Braid Spirit of Young Tableux

Viktor Lopatkin[†]

[†]Laboratory of Modern Algebra and Applications,
St.Petersburg State University

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The Plan

- Young Diagrams
- The Plactic Monoid
- The Schensted column algorithm comes from the Braid Universe
- The Braid Universe
- The End

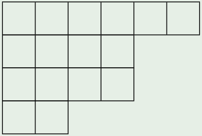
Definition of Young Diagrams

A Young diagram

is a collection of boxes, or cells, arranged in left-justified rows, with a (weakly) decreasing number of boxes in each row. Listing the number of boxes in each row gives a partition of the integer n corresponds to a Young diagram.

Example

The partition of 16 into $6 + 4 + 4 + 2$ corresponds to the Young diagram



Young Tableau

The purpose of writing a Young diagram instead of just the partition, of course, is to put something in the boxes (= numbering or filling).

A Young Tableau

is a filling that is

(1) weakly increasing across each row

(2) strictly increasing down each column

Example

1	2	2	3	3	5
2	3	5	5		
4	4	6	6		
5	6				

Row-insertion (the Schensted operation)

Let us row-insert 2 in

1	2	2	3
2	3	5	5
4	4	6	
5	6		

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$$\begin{array}{|c|c|c|c|}\hline 1&2&2&3\\\hline 2&3&5&5\\\hline 4&4&6\\\cline{1-3} 5&6\\\cline{1-2}\end{array}$$

$$\begin{array}{|c|c|c|c|}\hline 1&2&2&\bullet\\\hline 2&3&5&5\\\hline 4&4&6\\\cline{1-3} 5&6\\\cline{1-2}\end{array} \leftarrow \boxed{2} = \begin{array}{|c|c|c|c|}\hline 1&2&2&2\\\hline 2&3&\bullet&5\\\hline 4&4&6\\\cline{1-3} 5&6\\\cline{1-2}\end{array} \leftarrow \boxed{3}$$

$$\begin{array}{|c|c|c|c|}\hline 1&2&2&2\\\hline 2&3&3&5\\\hline 4&4&\bullet\\\cline{1-3} 5&6\\\cline{1-2}\end{array} \leftarrow \boxed{5} = \begin{array}{|c|c|c|c|}\hline 1&2&2&2\\\hline 2&3&3&5\\\hline 4&4&5\\\cline{1-3} 5&6&6\\\cline{1-3}\end{array}$$

The Product of Tableaux

Thus, the Schensted operation can be used to form a product tableau $\mathbb{Y}_1 \cdot \mathbb{Y}_2$ from any two tableaux $\mathbb{Y}_1$ and $\mathbb{Y}_2$.

Example

$$\begin{ytableau} 1 & 2 & 2 & 3 \\ 2 & 3 & 5 & 5 \\ 4 & 4 & 6 \\ 5 & 6 \end{ytableau} \cdot \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} = \left(\begin{ytableau} 1 & 2 & 2 & 3 \\ 2 & 3 & 5 & 5 \\ 4 & 4 & 6 \\ 5 & 6 \end{ytableau} \leftarrow \begin{ytableau} 2 \end{ytableau} \right) \cdot \begin{ytableau} 1 & 3 \end{ytableau}$$

$$= \begin{ytableau} 1 & 2 & 2 & 2 \\ 2 & 3 & 3 & 5 \\ 4 & 4 & 5 \\ 5 & 6 & 6 \end{ytableau} \cdot \begin{ytableau} 1 & 3 \end{ytableau} = \left(\begin{ytableau} 1 & 2 & 2 & 2 \\ 2 & 3 & 3 & 5 \\ 4 & 4 & 5 \\ 5 & 6 & 6 \end{ytableau} \leftarrow \begin{ytableau} 1 \end{ytableau} \right) \cdot \begin{ytableau} 3 \end{ytableau}$$

$$= \begin{ytableau} 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 5 \\ 3 & 4 & 5 \\ 4 & 6 & 6 \\ 5 \end{ytableau} \cdot \begin{ytableau} 3 \end{ytableau} = \begin{ytableau} 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 3 & 5 \\ 3 & 4 & 5 \\ 4 & 6 & 6 \\ 5 \end{ytableau}$$

Jeu de Taquin = 15 puzzle = (Loydova) Patnáctka

15—14—13.—THE GREAT PRESIDENTIAL PUZZLE.

Thus the set $\mathbb{Y}$ of all Young tableaux is a monoid with respect to Schensted operation.

A Knuth Point of View

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Knuth Transformations

$$\boxed{y}\boxed{z} \cdot \boxed{x} = \begin{array}{|c|c|}\hline x & z \\ \hline y & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad x < y \leq z$$

$$\boxed{x}\boxed{z} \cdot \boxed{y} = \begin{array}{|c|c|}\hline x & y \\ \hline z & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad x \leq y < z$$

The Plactic Monoid

There is a nice way to formalize the Knuth result.

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Let $A = \{1, 2, \ldots, n\}$ with $1 < 2 < \cdots < n$.

Then we call $\mathrm{Pl}(A) := A^*/\equiv$ the plactic monoid[a] on the alphabet set A, where A^* is the free monoid generated by A, $\equiv$ is the congruence of A^* generated by Knuth relations consist of

$$acb = cab\,(a \le b < c), \qquad bca = bac\,(a < b \le c).$$

[a]It was named the "monoпde plaxique" by Lascoux and Schützenberger (1981), who allowed any totally ordered alphabet in the definition. The etymology of the word "plaxique" is unclear; it may refer to plate tectonics ("tectonique des plaques" in French), as elementary relations that generate the equivalence allow conditional commutation of generator symbols: they can sometimes slide across each other (in apparent analogy to tectonic plates), but not freely.

Schensted's column algorithm

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A strictly decreasing word $\mathrm{I} \in \mathrm{A}^*$ is called a column.

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Example

$\mathrm{I} = 875421$. We also write $\mathrm{I} = (1; 1; 0; 1; 1; 0; 1; 1; 0; \ldots; 0)$.

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Let I be a column and let $x \in A$.

$$x \cdot I = \begin{cases} xI, \text{ if } xI \text{ is a column;} \\ I' \cdot y, \text{ otherwise} \end{cases}$$

where y is the rightmost letter in I and is larger than or equal to x, and I' is obtained from I by replacing y with x.

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Example

$$3 \cdot 24678 =$$

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Example

$$3 \cdot 24678 = 3 \cdot 24678 = 23678 \cdot 4.$$

"Lattice" Spirit of Schensted's column algorithm

Consider two columns I and J as ordered sets $\{\mathrm{I}\}$, $\{\mathrm{J}\}$ and set

$$\left\{\mathrm{J}^{\mathrm{I}}\right\} := \{\mathrm{x} \in \{\mathrm{J}\} : \mathrm{y} \rightleftarrows \mathrm{x} = 0 \text{ for any } \mathrm{y} \in \{\mathrm{I}\}\}$$

$$\{\mathrm{J}_{\mathrm{I}}\} := \{\mathrm{x} \in \{\mathrm{J}\} : \mathrm{y} \rightleftarrows \mathrm{x} = 1 \text{ for some } \mathrm{y} \in \{\mathrm{I}\}\}$$

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Let $\mathbb{I}$ be a set of all columns. Introduce binary operations $\vee, \wedge : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ as follows:

$$\{\mathrm{I} \vee \mathrm{J}\} := \mathrm{I} \cup \{\mathrm{J}^{\mathrm{I}}\}, \qquad \{\mathrm{I} \wedge \mathrm{J}\} := \{\mathrm{J}_{\mathrm{I}}\},$$

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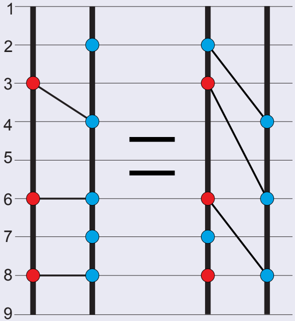
From Schensted's column algorithm it follows that

$$\mathrm{I} \cdot \mathrm{J} = (\mathrm{I} \vee \mathrm{J}) \cdot (\mathrm{I} \wedge \mathrm{J}).$$

Schensted's column algorithm

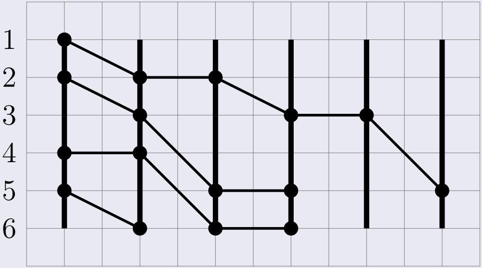


$$\color{red}{863}\cdot\color{blue}{87642} = \color{red}{87632}\cdot\color{blue}{864}.$$

i.e., $\{863\} \vee \{87642\} = \{87632\}$, $\{863\} \wedge \{87642\} = \{864\}$.

a Young Tableau = a Normal Form

1	2	2	3	3	5
2	3	5	5		
4	4	6	6		
5	6				



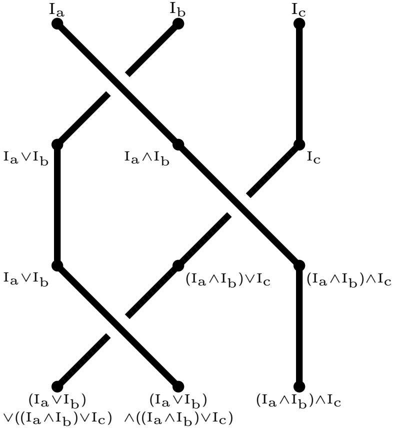
Theorem

[a] For any three columns $\mathrm{I_a}, \mathrm{I_b}, \mathrm{I_c}$, the following formulas are true

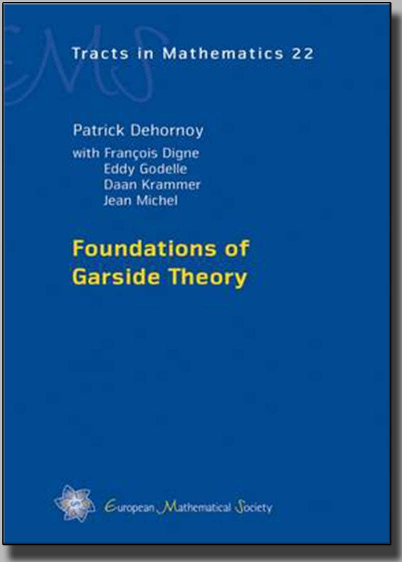
$$\mathrm{I_a} \vee (\mathrm{I_b} \vee \mathrm{I_c}) = (\mathrm{I_a} \vee \mathrm{I_b}) \vee ((\mathrm{I_a} \wedge \mathrm{I_b}) \vee \mathrm{I_c}),$$
$$(\mathrm{I_a} \wedge (\mathrm{I_b} \vee \mathrm{I_c})) \vee (\mathrm{I_b} \wedge \mathrm{I_c}) = (\mathrm{I_a} \vee \mathrm{I_b}) \wedge ((\mathrm{I_a} \wedge \mathrm{I_b}) \vee \mathrm{I_c}),$$
$$(\mathrm{I_a} \wedge (\mathrm{I_b} \vee \mathrm{I_c})) \wedge (\mathrm{I_b} \wedge \mathrm{I_c}) = (\mathrm{I_a} \wedge \mathrm{I_b}) \wedge \mathrm{I_c}.$$

[a] V. Lopatkin, Cohomology rings of the plactic monoid algebra via a Gröbner–Shirshov basis, Journal of Algebra and its Applications. 15(4), (2016), 30pp.

Let the Magic begin!



Patrick Dehornoy is real.



Braids; definitions and concepts.

Definition

The braid group on n strands (denoted B_n), also knows as the Artin braid group, is the group whose elements are equivalence classes of n-braids (e.g. under ambient isotopy), and whose group operation is composition of braids.

We will use as generators for B_n the set of positive crossings, that is, the crossings between two (necessary adjacent) strands, with the front strand having a positive slope. We denote these generators by $\sigma_1, \ldots, \sigma_{n-1}$.

These generators are subject to the following relations:

$$\begin{cases} \sigma_i \sigma_j = \sigma_j \sigma_i, \text{ if } |i-j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}. \end{cases}$$

One obvious invariant of an isotopy of a braid is the permutation it induces on the order of the strands

Example

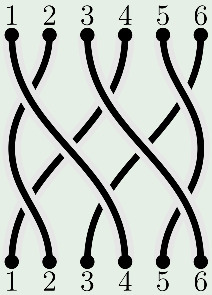


Figure: The simple braid R_π, where $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 6 & 1 & 5 & 3 \end{pmatrix}$

From permutations to braids

We thus have a homomorphism $p : B_n \to \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group. The generator σ_i is mapped to the transposition

$$s_i := (i, i+1) := \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & i+2 & \cdots & n \\ 1 & \cdots & i-1 & i+1 & i & i+2 & \cdots & n \end{pmatrix}$$

We want

to define an inverse map $p^{-1} : \mathfrak{S}_n \to B_n$

W. Thurston point of view, 0

Each permutation $\pi \in \mathfrak{S}_n$ gives rise to a total order relation $\leq_\pi$ on $\{1, \dots, n\}$ with $i \leq_\pi j$ if $\pi(i) < \pi(j)$.

We set

$R_\pi := \{(i,j) \in \{1, \dots, n\} \times \{1, \dots, n\} | i < j, \pi(i) > \pi(j)\}$.

Non-repeating braid.

We call a positive braid non-repeating (=simple) if any two of its strands cross at most once. We define $\mathrm{Div}(\Delta_n) \subset B_n^+$ to be the set of classes of non-repeating braids.

W. Thurston point of view, 1

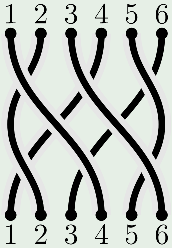
Example

Take the following permutation $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 6 & 1 & 5 & 3 \end{pmatrix} \in \mathfrak{S}_6$.

We have

$$\pi(1) > \pi(2), \pi(1) > \pi(4), \pi(1) > \pi(6), \pi(2) > \pi(4),$$
$$\pi(3) > \pi(4), \pi(3) > \pi(5), \pi(3) > \pi(6), \pi(5) > \pi(6).$$

hence $\mathrm{R}_\pi = \{(1,2),(1,4),(1,6),(2,4),(3,4),(3,5),(3,6),(5,6)\}$,



W.Thurston point of view, 2

Lemma

[a] A set R of pairs (i, j), with $i < j$, comes from some permutation if and only if the following two conditions are satisfied:

- If $(i, j) \in R$ and $(j, k) \in R$, then $(i, k) \in R$.
- If $(i, k) \in R$, then $(i, j) \in R$ or $(j, k) \in R$ for every j with $i < j < k$.

[a]Lemma 9.1.6, D.B.A. Epstein, I.W. Cannon, D.E. Holt, S.V.F. Levy, M.S. Paterson and W.P. Thurston, Word Processing in Groups, Jones and Bartlett Publishers, INC., 1992.

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The identity ϵ is the smallest element of $\mathfrak{S}_{\mathrm{n}}$.

The largest element is $\omega := \begin{pmatrix} 1 & \cdots & \mathrm{n} \\ \mathrm{n} & \cdots & 1 \end{pmatrix}$

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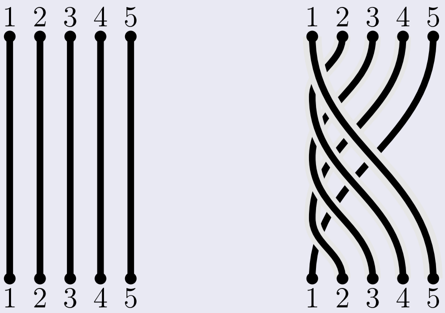
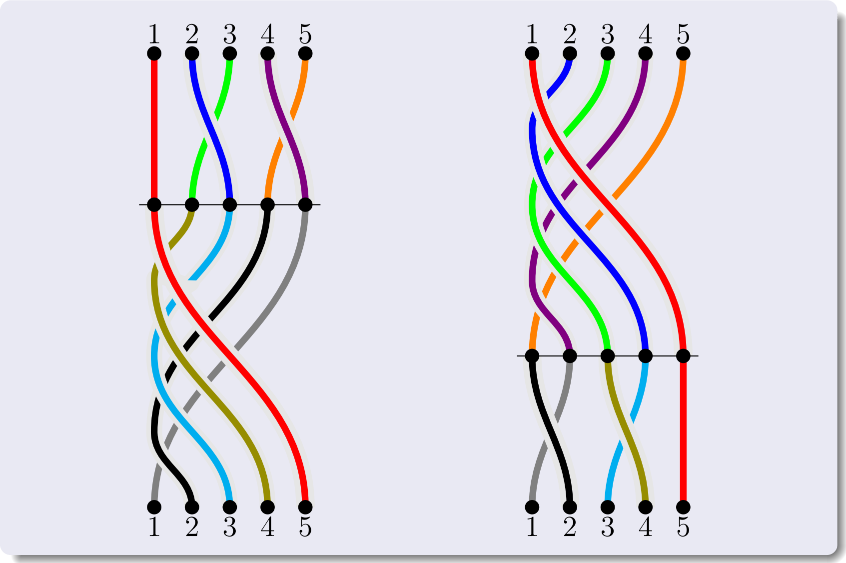


The Garside Braid and the Flip Involution.



Thurston Operations of non-repeating (=simple) braids.

The maximal common braid

For any two permutations $\pi_1, \pi_2 \in \mathfrak{S}_n$, and corresponding simple braid R_{π_1}, R_{π_2}, we define $R_{\pi_1} \wedge R_{\pi_2}$ as follows:

$$R_{\pi_1} \wedge R_{\pi_2} := \{(i,k) \in R_{\pi_1} \cap R_{\pi_2}, \mid (i,j) \in R_{\pi_1} \cap R_{\pi_2} \text{ or } (j,k) \in R_{\pi_1} \cap R_{\pi_2} \text{ for all } j \text{ with } i < j < k\}.$$

The Complement of a braid

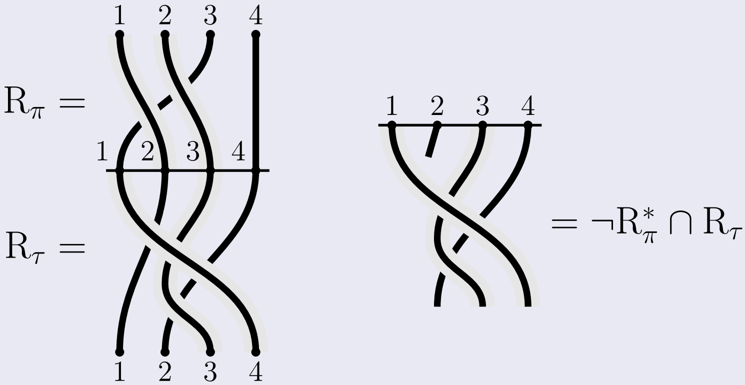
For a permutation $\pi \in \mathfrak{S}_n$, we set

$$\neg R_\pi := R_{\omega\pi} = \Delta \setminus R_\pi.$$

Example

$$\neg R_\epsilon = \Delta_n, \qquad \neg\Delta_n = R_\epsilon.$$

$R_a \cap R_b \neq \varnothing$ does not imply that $R_a \wedge R_b \neq \varnothing$



$$\neg R^*_\pi = \{(1,4),(2,3),(2,4),(3,4)\}$$
$$R_b = \{(1,2),(1,3),(1,4),(3,4)\}$$

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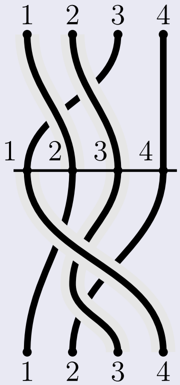
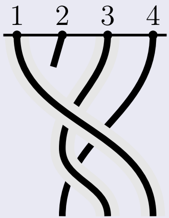
$R_\pi =$

$R_\tau =$





$= \neg R_\pi^* \cap R_\tau$

$$\neg R_\pi^* = \{(1,4),(2,3),(2,4),(3,4)\}$$
$$R_b = \{(1,2),(1,3),(1,4),(3,4)\}$$

The necessary and sufficient condition for a set of pairs to be a non-repeating (=simple) braid.

Lemma

[a] A set R of pairs (i, j), with $i < j$, comes from some permutation if and only if the following two conditions are satisfied:

(1) if $(i, j) \in R$ and $(j, k) \in R$, then $(i, k) \in R$,

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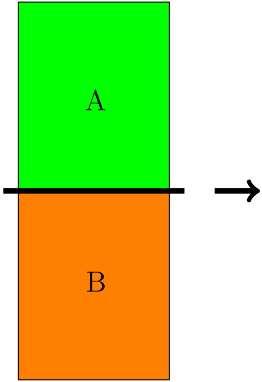
The Thurston Automaton (a sketch)



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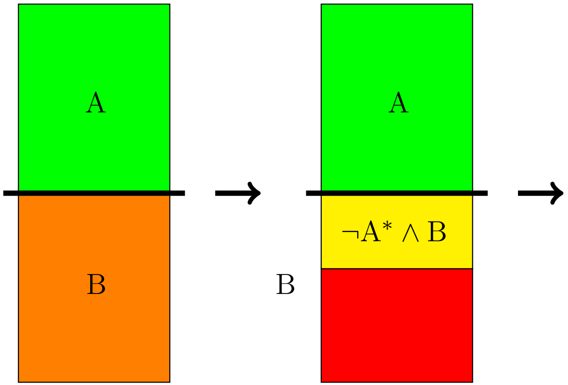

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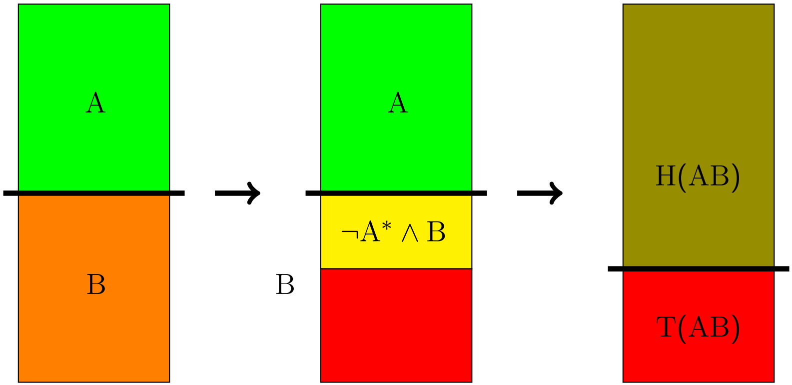

The Thurston Automaton (an example)

Let us consider the following permutations:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix}.$$

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We have

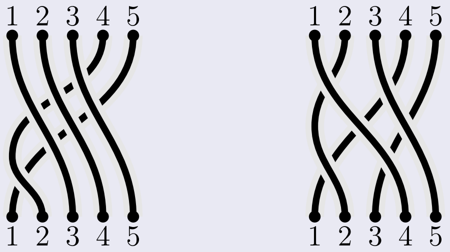
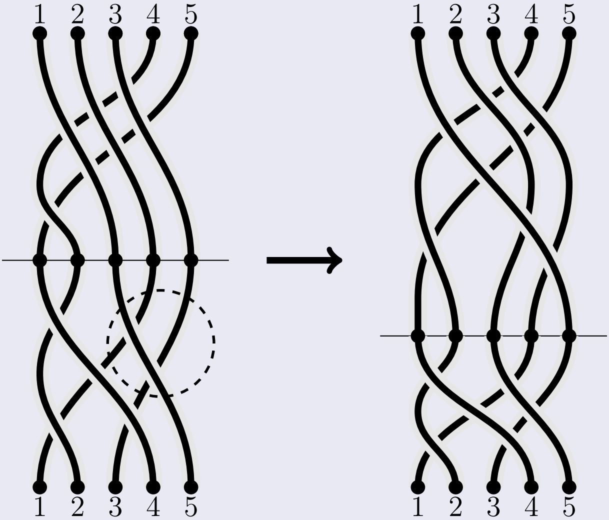


Figure: Simple braids R_π (left), R_τ (right).

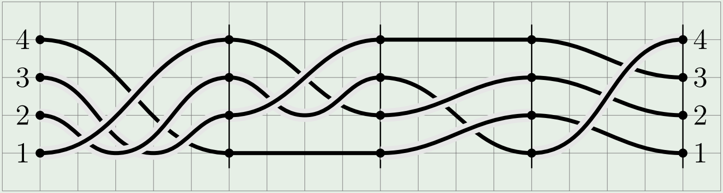
The Thurston Automaton (an example)



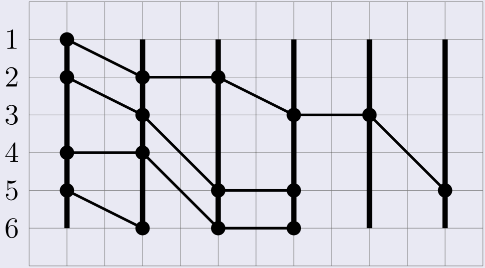
The Greedy Normal Form

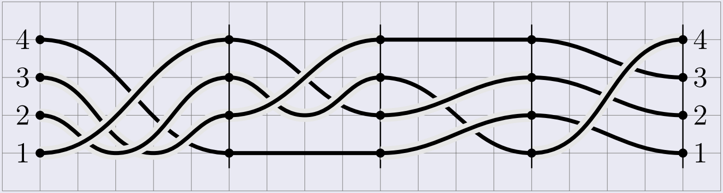
A braid w is in (left) greedy canonical form if it has a decomposition $w = \Delta^m R_{\pi_1} \cdots R_{\pi_k}$ where $\neg R^*_{\pi_{i-1}} \wedge R_{\pi_i} = \varnothing$. for all $1 \le i \le k-1$.

Example



The Greedy Normal Form and Young Tableaux





Thank you!!!

Velmi děkui za Vaši pozornost!!!