A is nilpotent if and only if it admits an invertible Leibniz-derivation.

Corollaries

Zinbiel algebras

An algebra A is called a *Zinbiel algebra*, if in it the following identity hold:

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Dzhumadildaev (2005) proved that every finite-dimensional Zinbiel algebra over a field of characteristic 0 is nilpotent. Now, the Theorem of Moens is true for Zinbiel algebras.

Counterexamples

Example (Dorofeev, 1970)

$$A = \mathrm{span}(a, b, c, d, e),$$

$$b = -ba = ae = -ea = db = -bd = -c,$$

$$ac = d, bc = e.$$

Counterexamples

Example (Dorofeev, 1970)

$$A = \text{span}(a, b, c, d, e),$$

$$b = -ba = ae = -ea = db = -bd = -c,$$

$$ac = d, bc = e.$$

Algebra A possesses an invertible derivation, but is not nilpotent!

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Right alternative algebras

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Right nilpotency

An algebra A is called *right nilpotent* if there exists $n \in \mathbb{N}$ such that

$$(\ldots(x_1x_2)\ldots)x_n) = 0, \;\; x_i \in A.$$

Theorem

A right alternative algebra is right nilpotent if it admits an invertible Leibniz-derivation.

Counterexamples

The case of prime characteristic

For any algebra A over a field of characteristic p the identity operator id_A is a Leibniz-derivation of order $(p+1)$:

$$(x_1, \ldots, x_{p+1})_q = \sum_{i=1}^{p+1} (x_1, \ldots, x_{p+1})_q$$

for any arrangement of brackets q.

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There are simple Lie algebras of prime characteristic with invertible derivations (Kuznetsov, Benkart, Kostrikin, 1995).

Restricted Lie algebras

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A Moens' theorem for restricted Lie algebras?

Counterexamples

The case of infinite dimension

Let $A = \mathbb{F}[x_1, \dots, x_k]$ be a free [associative, alternative, Lie, Malcev, Jordan, etc] algebra. We define a derivation d as $d(x_i) = x_i$. Then d is an invertible derivation of A, but A is not nilpotent.

n-Lie algebras

An n-ary anticommutative algebra is called Filippov (n-Lie) algebra if it satisfies the n-ary Jacobi identity:

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n].$$

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An example: a simple n-Lie algebra D_{n+1} of dimension $n+1$ with the product

$$[e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}] = (-1)^{n+i-1} e_i.$$

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For any $n > 2$ the algebra D_{n+1} has an invertible derivation.

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For any $n>2$ the algebra D_{n+1} has an invertible derivation.
A simple 8-dimensional ternary Malcev algebra (Pozhidaev) also admits an invertible derivation.

Thank you for your attention!