Entropy, Landauer's principle and category theory

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Outline

"The general struggle for existence of animate beings is not a struggle for raw materials – these, for organisms, are air, water and soil, all abundantly available – nor for energy which exists in plenty in any body in the form of heat, but a struggle for [negative] entropy, which becomes available through the transition of energy from the hot sun to the cold earth." L. Boltzmann, *The second law of thermodynamics* (Theoretical physics and philosophical problems). Springer-Verlag New York, LLC.

"Let me say first, that if I had been catering for them [physicists] alone I should have let the discussion turn on free energy instead. It is the more familiar notion in this context. But this highly technical term seemed linguistically too near to energy for making the average reader alive to the contrast between the two things." Erwin Schrödinger, *What is Life?*, 1944

Entropy in Thermodynamics

See [1].

Is entropy a measure of disorder?

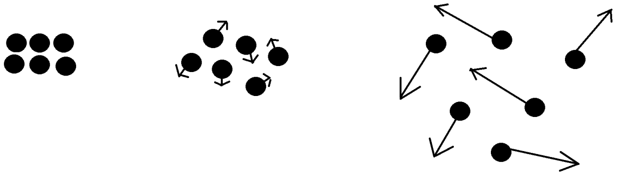
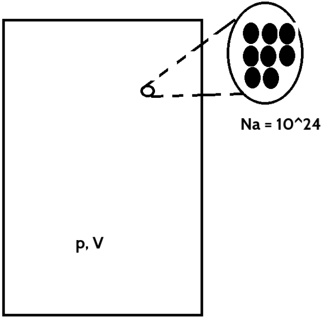


Figure: Where is more order?

Is entropy a measure of disorder?

Figure: Grave of Ludwig Boltzmann [Wikipedia].

Is entropy a measure of reduction?



- Classical description involves position and momenta of $N_A \approx 10^{23}$ particles.
- Thermodynamics reduces the number of parameters (dimension of space of states) to a few p (pressure), V (volume), etc.
- How to describe evolution of the system with such coarse level of description?

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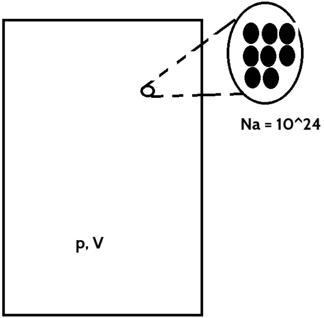

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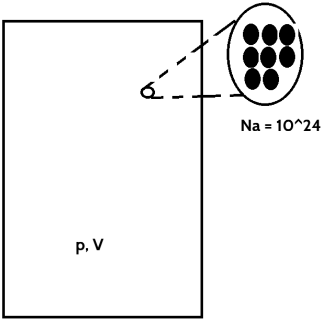

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The first law of thermodynamics

- There is internal energy U as a parameter of the system.
- The state space is described by p_i and v_i and equations of states, e.g., $p_i v_i = n_i R T_i$, that is U and v_i after using all relations.
- There exists a work 1-form $W = \sum_i p_i(U, v_i) dv_i$.
- There exists a heat 1-form $Q = \sum_i Q_i(U, v_i)$.

The First Law of Thermodynamics

The difference $Q - W$ is exact:

$$dU = Q - W. \qquad (2)$$

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Types of processes

Process in Thermodynamics

- A process is a change of sate $x \to y$.
- A *quasi-static* process is represented as a path in a state space.
- A *non-quasi-static* process cannot be represented as a path in a state space.
- An *adiabatic quasi-static* process $Q(\dot{\gamma}) = 0$.
- An *adiabatic non-quasi-static* process Q and W has no sense since there is no path in a state space. $U(x) - U(y)$ has sense.

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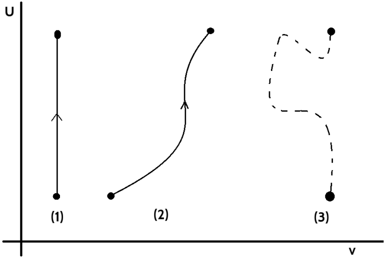

- (1) - heating at constant volume $W(\dot{\gamma}) = 0$, $dU = Q$,
- (2) - quasi-static adiabatic process $Q(\dot{\gamma}) = 0$, $dU = -W$,
- (3) - stirring at constant volume, adiabatic but not quasi-static (no curve in a state space).

The Second Law of Thermodynamics (Caratheodory)

In every neighbourhood of every state x there are states y that are not accessible from x via **quasi-static adiabatic** paths (along which $Q = 0$).

...as a corollary we get:

Existence of entropy

The adiabatic distribution $Q = 0$ is integrable, that is,

$$\frac{Q}{T} = dS, \tag{3}$$

where T is an integrating factor called the absolute temperature.

What is more: The function S is globally defined! (Not only locally integrable.)

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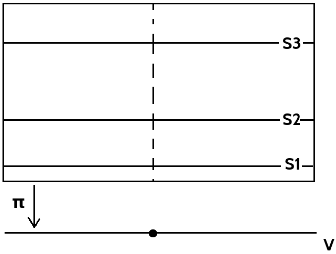

- State space is foliated by constant-entropy (adiabatic) leaves (integral manifolds of distribution $Q = 0$).
- Heating/cooling in constant volume paths are orthogonal $dU = Q$, $W = 0$ to the adiabatic leaves.
- There is a straight way to introduce contact geometry.

The Second Law of Thermodynamics

Theorem

If state y results from x by an adiabatic (quasi-static or not) process, then $S(y) \geq S(x)$.

This takes us closer to the more fundamental meaning of entropy...

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Entropy as an ordering relation

See [2].

The stage

Thermodynamic system

Thermodynamic system is described by a points (equilibrium states) $X, Y, Z, \ldots$ in a state space Γ.

Additional assumptions

Thermodynamic system also fulfils:

- Composition: $(X, Y) \in \Gamma_1 \times \Gamma_2$,
- Scaling: $\mathcal{R}_+ \times \Gamma \to \Gamma$, that is $\lambda\Gamma = \Gamma^{(\lambda)}$, $X \to \lambda X$. (extensive properties like volume, mass are scaled)

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Adiabatic accessibility

Adiabatic accessibility

Y is adiabatically accessible from X, ($X \prec Y$) when there is an adiabatic process that transforms X into Y.

- $X \prec\prec Y$ if $X \prec Y$ and not $Y \prec X$,
- $X \sim Y$ if $X \prec Y$ and $Y \prec X$.

Adiabatic process (Planck) [2]

A state Y is adiabatically accessible from a state X, in symbols $X \prec Y$, if it is possible to change the state from X to Y by means of an interaction with some device consisting of some auxiliary system and a weight, in such a way that the auxiliary system returns to its initial state at the end of the process whereas the weight may have risen or fallen.

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Properties of ordering

- Monotonicity: $X \sim X$
- Transitivity: $X \prec Y$ and $Y \prec Z$ then $X \prec Z$
- Consistency: $X \prec X'$ and $Y \prec Y'$ implies $(X, Y) \prec (X', Y')$
- Scaling invariance: $\lambda > 0$ and $X \prec Y$ implies $\lambda X \prec \lambda Y$
- Splitting recombination: $X \sim (\lambda X, (1 - \lambda) X)$
- Stability: if $(X, \epsilon Z) \prec (Y, \epsilon Z')$ then $X \prec Y$ for $\epsilon \to 0^+$.

The ordering is a 'pullback' of ordering from the ordering of the real numbers...

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Entropy

$S : \Gamma \to \mathcal{R}$ is called entropy if it fulfils

- Monotonicity: $X \prec Y \quad \Leftrightarrow \quad S(X) \leq S(Y)$
- Additivity: $S(X, Y) = S(X) + S(Y)$
- Extensibility: $S(\lambda X) = \lambda S(X)$

...as a conclusion:

- If $X \sim Y$ then $S(X) = S(Y)$.
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Theorem 4 [2]

The relation $\prec$ defines uniquely entropy S up to multiplicative and additive constant.

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Entropy in category theory

See [3].

Poset

Poset (as a category)

Poset (pre-ordered set) $(P, \prec)$ is a set P with total order relation $\prec$ with arrow $x \to y$ when $x \prec y$.

If there is a group acting on poset then relating structure we call G-poset.

Mappings that preserve structure are monotone mappings. More strictly:

Order-preserving mappings

Let $\mathcal{C} = (C, \preccurlyeq)$ and $\mathcal{D} = (D, \sqsubseteq)$ are two posets then the mapping (functor) $F : \mathcal{C} \to \mathcal{D}$ is

- *monotone* if for any $x, y \in C$, if $x \preccurlyeq y$, then $Fx \sqsubseteq Fy$;
- *order-embedding* if for all $x, y \in C$, $x \preccurlyeq y \Leftrightarrow Fx \sqsubseteq Fy$;
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Galois' connection

Galois' connection [4]

Suppose that $\mathcal{C} = (C, \preccurlyeq)$ and $\mathcal{D} = (D, \sqsubseteq)$ are two posets, and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be a pair of functors such that for all $c \in C$, $d \in D$,

$$Fc \sqsubseteq d \quad \Leftrightarrow \quad c \preccurlyeq Gd. \tag{4}$$

Then F and G form a *Galois connection* between $\mathcal{C}$ and $\mathcal{D}$. When this holds, we write $F \dashv G$, and F is said to be the *left adjoint* of G, and G is the *right adjoint* of F.

In other words, the Galois connection is 'a minimal posets mapping that respects their order structure'.

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The Plan

1. state-space (G-Set) + entropy $\to$ total ordering,
2. total ordering $\to$ poset (G-poset) structure,
3. two posets $\to$ Galios (Landauer's) connection between them.

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Ladnauer's functors

Entropy system

The *entropy system* is the object of G-Pos category, which objects are $\mathcal{G} = (\Gamma, \preccurlyeq)$, with preserving ordering group $(\mathbb{R}^{+}, \cdot, 1)$ action, where the (partial or) total order is given by the entropy function $S : \Gamma \to \mathbb{R}$.

Galois connection in terms of entropy

In terms of the entropy the condition

$$Fc \sqsubseteq d \quad \Leftrightarrow \quad c \preccurlyeq Gd \tag{5}$$

is given as

$$S_2(Fc) \leq S_2(d) \Leftrightarrow S_1(c) \leq S_1(Gd). \tag{6}$$

We name the functors F and G the Landauer's functors.

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Processes

Adiabatic reversible/irreversible processes

An entropy system map, that is a poset map $f : \Gamma \to \Gamma$ is reversible at $p \in \Gamma$, if $p = f(p)$, that is $S(p) = S(f(p))$, i.e. f at p preserves entropy. Otherwise f is irreversible at p.

Note:

- This is definition for ANY poset which is induced from 'entropy' structure.
- It should work for any system, not necessary thermodynamic one.

Main Theorem

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For two entropy systems $\mathcal{G}_1 = (\Gamma_1, \preccurlyeq)$ and $\mathcal{G}_2 = (\Gamma_2, \sqsubseteq)$, and functors $F : \mathcal{G}_1 \to \mathcal{G}_2$ and $G : \mathcal{G}_2 \to \mathcal{G}_1$, we have following possibilities for Landauer-Galois' connections

1.

Possibilities	Γ_2 reversible	Γ_2 irreversible
Γ_1 reversible	YES	YES
Γ_1 irreversible	NO	YES

for which $F \dashv G$,

2. transpose above table for $G \dashv F$,

3.

Possibilities	Γ_2 reversible	Γ_2 irreversible
Γ_1 reversible	YES	NO
Γ_1 irreversible	NO	YES

for which F, G are order-embeddings; If the functors are surjective, then they are order-isomorphisms.

Applications

See [3].

Toy example

- Two systems: $\Gamma_1 = (\mathbb{R}_{\geqslant 0}, S)$ and $\Gamma_2 = (\mathbb{N}_{\geqslant 0}, S)$ with $S(x) = x$.
- Consider $F : \Gamma_1 \to \Gamma_2$ defined as $F(z) = \lceil \frac{z}{3} \rceil$ and $G : \Gamma_2 \to \Gamma_1$ given by $G(z) = 3z$.
- We have obviously $F \dashv G$, i.e.

$$\left\lceil \frac{x}{3} \right\rceil \leq y \quad \Leftrightarrow \quad x \leq 3y. \tag{7}$$

  - Take $f : \Gamma_1 \to \Gamma_1$ given by a simple shift $f(z) = z + 0.2$.
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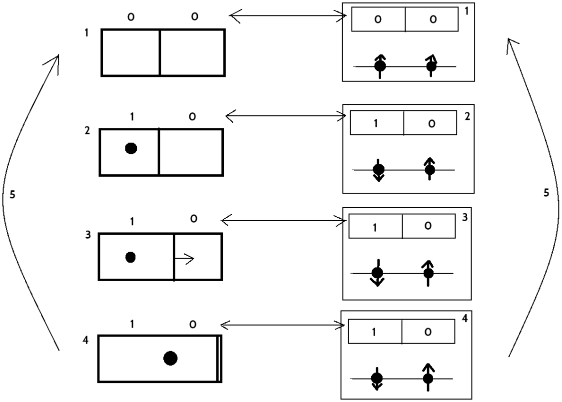
Landuauer's explanation of Maxwell's Demon



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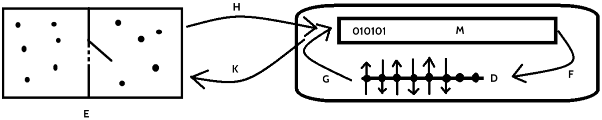
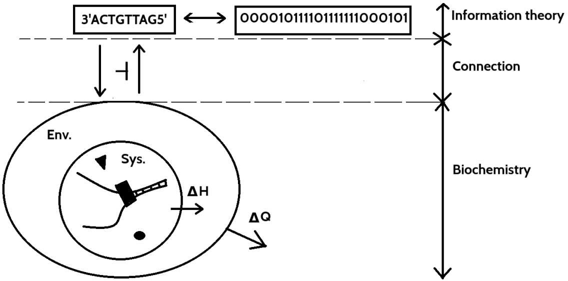


The heat (Landauer's heat) expelled is greater or equal to the work done during expansion:

$$W = k_B T \int_{V/2}^{V} \frac{dV}{V} = k_B T \ln 2. \tag{8}$$

DNA computing



Evolution

- $(P, \subseteq)$ - population with $p \subseteq q$ if the animal species p is also the animal species q in the sense of specificity on the Tree of life;
- $(G, \leq)$ describes gene polls and the ordering has the following meaning: $a \leq b$ when the gene pool b can be generated by the gene pool a.
- $i : P \to G$ sends each population to the gene pool that defines it.
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- Reversing the process, we can define entropy of genes and populations. We can even define Landauer's heat of evolution.
- For more interesting examples, see [5].

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Acknowledgement

I would like to thanks those who made me interested in category theory and abstract approach to science - those who I met in person or know only by their magnificent books and papers. This research was supported by the GACR grant 17-19437S, and the grant MUNI/A/1138/2017 of Masaryk University.

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