An IDEAL characterization of cosmological and black hole spacetimes

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Igor Khavkine

Institute of Mathematics
Czech Academy of Sciences, Prague

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University of Ostrava
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Motivation

- The **fundamental symmetries** in General Relativity (GR) are **diffeomorphisms**.
- Two (Lorentzian) spacetime geometries (M, g) and (M, g') may appear to be very different but still be related by a diffeomorphism. The geometries are **isometric**.
- A lot of effort can go into deciding whether two geometries belong to the same (local) isometry class.

Definition (locally isometric)

(M, g) is **locally isometric** to (N, h) if $\forall x \in M \; \exists y \in N$ such that a neighborhood of x is isometric to a neighborhood of y. All such (M, g) constitute the local isometry class of (N, h).

IDEAL Characterization

- **Q:** Given a model geometry (N, h), is it possible to verify when (M, g) belongs to its local isometry class by checking a list of equations
$$T_a[g] = 0 \quad (a = 1, 2, \cdots, A),$$
where each $T_a[g]$ is a **tensor covariantly constructed** from g and its derivatives?
- If Yes, we call this an **IDEAL** (Intrinsic, Deductive, Explicit, ALgorithmic) characterization of the local isometry class of (N, h). Sometimes, also called **Rainich-like**.
- Generalizes to (M, g, Φ), including matter (tensor) fields, if we use covariant tensor equations of the form $T_a[g, \Phi] = 0$.
- An alternative to the Cartan-Karlhede moving-frame-based characterization.
- Also, the **linearizations** $T_a[g + \varepsilon p] = T_a[g] + \varepsilon \dot{T}_a[g; p] + O(\varepsilon^2)$ constitute a **complete list of local gauge invariant observables** $\dot{T}_a[h; -]$ for linearized GR on (N, h).

Examples:

- **Very few examples** of IDEAL characterizations are actually known. To my knowledge, they are either classic or due to the work of Ferrando & Sáez (València).
- Examples:
  - **Constant curvature** (1800s): $R = R[g]$ — Riemann tensor,

$$R_{ijkh} = k(g_{ik}g_{jh} - g_{jk}g_{ih})$$

  - **Schwarzschild** of mass M in 4D (1998): $W = W[g]$ — Weyl tensor,

$$\begin{aligned} R_{ij} &= 0, & S_{ijlm}S^{lm}{}_{kh} + 3\rho S_{ijkh} &= 0, \\ P_{ab} &= 0, & \rho/\alpha^{3/2} - \mathrm{M} &= 0, \end{aligned}$$

$$\text{where} \quad \begin{aligned} \rho &= -(\tfrac{1}{12}\operatorname{tr} W^3)^{1/3}, & S_{ijkh} &= W_{ijkh} - \tfrac{1}{6}(g_{ik}g_{jh} - g_{jk}g_{ih}), \\ \alpha &= \tfrac{1}{9}(\nabla \ln \rho)^2 - 2\rho, & P_{ij} &= ({}^*W)_i{}^k{}_j{}^h \nabla_k \rho \nabla_h \rho. \end{aligned}$$

  - Reissner-Nordström (2002), Kerr (2009), few more (2010, 2017)
- **NEW:** FLRW, inflationary, Schwarzschild-Tangherlini

Current General Strategy

- Fix a class of reference geometries $(M, g_0(\beta))$, with parameters β.
- Suppose there already exists a characterization of this class by the **existence** of tensor fields σ satisfying equations

$$S_a[g, \sigma] = 0,$$

  covariantly constructed from σ, g_{ij}, R_{ijkl} and their covariant derivatives.
- Exploiting the geometry of $(M, g_0(\lambda))$, we try to **find formulas** for $\sigma = \Sigma[g_0]$ covariantly constructed from g_{ij}, R_{ijkl} and their covariant derivatives. If successful, we get an IDEAL characterization of **this class** by

$$T_a[g] := S[g, \Sigma[g]] = 0.$$

- If necessary, find **further covariant expressions** for the parameters $\beta = B[g_0]$, adding equations $B[g] - \beta = 0$ to the above list, until we can IDEALly characterize **individual isometry classes**.

FLRW and Inflationary Spacetimes

(cf. arXiv:1704.05542, with Canepa & Dappiaggi) Let $\dim M = m + 1$.

- (M, g) is (locally) **FLRW** when around every point of M there exist local coordinates $(t, x_1, \ldots, x_m)$, such that
  (a) $g_{ij}(t, x_1, \ldots, x_m) = -(\mathrm{d}t)^2_{ij} + f^2(t) h_{ij}(x_1, \ldots, x_m)$ (**warped product**),
  (b) h_{ij} is of constant curvature (**homogeneous** and **isotropic**), e.g.

$$h_{ij} = \frac{1}{(1 - \alpha r^2)} (\mathrm{d}r)^2_{ij} + r^2 \mathrm{d}\Omega^2_{ij}, \quad \text{with} \quad \mathcal{R}[h] = m(m-1)\alpha.$$

- (M, g, ϕ) is (locally) **inflationary** when it is locally **FLRW** and the local coordinates $(t, x_1, \ldots, x_m)$ can be chosen so that the scalar $\phi = \phi(t)$, while also satisfying the **Einstein-Klein-Gordon** equations

$$R_{ij} - \frac{1}{2}\mathcal{R} g_{ij} = \kappa \left(\nabla_i \phi \nabla_j \phi - \frac{1}{2} g_{ij} [(\nabla\phi)^2 + V(\phi)] \right)$$

with some potential $V(\phi)$ and $\kappa \sim$ Newton's constant.

Warped $m + 1$ Products

Without constant spatial curvature, an FLRW geometry is called a Generalized Robertson Walker (**GRW**) geometry.

Theorem (Sánchez, 1998)

(M, g) is locally GRW iff $\exists U$ — unit timelike vector field satisfying

$$\mathfrak{P}_{jk} := U_{[j}\nabla_{k]}\frac{\nabla^i U_i}{m} = 0, \quad \mathfrak{D}_{ij} := \nabla_i U_j - \frac{\nabla_k U^k}{m}(g_{ij} + U_i U_j) = 0.$$

Theorem (Chen, 2014)

(M, g) is locally GRW iff $\exists v, \mu$ — timelike vector field and scalar satisfying $\nabla_i v_j = \mu g_{ij}$.

In coordinates, $U^i = (\partial_t)^i$ and $v^i = f(t)U^i$, meaning $U = v/\sqrt{-v^2}$.

In GRW **pre-history**, Sánchez's conditions were know and stated as follows: U is unit, geodesic, shear-free, twist-free and has spatially-constant expansion (Ehlers, 1961), (Easley, 1991).

Constant Spatial Curvature

- Convenient to define the Kulkarni-Nomizu product:

$$(A \odot B)_{ijkh} = A_{ik}B_{jh} - A_{jk}B_{ih} - A_{ih}B_{jk} + A_{jh}B_{ik}.$$

- Given Sánchez's U^i, define $\xi := \frac{\nabla^i U_i}{m}$, $\boldsymbol{\eta} := -U^i \nabla_i \xi$.
  Eventually, U is one of normalized $\nabla\mathcal{R}$, $\nabla(\mathcal{B} := R_{ij}R^{ij})$ or $\nabla\phi$.
- Spatial Zero Curvature Deviation (**ZCD**) tensor:

$$\mathfrak{Z}_{ijkh} := R_{ijkh} - \left(g \odot \left[\frac{\xi^2}{2} g - \eta UU\right]\right)_{ijkh}, \quad \zeta := \frac{\mathfrak{Z}_i{}^i{}_k{}^k}{m(m-1)}.$$

- Spatial Constant Curvature Deviation (**CCD**) tensor:

$$\mathfrak{C}_{ijkh} := R_{ijkh} - \left(g \odot \left[\frac{(\xi^2 + \zeta)}{2} g - (\eta - \zeta) UU\right]\right)_{ijkh}.$$

- $\mathfrak{Z}_{ijkh} = 0 \implies$ **flat FLRW.**
  $\mathfrak{C}_{ijkh} = 0$, $U_{[i}\nabla_{j]}\zeta = 0 \implies$ **generic FLRW** (any curvature).

FLRW Scale Factor

- Scale factor derivatives: $\xi = \frac{f'}{f}, \quad \boldsymbol{\eta} = \frac{f''}{f} - \frac{f'^2}{f^2}, \quad \zeta = \frac{\alpha}{f^2}.$
- **Perfect fluid** interpretation: p — pressure, ρ — energy density,

$$R_{ij} - \frac{1}{2}\mathcal{R}g_{ij} + \Lambda g_{ij} = \kappa(\rho + p)U_i U_j + \kappa p g_{ij},$$

  reduces to the **Friedmann** and **acceleration** equations

$$\xi^2 + \zeta = \frac{2}{m(m-1)}\kappa\rho, \quad \boldsymbol{\eta} - \zeta = -\frac{1}{m-1}\kappa(\rho + p).$$

- Flat FLRW with $\zeta = 0$, $(f'/f)' \neq 0$: can find $P(u)$ such that

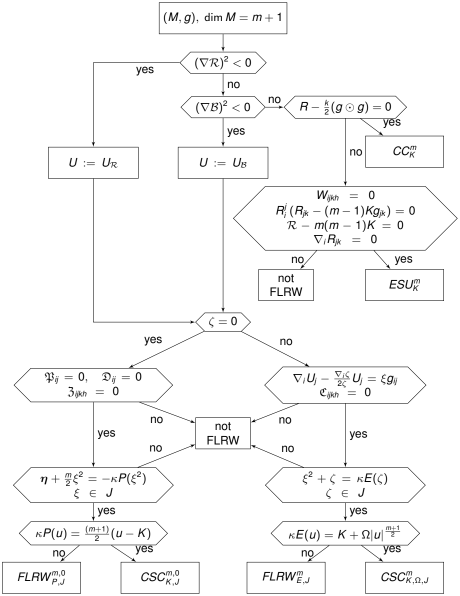
$$\boldsymbol{\eta} + \frac{m}{2}\xi^2 = -\kappa P(\xi^2).$$

- Generic FLRW with $f'/f \neq 0$: can find $E(u)$ such that

$$\xi^2 + \zeta = \kappa E(\zeta).$$

- ODEs in f (with parameter α) **fix scale factor** up to $(f(t), \alpha) \mapsto (Af(t + t_0), A\alpha)$, exhausting isometric $(f(t), \alpha)$ pairs.

Flowchart: FLRW Characterization



Inflationary Scale Factor

- Scale factor derivatives: $\xi = \frac{f'}{f}, \quad \eta = \frac{f''}{f} - \frac{f'^2}{f^2}, \quad \zeta = \frac{\alpha}{f^2}.$
- **Einstein-Klein-Gordon** equations reduce to

$$\xi^2 + \zeta = \kappa \frac{\phi'^2 + V(\phi)}{m(m-1)}, \quad \eta - \zeta = -\kappa \frac{\phi'^2}{(m-1)}.$$

- Flat inflationary with $\zeta = 0$, $\phi' \neq 0$: can find $\Xi(u)$ such that

$$\text{(“Hamilton-Jacobi” eq.)} \quad (\partial_u \Xi(u))^2 - \kappa \frac{m\Xi^2(u)}{(m-1)} + \kappa^2 \frac{V(u)}{(m-1)^2} = 0,$$

$$\phi' = -\frac{(m-1)}{\kappa} \partial_\phi \Xi(\phi), \quad \xi = \Xi(\phi).$$

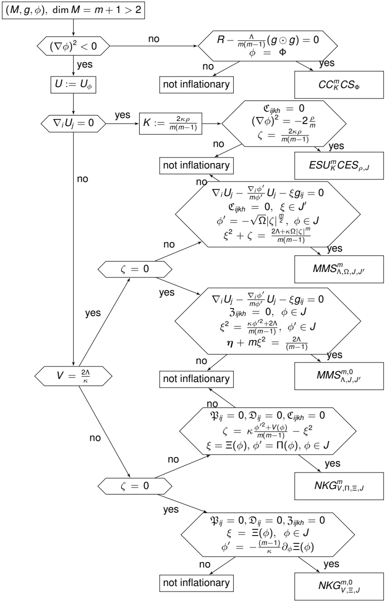
- Generic inflationary with $\phi' \neq 0$: can find $\Xi(u)$, $\Pi(u)$ such that

$$\text{(new?)} \quad \begin{aligned} \Pi \left(\partial_u \Xi + \kappa \frac{\Pi}{(m-1)} \right) - \left(\kappa \frac{\Pi^2 + V}{m(m-1)} - \Xi^2 \right) &= 0, \\ \partial_u \left(\kappa \frac{\Pi^2 + V}{m(m-1)} - \Xi^2 \right) + 2\frac{\Xi}{\Pi} \left(\kappa \frac{\Pi^2 + V}{m(m-1)} - \Xi^2 \right) &= 0, \end{aligned}$$

$$\phi' = \Pi(\phi), \quad \xi = \Xi(\phi).$$

- ODEs in (f, ϕ) **fix scale factor and inflaton** up to $(f(t), \phi(t)) \mapsto (Af(t + t_0), \phi(t + t_0))$, exhausting isometric $(f(t), \phi(t))$ pairs.

Flowchart: Inflationary Characterization



Schwarzschild-Tangherlini Spacetimes

(cf. arXiv:1807.09699) Let $n = m + 2$.
$(M, \bar{g})$ is locally **gST** (generalize Schwarzschild-Tangherlini) when a neighborhood of every point of M is isometric to a portion of the maximal analytic extension of $\bar{g}_{ij}$, where

(a) $\bar{g}_{ij} = g_{ij} + r^2\Omega_{ij}$ (**warped product**),

(b) Ω_{ij} is an m-dimensional **constant curvature** metric, with **sectional curvature** α,

(c) $g_{ij} = -f(r)dt_i dt_j + \frac{1}{f(r)}dr_i dr_j$, with $f(r) = \alpha - \frac{2\mu}{r^{n-3}} - \frac{2\Lambda}{(n-1)(n-2)}r^2$.

These are **higher dimensional** versions of the spherically symmetric Schwarzschild black hole of **mass** μ, **cosmological constant** Λ.

Warped $m+2$ products

Without any constant curvature condition, such warped products can be characterized as follows.

Theorem (García-Parrado, 2006; Ferrando-Sáez, 2010)

$(M,\bar{g})$ is locally $\bar{g}_{ij} = g_{ij} + r^2\Omega_{ij}$ iff $\exists \ell_i, \bar{\Omega}_{ij}$ satisfying

$$\bar{\nabla}_{[i}\ell_{j]} = 0, \quad \ell^i\bar{\Omega}_{ij} = 0,$$

$$\bar{\Omega}_i{}^j\bar{\Omega}_{jk} = \bar{\Omega}_{ik}, \quad \bar{\Omega}_i{}^i = m, \quad \bar{\nabla}_i\bar{\Omega}_{jk} = -2\bar{\Omega}_{i(j}\ell_{k)}.$$

Then $\ell_i = \bar{\nabla}_i \log|r|$, $\Omega_{ij} = r^{-2}\bar{\Omega}_{ij}$ and $g_{ij} = \bar{g}_{ij} - \bar{\Omega}_{ij}$.

In addition, it is well known that a **spherically symmetric vacuum** is locally gST. More generally, we have

Theorem (extended Birkhoff's theorem)

If $(M,\bar{g})$ is a warped $2+m$ product and $\bar{R}_{ij} - \frac{1}{2}\bar{\mathcal{R}}\bar{g}_{ij} + \Lambda\bar{g}_{ij} = 0$, then it is locally gST.

A detailed proof is given in Prop.6 of my arXiv:1807.09699.

gST Characterization

It remains only to express the tensors $\ell_i, \bar{\Omega}_{ij}$ and the parameters Λ, μ, α in terms of the curvature of a gST geometry $(M, \bar{g})$.

Theorem (IK)

Let $(M, \bar{g})$ be a given gST geometry. Then, letting

$$\bar{T}_{ijkl} := \bar{R}_{ijkl} - \frac{2\Lambda}{(n-1)(n-2)}(\bar{g}_{i[k}\bar{g}_{l]j}), \quad \rho := \left[\frac{(\bar{T}\cdot\bar{T}\cdot\bar{T})_{ij}{}^{ij}}{8(n-1)(n-2)(n-3)[(n-2)(n-3)(n-4)+2]}\right]^{\frac{1}{3}},$$

$$\ell_i := -\frac{1}{(n-1)}\frac{\bar{\nabla}_i\rho}{\rho}, \quad A := \ell_i\ell^i + 2\rho + \frac{2\Lambda}{(n-1)(n-2)},$$

$$\bar{\Omega}_{ij} := \frac{2(n-2)^2}{(n-1)(n-4)}\frac{(\bar{T}\cdot\bar{T})_{ik}{}^k{}_j}{(\bar{T}\cdot\bar{T})_{kl}{}^{kl}} + \frac{(n-2)(n-3)}{(n-1)(n-4)}\bar{g}_{ij}, \quad g_{ij} := \bar{g}_{ij} - \bar{\Omega}_{ij},$$

$$Z_{ijkl} := \bar{T}_{ijkl} - \rho\left[\frac{(n-2)(n-3)}{2}(g\odot g)_{ijkl} + (\bar{\Omega}\odot\bar{\Omega})_{ijkl} - (n-3)(g\odot\bar{\Omega})_{ijkl}\right],$$

we have $Z_{ijkl} = 0$, $\ell_i, \bar{\Omega}_{ij}$ satisfy the warped product conditions, while $\Lambda = \frac{2n\bar{\mathcal{R}}}{(n-2)}$, and $A^{n-1}\rho^{-2} = \alpha^{n-1}\mu^{-2}$.

Here, $\alpha^{n-1}\mu$ is an invariant combination of α and μ.

Discussion

- An IDEAL characterization of the (local) isometry class of a physically interesting spacetime is a **natural problem** of geometric interest.
- **FLRW**, **inflationary** and **gST** spacetimes are now on the (currently short) list of IDEAL-ly characterized geometries.
- Does an IDEAL characterization **exist** whenever a Cartan-Karlhede characterization exists?
- Next steps:
  - **Bianchi** (homogeneous) cosmologies?
  - **Kasner** singular solutions?
  - Higher dimensional rotating **Myers-Perry** black holes?

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Thank you for your attention!