Non-Weyl Quantum and Microwave Graphs

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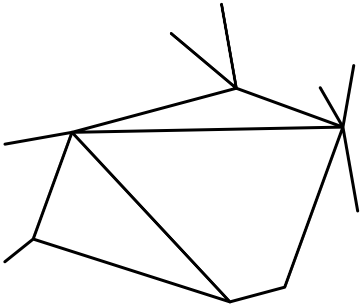
Description of the Model

- configuration space is a graph Γ described by the set of vertices $\mathcal{V}$ and the set of internal edges $\mathcal{L}$ and halflines $\mathcal{L}_\infty$
- corresponding Hilbert space

$$\mathcal{H} = \bigoplus_{(j,n)\in I_{\mathcal{L}}} L^2([0, l_{jn}]) \oplus \bigoplus_{j\in I_{\mathcal{C}}} L^2([0,\infty)).$$

- elements of Hilbert space
$\psi = (f_{jn} : \mathcal{L}_{jn} \in \mathcal{L},\, f_{j\infty} : \mathcal{L}_{j\infty} \in \mathcal{L}_\infty)^T.$



Coupling Conditions

- second order differential operator $H = -\mathrm{d}^2/\mathrm{d}x^2 + V(x)$ – corresponds to the Hamiltonian of a quantum particle (using $\hbar = 1,\ m = 1/2$)
- domain of the Hamiltonian: functions in corresponding Sobolev space $f_{jn} \in W^{2,2}([0, l_{jn}]), f_{j\infty} \in W^{2,2}([0,\infty))$, which fulfil the coupling conditions at the vertices
- for each vertex $\mathcal{X}_j$ one denotes the vector of functional values

$$\Psi_j = (f_1(\mathcal{X}_j), f_2(\mathcal{X}_j), \dots, f_{d_j}(\mathcal{X}_j))^T, \quad d_j = \mathrm{card}\,\mathcal{N}(\mathcal{X}_j)$$

- vector of outgoing derivatives

$$\Psi_j' = (f_1'(\mathcal{X}_j), f_2'(\mathcal{X}_j), \dots, f_{d_j}'(\mathcal{X}_j))^T, \quad d_j = \mathrm{card}\,\mathcal{N}(\mathcal{X}_j)\,.$$

- all possible couplings can be written as

$$(U_j - I)\Psi_j + i(U_j + I)\Psi_j' = 0\,,$$

where U_j's are square $d_j \times d_j$ unitary matrices

- alternative description of the coupling conditions using $d_j \times d_j$ matrices A_j, B_j

$$A_j\Psi_j + B_j\Psi_j' = 0\,,$$

where $d_j \times 2\,d_j$ rectangular matrix (A_j, B_j) has maximal rank and matrix $A_jB_j^*$ is self-adjoint for each vertex

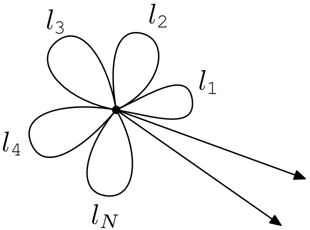
- relationship between A, B and U

$$A = C(U - I)\,, \quad B = iC(U + I)$$

$$U = -(A - iB)^{-1}(A + iB)$$

Flower-like Model

- graph with finitely many internal edges can be described by a model with only one vertex (Kuchment, 2008)
- the actual topology of the graph is encoded in the matrix U



- coupling is described by a $(2N+M)\times(2N+M)$ ($N = \mathrm{card}\,\mathcal{L}$ and $M = \mathrm{card}\,\mathcal{L}_\infty$) unitary block diagonal matrix U consisting of blocks U_j as

$$(U-I)\Psi + i(U+I)\Psi' = 0$$

Example of the Coupling: Standard Coupling Conditions

- also known as Kirchhoff coupling conditions, coupling conditions, sometimes even called Neumann
- continuity of the functional value and sum of outward derivatives disappears

$$f_j := f_{jn}(j) = f_{jm}(j) \quad \text{for all } n, m \in \nu(j)\,,$$
$$\sum_{n \in \nu(j)} f'_{jn}(j) = 0.$$

- corresponding coupling matrices

$$U_j = \frac{2}{d_j} J - I\,,$$

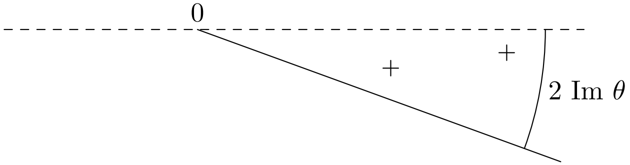
where I is unit matrix and J has all entries equal to one

Resolvent Resonances

- poles of meromorphic continuation of $(H - \lambda\,\mathrm{id})^{-1}$
- another definition: $\lambda = k^2$ is a resolvent resonance if there exists a generalized eigenfunction $f \in L^2_{\mathrm{loc}}(\Gamma)$, $f \not\equiv 0$ satisfying $-f''(x) = k^2 f(x)$ on all edges of the graph and fulfilling the coupling conditions, which on all external edges behaves as $c_j\,\mathrm{e}^{ikx}$.
- external complex scaling: transformation $g_j(x) \to U_\theta g_j(x) = \mathrm{e}^{\theta/2} g_j(x\mathrm{e}^\theta)$ with an imaginary θ
- non-selfadjoint operator H_θ with the domain $f_{jn} \in W^{2,2}([0, l_{jn}])$ and $g_{j\theta} = U_\theta g_j$ with $g_j \in W^{2,2}(\mathcal{L}_{j\infty})$

$$H_\theta \begin{pmatrix} \{g_j\} \\ \{f_{jn}\} \end{pmatrix} = \begin{pmatrix} \{-\mathrm{e}^{-2\theta} g_j''\} \\ \{-f_{jn}'' + V_{jn} f_{jn}\} \end{pmatrix}$$

- essential spectrum of the transformed Hamiltonian rotates into the lower complex halfplane ($\mathrm{e}^{-2\mathrm{Im}\,\theta}[0,\infty)$) and "uncovers" the poles of the resolvent on the second sheet



- resonances – eigenvalues of H_θ

Scattering Resonances

- parametrizing all external edges by $x \in [0, \infty)$
- solutions on the external edges: superposition of incoming and outgoing waves $g_j(x) = c_j \mathrm{e}^{-ikx} + d_j \mathrm{e}^{ikx}$
- motivation for incoming and outgoing waves: time-dependent Schrödinger equation $(-\partial_x^2 - i\partial_t) u_j(x,t) = 0$
- separating variables we can write: $u_j(x,t) = \mathrm{e}^{-itk^2} g_j(x)$, where $g_j(x)$ solves time-independent Schrödinger equation
- hence $u_j(x) = c_j \mathrm{e}^{-ik(x+kt)} + d_j \mathrm{e}^{ik(x-kt)}$
- $S = S(k)$ which maps the vector of amplitudes of the incoming waves $c = \{c_n\}$ into the vector of the amplitudes of the outgoing waves $d = \{d_n\}$
- scattering resonances – complex energies where S diverges

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Equivalence of Resonances

Theorem (Exner, J. L.)

k with k^2 not being eigenvalue of H is a resolvent resonance if and only if it is a scattering resonance.

- there can be eigenvalues with eigenfunctions supported on the compact part of the grap which "cannot be seen by a scattering matrix"
- idea of the proof: condition of solvability of set of equation for resolvent resonances and condition when the system for S-matrix is not solvable are equivalent
- $S(-k) = S(k)^{-1}$ ensures that k with $k^2 \notin \mathbb{R}$ is not at once zero of the denominator of the S-matrix and zero of its nominator

Effective Coupling on the Finite Graph

- graph with semi-infinite edges with coupling given by matrix

$$U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$$

  where U_1 is the $2N \times 2N$ square matrix referring to the compact subgraph, U_4 is the $M \times M$ square matrix related to the exterior part, and U_2 and U_3 are rectangular matrices
- idea: replace coupling at the vertex with semi-infinite edges by an effective coupling eliminating the external variables
- after performing external complex scaling we obtain equations for an effective coupling matrix:

$$\tilde{U}(k) = U_1 - (1-k)U_2[(1-k)U_4 - (k+1)I]^{-1}U_3$$

- $\tilde{U}(k)$ is energy-dependent

Weyl's law for Asymptotics of Eigenvalues

- assume a Laplacian on a Riemannian manifold; the number of eigenvalues with the absolute value smaller than Q in the energy plane is

$$N(Q) = \frac{1}{(2\pi)^n}\omega_n V Q^{n/2} + \mathcal{O}\left(Q^{\frac{n-1}{2}}\right)$$

- for one dimension:

$$N(Q) = \frac{1}{\pi} V Q^{1/2} + \mathcal{O}(1)$$

- for our purposes is better to study the situation in the k-plane ($E = k^2$)
- every eigenvalue is counted twice

$$N(R) = \frac{2}{\pi} V R + \mathcal{O}(1)$$

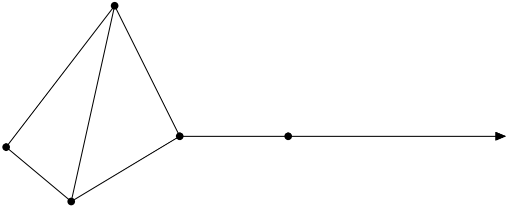
Asymptotics of Resonances

- asymptotical behaviour of the number of resonances (including the eigenvalues) in the circle of radius R for $R \to \infty$
- Weyl's law

$$N(R) = \frac{2V}{\pi} R + \mathcal{O}(1)\,,$$

where V is the size of the graph $V := \sum_{j=1}^{N} l_j$

- simple example of a non-Weyl graph: standard condition on the line



Asymtotics for Standard Conditions

Theorem (Davies, Pushnitski)

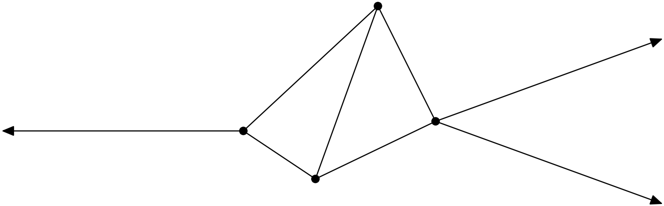
Suppose that Γ *is the graph with standard (Kirchhoff) coupling conditions at all the vertices. Then*

$$N(R) = \frac{2}{\pi} W R + O(1)\,, \quad \text{as } R \to \infty,$$

where the coefficient W *satisfies* $0 \leq W \leq \mathrm{vol}\,(\Gamma)$. *The behaviour is non-Weyl* ($0 \leq W < \mathrm{vol}\,(\Gamma)$) *iff there is a balanced vertex.*

- balanced vertex: number of internal edges is equal to the number of external edges

Asymtotics for Standard Conditions



- idea of the proof: resonances are given by zeros of exponential polynomials $F(k) = \sum_{r=1}^{n} a_r \mathrm{e}^{i\sigma_r k}$
- zeros of such functions are situated in the strips parallel to the real axis and their number is proportional to the difference of the imaginary parts of the exponents
- the graph has Weyl asymptotics iff coefficients of terms $\mathrm{e}^{\pm iVk}$ are nontrivial

Asymptotics for General Conditions

Theorem (Davies, Exner, J.L.)

Graph is non-Weyl iff the effective coupling matrix

$$\tilde{U}(k) = U_1 - (1-k)U_2[(1-k)U_4 - (k+1)I]^{-1}U_3$$

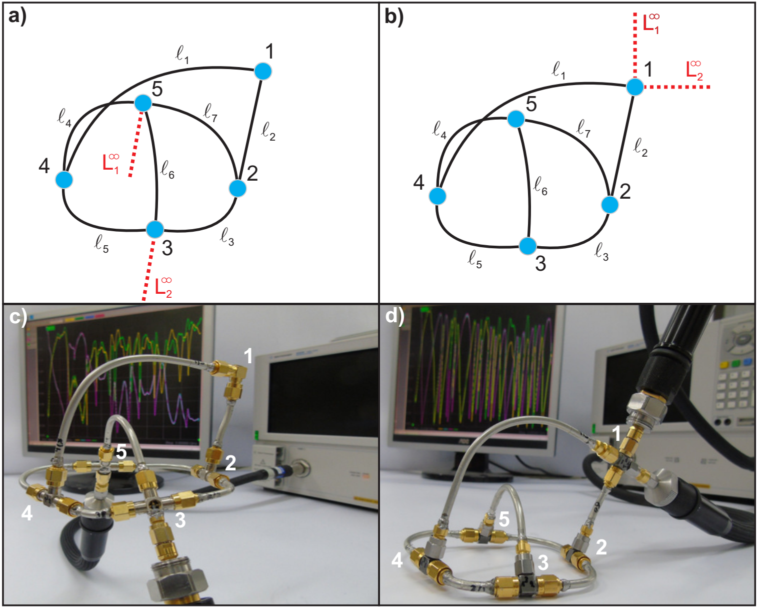
has eigenvalue $\frac{1+k}{1-k}$ *or* $\frac{1-k}{1+k}$.

- the idea of the proof: determining whether the coefficient by $\mathrm{e}^{\pm ikV}$ is zero
- permutation-symmetric coupling: $U = aJ + bI$, with $|b_j| = 1$ and $|b_j + a_j \deg \mathcal{X}_j| = 1$
- the graph with permutation-symmetric coupling is non-Weyl only with standard or "antiKirchhoff" conditions and at least one vertex balanced

Microwave Graphs

- quantum graphs can be simulated by microwave graphs
- the same equation: the telegrapher's equation for microwave graphs is formally the same as Schrödinger equation for quantum graphs
- coaxial cables, inner and outer conductor, the space between them filled with teflon of dielectric constant $\varepsilon \approx 2.06$
- the optical length $\sqrt{\varepsilon}$-multiple of the geometrical length
- losses: $k = k_{\mathrm{R}} + i\beta\sqrt{k_{\mathrm{R}}}$, $k_{\mathrm{R}} = \frac{2\pi}{c} f$, f is frequency, $\beta = 0.00762$, c is the speed of light

Experiment



a)
b)
c)
d)
ℓ_1
ℓ_2
ℓ_3
ℓ_4
ℓ_5
ℓ_6
ℓ_7
L_1^∞
L_2^∞
1
2
3
4
5

Why Has the Non-Weyl Graph Smaller Effective Size?

- let ℓ_2 be the length of the shortest edge emanating from the balanced vertex 1
- let us introduce a fictitious vertex of the degree two with standard coupling at the edge (1,4) at the distance ℓ_2 from the vertex 1 and denote it by 6
- we denote the wavefunctions on the edges (1,6) and (1,2) by $u_1(x)$ and $u_2(x)$, respectively, with $x = 0$ at the vertex 1 and the wavefunctions on the leads L_1^∞ and L_2^∞ by $f_1(x)$ and $f_2(x)$, again with $x = 0$ at the vertex 1
- the coupling condition yields

$$u_1(0) = u_2(0) = f_1(0) = f_2(0)\,,$$
$$u_1'(0) + u_2'(0) + f_1'(0) + f_2'(0) = 0\,. \qquad (1)$$

- introduce symmetrization and antisymmetrization of the previously defined components of wavefunctions

$$\begin{aligned} v_+ &= \tfrac{1}{\sqrt{2}}(u_1 + u_2), \quad v_- = \tfrac{1}{\sqrt{2}}(u_1 - u_2), \\ g_+ &= \tfrac{1}{\sqrt{2}}(f_1 + f_2), \quad g_- = \tfrac{1}{\sqrt{2}}(f_1 - f_2). \end{aligned} \tag{2}$$

- from the coupling conditions at the vertex 1 it follows using $u_1(0) = u_2(0)$ and $f_1(0) = f_2(0)$ that

$$\begin{aligned} v_+(0) &= \tfrac{1}{\sqrt{2}}(u_1(0) + u_2(0)) = \sqrt{2}\, u_1(0), \\ g_+(0) &= \tfrac{1}{\sqrt{2}}(f_1(0) + f_2(0)) = \sqrt{2}\, f_1(0)\,, \\ v_-(0) &= \tfrac{1}{\sqrt{2}}(u_1(0) - u_2(0)) = \tfrac{1}{\sqrt{2}}(u_1(0) - u_1(0)) = 0\,, \\ g_-(0) &= \tfrac{1}{\sqrt{2}}(f_1(0) - f_2(0)) = \tfrac{1}{\sqrt{2}}(f_1(0) - f_1(0)) = 0\,. \end{aligned} \tag{3}$$

- the coupling condition can be in the new functions written (using $u_1(0) = f_1(0)$) as

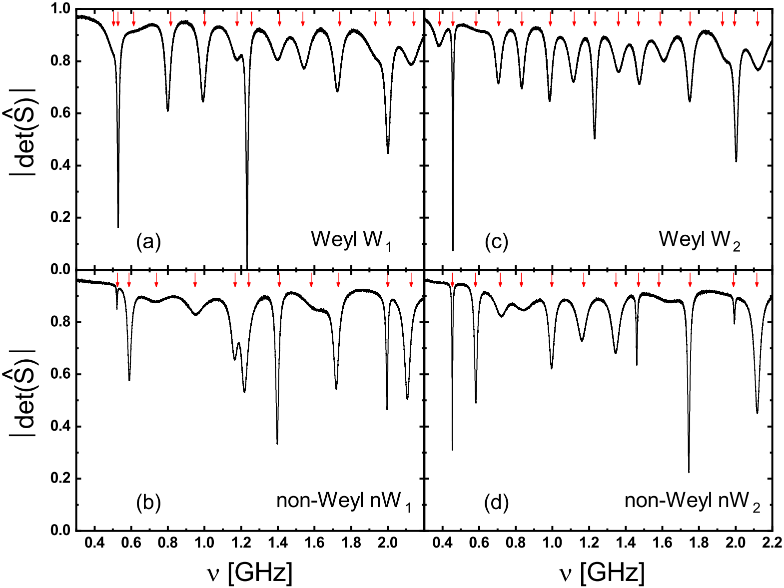
$$v_+(0) = g_+(0)\,, \quad v_+'(0) + g_+'(0) = 0\,, \quad v_-(0) = g_-(0) = 0\,. \tag{4}$$

- the symmetric subspace (v_+ and g_+): standard condition connecting an internal edge and a lead
- in the antisymmetric subspace: Dirichlet condition
- denote by h the wavefunction component on the rest of the graph
- then the map

$$U : (u_1, u_2, f_1, f_2, h)^{\mathrm{T}} \mapsto (v_+, v_-, g_+, g_-, h)^{\mathrm{T}} \tag{5}$$

  is unitary and transforms the "old" Hamiltonian H to the "new" Hamiltonian $H_U = UHU^{-1}$
- the graph for the Hamiltonian H_U connects an internal edge of length ℓ_2 with an external lead by the standard condition
- no interaction – these two edges may be replaced by one external lead – reducing the effective size of the graph by ℓ_2
- a new, more complicated, coupling condition at the real vertex 2 and the fictitious vertex 6 which joins these two vertices; it assures that the effective size is not smaller

Results



|det(Ŝ)|
1.0
0.8
0.6
0.4
0.2
0.0
(a)
Weyl W_1
(c)
Weyl W_2
(b)
non-Weyl nW_1
(d)
non-Weyl nW_2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
ν [GHz]

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Thank you for your attention!

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