Lagrangians with reduced-order Euler–Lagrange equations

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Abstract

Any Lagrangian form of order k obtained by horizontalization of a form of order $k-1$ gives rise to Euler–Lagrange equations of order strictly less than $2k$.

But these are not the only possibilities. For example, with two independent variables, the horizontalization of a first-order 2-form gives a Lagrangian quadratic in the second-order variables; but there are also cubic second-order Lagrangians with third-order Euler–Lagrange equations.

Abstract (continued)

In this talk I shall show first that any Lagrangian of order k with Euler–Lagrange equations of order less than $2k$ must be a polynomial in the k-th order variables of order not greater than the number of different symmetric multi-indices of length k.

I shall then describe a geometrical construction, based on Peter Olver's idea of differential hyperforms, which gives rise to Lagrangians with reduced-order Euler–Lagrange equations.

A version of this talk was given at Ostrava in June 2017. The work has been published in *SIGMA* **14** (2018), 089, 13 pages.

The Euler–Lagrange equations

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The Euler–Lagrange equations are

$$\frac{\partial L}{\partial u^\beta} - \frac{d}{dx}\frac{\partial L}{\partial u^\beta_x} = 0$$

and expanding the total derivative d/dx gives

$$\frac{\partial L}{\partial u^\beta} - \frac{\partial^2 L}{\partial x \partial u^\beta_x} - u^\alpha_x \frac{\partial^2 L}{\partial u^\alpha \partial u^\beta_x} - u^\alpha_{xx}\frac{\partial^2 L}{\partial u^\alpha_x \partial u^\beta_x}$$

In general these equations are second-order, but if L is linear in the variables u^α_x then they are first-order.

The Euler–Lagrange equations (2)

Now suppose there are m independent variables x^i, n dependent variables u^α, and mn derivative variables u_i^α.

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The Euler–Lagrange equations are now

$$\frac{\partial L}{\partial u^\beta} - \frac{d}{dx^j} \frac{\partial L}{\partial u_j^\beta} = 0$$

and expanding the total derivative d/dx^j now gives

$$\frac{\partial L}{\partial u^\beta} - \frac{\partial^2 L}{\partial x^j \partial u_j^\beta} - u_j^\alpha \frac{\partial^2 L}{\partial u^\alpha \partial u_j^\beta} - u_{ij}^\alpha \frac{\partial^2 L}{\partial u_i^\alpha \partial u_j^\beta}$$

In general these equations are second-order, but if L is linear in the variables u_i^α then they are first-order. But ...

The Euler–Lagrange equations (3)

$$\frac{\partial L}{\partial u^\beta} - \frac{\partial^2 L}{\partial x^j \partial u^\beta_j} - u^\alpha_j \frac{\partial^2 L}{\partial u^\alpha \partial u^\beta_j} - u^\alpha_{ij} \frac{\partial^2 L}{\partial u^\alpha_i \partial u^\beta_j}$$

The equations can be first-order even when L is not linear: for example $L = f(x,u)\big(u^\alpha_i u^\beta_j - u^\alpha_j u^\beta_i\big)$

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These Lagrangians come from the geometric construction of *horizontalization* on jet bundles:

> with a fibred manifold $\pi : E \to M$,
> any differential form ω on E
> gives a horizontal differential form $\mathrm{h}(\omega)$ on $J^1\pi$

For instance, $\mathrm{h}\big(du^\alpha \wedge du^\beta\big) = \big(u^\alpha_i u^\beta_j - u^\alpha_j u^\beta_i\big) dx^i \wedge dx^j$

The Euler–Lagrange equations (4)

The same applies for higher-order Lagrangians.

If the Lagrangian L has order k, the Euler–Lagrange equations are generically of order $2k$

$$\sum_{|I|=0}^{k} (-1)^{|I|} \frac{d^{|I|}}{dx^I} \frac{\partial L}{\partial u^\beta_I} = 0$$

where $I \in \mathbb{N}^k$ is a *symmetric multi-index*:

$$\text{if } u^\beta_I = u^\beta_{i_1 i_2 \cdots i_k} \text{ then } I(i) = |\{i_r : i_r = i\}|$$

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The geometry of the multi-index space is important:

$|I| = \sum_{i=1}^m I(i)$ is the *length* of I;

$\|I\|^2 = \sum_{i=1}^m \big(I(i)\big)^2$ is the *square Euclidean norm* of I

Reduced-order Euler–Lagrange equations

$$\sum_{|J|=0}^{k} (-1)^{|J|} \frac{d^{|J|}}{dx^J} \frac{\partial L}{\partial u_J^\beta} = 0$$

Each total derivative d/dx^j increases the order of its argument by one, so that the terms of order $2k$ come from

$$\sum_{|J|=k} (-1)^k \frac{d^{|J|}}{dx^J} \frac{\partial L}{\partial u_J^\beta} \quad \text{and equal} \quad \sum_{|I|=|J|=k} (-1)^k u_{I+J}^\alpha \frac{\partial^2 L}{\partial u_I^\alpha \partial u_J^\beta}$$

The equations will have order less than $2k$ if, and only if, for each multi-index H of length $2k$,

$$\sum_{I+J=H} \frac{\partial^2 L}{\partial u_I^\alpha \partial u_J^\beta} = 0$$

The polynomial condition

Euler–Lagrange equations:

$$\sum_{|J|=0}^{k} (-1)^{|J|} \frac{d^{|J|}}{dx^J} \frac{\partial L}{\partial u_J^\beta} = 0$$

Condition for lower order equations: whenever $|H| = 2k$ then

$$\sum_{I+J=H} \frac{\partial^2 L}{\partial u_I^\alpha \partial u_J^\beta} = 0$$

Theorem

A necessary condition for the Euler–Lagrange equations to have order less than $2k$ is that L is a polynomial in the highest-order derivatives u_I^α, $|I| = k$, of order at most p_k

where p_k is the number of distinct multi-indices of length k

Special case: $k = 2$

$$\sum_{|K|=2} \frac{d^2}{dx^K} \frac{\partial L}{\partial u^\beta_K}$$

has order strictly less than 4, so that

$$\sum_{|J|=|K|=2} u^\alpha_{J+K} \frac{\partial^2 L}{\partial u^\alpha_J \partial u^\beta_K} + \cdots$$

has order strictly less than 4. That means

$$\sum_{J+K=H} \frac{\partial^2 L}{\partial u^\alpha_J \partial u^\beta_K} = 0$$

whenever $|H| = 4$ and $|J| = |K| = 2$.

Special case: $k = 2$ (2)

$$\sum_{J+K=H} \frac{\partial^2 L}{\partial u^\alpha_J \partial u^\beta_K} = 0$$

whenever $|H| = 4 = 2k$ and $|J| = |K| = 2 = k$

Put $K_i = (0, \ldots, 0, 2, 0, \ldots 0)$ and $H_i = (0, \ldots, 0, 4, 0 \ldots 0)$
so that $H_i = K_i + K_i$ ('pure' multi-indices); then

$$\frac{\partial^2 L}{\partial u^\alpha_{K_i} \partial u^\beta_{K_i}} = 0$$

so L is at most linear in $u^\alpha_{K_i}$

Special case: $k = 2$ (3)

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whenever $|H| = 4$ and $|J| = |K| = 2$.

If $K_{ij} = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$ ('mixed' multi-indices)
so that $H = K_{ij} + K_{ij} = (0, \ldots, 2, 0, \ldots, 0, 2, 0, \ldots, 0)$, then

$$\frac{\partial^2 L}{\partial u^\alpha_{K_{ij}} \partial u^\beta_{K_{ij}}} = -\frac{\partial^2 L}{\partial u^\alpha_{K_i} \partial u^\beta_{K_j}} - \frac{\partial^2 L}{\partial u^\alpha_{K_j} \partial u^\beta_{K_i}}$$

(we have turned 'mixed' into 'pure'!)

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$$\frac{\partial^2 L}{\partial u_{K_{ij}}^\alpha \partial u_{K_{ij}}^\beta} = -\frac{\partial^2 L}{\partial u_{K_i}^\alpha \partial u_{K_j}^\beta} - \frac{\partial^2 L}{\partial u_{K_j}^\alpha \partial u_{K_i}^\beta}$$

(we have turned 'mixed' into 'pure'!) so that

$$\frac{\partial^4 L}{\partial u_{K_{ih}}^\gamma \partial u_{K_{ih}}^\delta \partial u_{K_{ij}}^\alpha \partial u_{K_{ij}}^\beta} = 0$$

(with $i \neq h, j$); L is at most cubic in 'overlapping' terms $u_{K_{ij}}^\alpha$

Special case: $k = 2$ (4)

So we know

$$\frac{\partial^2 L}{\partial u^\alpha_{K_i} \partial u^\beta_{K_i}} = 0\,, \quad \frac{\partial^4 L}{\partial u^\gamma_{K_{ih}} \partial u^\delta_{K_{ih}} \partial u^\alpha_{K_{ij}} \partial u^\beta_{K_{ij}}} = 0 \quad (i \neq h, j)$$

If

$$\frac{\partial^r L}{\partial u^{\alpha_1}_{J_1} \partial u^{\alpha_2}_{J_2} \cdots \partial u^{\alpha_r}_{J_r}} \neq 0$$

then the list of multi-indices $(J_1 J_2 \cdots J_r)$ is constrained.

This implies $r \leq \frac{1}{2} m(m+1) = p_2$,
the number of distinct multi-indices of length 2

Proof of the polynomial condition

A necessary condition for the Euler–Lagrange equations to have order less than $2k$ is that L is a polynomial in the highest-order derivatives u^α_I, $|I| = k$, of order at most p_k

Consider

$$\frac{\partial^{p_k+1} L}{\partial u^{\alpha_1}_{J_1} \cdots \partial u^{\alpha_{p_k}}_{J_{p_k}} \partial u^{\alpha_{p_k+1}}_{J_{p_k+1}}}$$

so at least two of the multi-indices must be the same — say $J_1 = J_2$

Use the condition $\displaystyle\sum_{I+J=H} \frac{\partial^2 L}{\partial u^\alpha_I \partial u^\beta_J} = 0$ to put

$$\frac{\partial^{p_k+1} L}{\partial u^{\alpha_1}_{J_1} \partial u^{\alpha_2}_{J_2} \partial u^{\alpha_3}_{J_3} \cdots} = \sum_{\substack{K_1+K_2=J_1+J_2 \\ (K_1,K_2)\neq(J_1,J_2)}} - \frac{\partial^{p_k+1} L}{\partial u^{\alpha_1}_{K_1} \partial u^{\alpha_2}_{K_2} \partial u^{\alpha_3}_{J_3} \cdots}$$

Proof of the polynomial condition (2)

$$\frac{\partial^{p_k+1} L}{\partial u_{J_1}^{\alpha_1} \partial u_{J_2}^{\alpha_2} \partial u_{J_3}^{\alpha_3} \cdots} = \sum_{\substack{K_1+K_2=J_1+J_2 \\ (K_1,K_2)\neq(J_1,J_2)}} - \frac{\partial^{p_k+1} L}{\partial u_{K_1}^{\alpha_1} \partial u_{K_2}^{\alpha_2} \partial u_{J_3}^{\alpha_3} \cdots}$$

But each term on the RHS also has a repeated multi-index! So we can continue . . .

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But eventually, *every* term will have a repeated ‘pure’ multi-index J (where $J(j) = k$ for some j, and $J(j) = 0$ for $i \neq j$)

and then $\displaystyle\sum_{J+J=H} \frac{\partial^2 L}{\partial u_J^\alpha \partial u_J^\beta} = 0$ implies that

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But how do we know?

Proof of the polynomial condition (3)

We use the parallellogram rule for Euclidean norms!

$$2\|x\|^2 \leq 2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2$$

with equality when $y = 0$,

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we have $\|J\|^2 + \|J\|^2 = 2\|J\|^2 < \|K_1\|^2 + \|K_2\|^2$

The sum of the square Euclidean norms in the terms keeps increasing ... and $\|K\|^2 = \sum \big(K(i)\big)^2$ is maximal when K is pure!

So eventually we get $k+1$ pure multi-indices per term

Proof of the polynomial condition (4)

Therefore

$$\frac{\partial^{p_k+1} L}{\partial u_{J_1}^{\alpha_1} \cdots \partial u_{J_{p_k}}^{\alpha_{p_k}} \partial u_{J_{p_k+1}}^{\alpha_{p_k+1}}} = 0$$

so that L is a polynomial in the u_J^α, $|J| = k$, of degree at most p_k.

$\square$

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so that L is a polynomial in the u^α_J, $|J| = k$, of degree at most p_k. $\square$

But this necessary condition is not sufficient: for instance, $L = (u_{xy})^2$ has Euler–Lagrange equations $2u_{xxyy} = 0$

All the Lagrangians with lower-order equations appear to be determinants

Geometrically, determinants arise as the coefficients of wedge products $dx \wedge dy \wedge dz \wedge \cdots$

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Geometrically, determinants arise as the coefficients of wedge products $dx \wedge dy \wedge dz \wedge \cdots$

... but also as coefficients of $dx^2 \wedge dxdy \wedge dy^2 \wedge \cdots$

Differential hyperforms

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Consider hyperforms on jet manifolds $J^k\pi$ that are

- horizontal over M, and
- wedge products of symmetric tensors (all of the same rank)

A (p, q) *hyperform* is a section of $\bigwedge^p S^q T^* M$, pulled back to $J^k\pi$

These are generated over $C^\infty(J^k\pi)$ by $dx^{I_1} \wedge dx^{I_2} \wedge \cdots \wedge dx^{I_p}$ where $dx^I = dx^{i_1} dx^{i_2} \cdots dx^{i_q}$ with $I = (i_1, i_2, \cdots, i_q)$

Affine $(1, q)$ hyperforms

A $(1, q)$ hyperform $(1 \leq q \leq k)$ is a horizontal symmetric tensor $\theta : J^k\pi \to S^q T^* M$

As $J^k\pi \to J^{k-1}\pi$ is an affine bundle, we say that θ is an *affine* $(1, q)$ *hyperform* if its restriction to each fibre of the bundle is an affine map: in coordinates

$$\theta = \sum_{\substack{|I|=k \\ |\mathcal{J}|=q}} \left(\theta^I_{\alpha\mathcal{J}} u^\alpha_I + \theta_{\mathcal{J}}\right) dx^{\mathcal{J}}$$

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These affine $(1, q)$ hyperforms are too general. We shall restrict attention to *special affine* $(1, q)$ *hyperforms*

Special affine $(1, q)$ hyperforms

The affine bundle $J^k\pi \to J^{k-1}\pi$ has associated vector bundle $V\pi \otimes S^kT^*M \to J^{k-1}\pi$

The fibre-affine map θ has an associated fibre-linear 'difference map' $\bar{\theta} : V\pi \otimes S^kT^*M \to S^qT^*M$

We say that θ is a *special affine* $(1, q)$ *hyperform* if there is a tensor $\tilde{\theta} \in V\pi^* \otimes S^{k-q}TM$ such that the difference map $\bar{\theta}$ is given by contraction of elements of its domain $V\pi \otimes S^kT^*M$ with $\tilde{\theta}$.

In coordinates (where θ^I_α are the coordinates of $\tilde{\theta}$)

$$\theta = \sum_{\substack{|I|=k-q \\ |\mathcal{J}|=q}} (\theta^I_\alpha u^\alpha_{I+\mathcal{J}} + \theta_\mathcal{J}) dx^\mathcal{J}$$

Special affine $(1, q)$ hyperforms — example

$$\theta = \sum_{\substack{|I|=k-q \\ |\mathcal{J}|=q}} \left(\theta^I_\alpha u^\alpha_{I+\mathcal{J}} + \theta_{\mathcal{J}}\right) dx^{\mathcal{J}}$$

In the special case where $q = 1$ we have

$$\theta = \sum_{|I|=k-1} \left(\theta^I_\alpha u^\alpha_{I+1_j} + \theta_j\right) dx^j$$

the ordinary horizontalization of the 1-form $\sum_{|I|=k-1} \theta^I_\alpha du^\alpha_I + \theta_j dx^j$

There is no invariant operation of horizontalization for hyperforms when $q \geq 2$; but special affine $(1, q)$ hyperforms generalize the images of the horizontalization operator on ordinary 1-forms

Hyperaffine (p_q, q) hyperforms

A (p_q, q) hyperform ω is a section of the line bundle $\bigwedge^{p_q} S^q T^* M$, pulled back to $J^k\pi$

It is *hyperaffine* if it is generated by wedge products of special affine hyperforms $\theta = \sum_{|I|=k-q, |\mathcal{J}|=q} \left(\theta^I_\alpha u^\alpha_{I+\mathcal{J}} + \theta_{\mathcal{J}}\right) dx^{\mathcal{J}}$

If $\omega = \omega_q dx^{\mathcal{J}_1} \wedge dx^{\mathcal{J}_2} \wedge \cdots \wedge dx^{\mathcal{J}_{p_q}}$ then ω_q is a linear combination of determinants (or their minors)

$$\begin{vmatrix} u^{\alpha_1}_{I_1+\mathcal{J}_1} & u^{\alpha_1}_{I_1+\mathcal{J}_2} & \cdots & u^{\alpha_1}_{I_1+\mathcal{J}_{p_q}} \\ u^{\alpha_2}_{I_2+\mathcal{J}_1} & u^{\alpha_2}_{I_2+\mathcal{J}_2} & \cdots & u^{\alpha_2}_{I_2+\mathcal{J}_{p_q}} \\ \vdots & \vdots & \ddots & \vdots \\ u^{\alpha_{p_q}}_{I_{p_q}+\mathcal{J}_1} & u^{\alpha_{p_q}}_{I_{p_q}+\mathcal{J}_2} & \cdots & u^{\alpha_{p_q}}_{I_{p_q}+\mathcal{J}_{p_q}} \end{vmatrix}$$

What does this have to do with Lagrangians?

A Lagrangian m-form λ defines local Lagrangian functions L by

$$\lambda = L\,dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m$$

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A Lagrangian m-form λ defines local Lagrangian functions L by

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Say that λ is *hyperaffine* if, in any coordinate system,

$$L = \omega_1 + \omega_2 \cdots + \omega_k$$

where each ω_q is the coefficient of a hyperaffine hyperform

$$\omega = \omega_q dx^{\mathcal{J}_1} \wedge dx^{\mathcal{J}_2} \wedge \cdots \wedge dx^{\mathcal{J}_{pq}}$$

This is independent of the coordinate system
In new coordinates $(\tilde{x}, \tilde{u})$, the volume $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m$ changes by the Jacobian determinant $J(\tilde{x}, x)$, whereas each hypervolume $dx^{\mathcal{J}_1} \wedge dx^{\mathcal{J}_2} \wedge \cdots \wedge dx^{\mathcal{J}_{pq}}$ changes by a power of $J(\tilde{x}, x)$

Euler–Lagrange equations of hyperaffine Lagrangians

Theorem

If L is the Lagrangian function of a hyperaffine Lagrangian then the Euler-Lagrange equations have reduced order

Euler–Lagrange equations of hyperaffine Lagrangians

Theorem

If L is the Lagrangian function of a hyperaffine Lagrangian then the Euler-Lagrange equations have reduced order

It is sufficient to show this for a determinant

$$\Delta = \begin{vmatrix} u^{\alpha_1}_{I_1+\mathcal{J}_1} & u^{\alpha_1}_{I_1+\mathcal{J}_2} & \cdots & u^{\alpha_1}_{I_1+\mathcal{J}_{pq}} \\ u^{\alpha_2}_{I_2+\mathcal{J}_1} & u^{\alpha_2}_{I_2+\mathcal{J}_2} & \cdots & u^{\alpha_2}_{I_2+\mathcal{J}_{pq}} \\ \vdots & \vdots & \ddots & \vdots \\ u^{\alpha_{pq}}_{I_{pq}+\mathcal{J}_1} & u^{\alpha_{pq}}_{I_{pq}+\mathcal{J}_2} & \cdots & u^{\alpha_{pq}}_{I_{pq}+\mathcal{J}_{pq}} \end{vmatrix}$$

so write Δ as

$$\Delta = \sum_{\sigma \in \mathfrak{S}_h} \varepsilon_\sigma u^{\alpha_1}_{I_1+\mathcal{J}_{\sigma(1)}} u^{\alpha_2}_{I_2+\mathcal{J}_{\sigma(2)}} \cdots u^{\alpha_h}_{I_h+\mathcal{J}_{\sigma(h)}}$$

Euler–Lagrange equations of hyperaffine Lagrangians (2)

$$\Delta = \sum_{\sigma \in \mathfrak{S}_h} \varepsilon_\sigma u^{\alpha_1}_{I_1+\mathcal{J}_{\sigma(1)}} u^{\alpha_2}_{I_2+\mathcal{J}_{\sigma(2)}} \cdots u^{\alpha_h}_{I_h+\mathcal{J}_{\sigma(h)}}$$

Substituting in the Euler–Lagrange equations gives

$$\sum_{|\mathrm{K}|=k} \frac{d^{|\mathrm{K}|}}{dx^{\mathrm{K}}} \frac{\partial L}{\partial u^\beta_{\mathrm{K}}} = \sum_{\substack{1 \le r,s \le h \\ s \ne r}} \sum_{\sigma \in \mathfrak{S}_h} \delta^{\alpha_r}_\beta \varepsilon_\sigma \Phi_{rs\sigma} u^{\alpha_s}_{I_r+I_s+\mathcal{J}_{\sigma(r)}+\mathcal{J}_{\sigma(s)}}$$

where the coefficients $\Phi_{rs\sigma}$ are

$$\Phi_{rs\sigma} = u^{\alpha_1}_{I_1+\mathcal{J}_{\sigma(1)}} u^{\alpha_2}_{I_2+\mathcal{J}_{\sigma(2)}} \cdots \widehat{r} \cdots \widehat{s} \cdots u^{\alpha_h}_{I_h+\mathcal{J}_{\sigma(h)}}$$

Euler–Lagrange equations of hyperaffine Lagrangians (2)

$$\Delta = \sum_{\sigma \in \mathfrak{S}_h} \varepsilon_\sigma u^{\alpha_1}_{I_1 + \mathcal{J}_{\sigma(1)}} u^{\alpha_2}_{I_2 + \mathcal{J}_{\sigma(2)}} \cdots u^{\alpha_h}_{I_h + \mathcal{J}_{\sigma(h)}}$$

Substituting in the Euler–Lagrange equations gives

$$\sum_{|\mathrm{K}|=k} \frac{d^{|\mathrm{K}|}}{dx^{\mathrm{K}}} \frac{\partial L}{\partial u^\beta_{\mathrm{K}}} = \sum_{\substack{1 \le r,s \le h \\ s \ne r}} \sum_{\sigma \in \mathfrak{S}_h} \delta^{\alpha_r}_\beta \varepsilon_\sigma \Phi_{rs\sigma} u^{\alpha_s}_{I_r + I_s + \mathcal{J}_{\sigma(r)} + \mathcal{J}_{\sigma(s)}}$$

where the coefficients $\Phi_{rs\sigma}$ are

$$\Phi_{rs\sigma} = u^{\alpha_1}_{I_1 + \mathcal{J}_{\sigma(1)}} u^{\alpha_2}_{I_2 + \mathcal{J}_{\sigma(2)}} \cdots \widehat{r} \cdots \widehat{s} \cdots u^{\alpha_h}_{I_h + \mathcal{J}_{\sigma(h)}}$$

Fix $r \ne s$. Given $\sigma \in \mathfrak{S}_h$ put $\tilde{\sigma} = \sigma \circ (r,s) \ne \sigma$.
$\Phi_{rs\sigma} = \Phi_{rs\tilde{\sigma}}$ and $\varepsilon_\sigma = -\varepsilon_{\tilde{\sigma}}$ so all the terms cancel. $\square$

Determinants

Established so far:

- If a Lagrangian function of order k has reduced-order Euler–Lagrange equations then it is a polynomial of order at most p_k in the variables u^α_H $(|H| = k)$;
- Every hyperaffine Lagrangian has reduced-order Euler–Lagrange equations (and is a polynomial with a particular determinant structure)

I conjecture that every Lagrangian with reduced-order Euler–Lagrange equations has this particular determinant structure, and so is hyperaffine.

Determinants (2)

A general polynomial Lagrangian function of order k and degree p_k is

$$L = \sum_{r=0}^{p_k} A^{H_1 H_2 \cdots H_r}_{\alpha_1 \alpha_2 \cdots \alpha_r} u^{\alpha_1}_{H_1} u^{\alpha_2}_{H_2} \cdots u^{\alpha_r}_{H_r}$$

with implicit sums over the indices and multi-indices, and with $|H| = k$

Can this be written as a linear combination of determinants

$$\begin{vmatrix} u^{\alpha_1}_{I_1+\mathcal{J}_1} & u^{\alpha_1}_{I_1+\mathcal{J}_2} & \cdots & u^{\alpha_1}_{I_1+\mathcal{J}_r} \\ u^{\alpha_2}_{I_2+\mathcal{J}_1} & u^{\alpha_2}_{I_2+\mathcal{J}_2} & \cdots & u^{\alpha_2}_{I_2+\mathcal{J}_r} \\ \vdots & \vdots & \ddots & \vdots \\ u^{\alpha_r}_{I_r+\mathcal{J}_1} & u^{\alpha_r}_{I_r+\mathcal{J}_2} & \cdots & u^{\alpha_r}_{I_r+\mathcal{J}_r} \end{vmatrix} \qquad \begin{array}{l} |\mathcal{J}| = q, \quad |I| = k - q \\ 1 \le q \le k, \quad 0 \le r \le p_q\,? \end{array}$$

Determinants (3)

Consider homogeneous polynomials $A^{H_1 H_2 \cdots H_r}_{\alpha_1 \alpha_2 \cdots \alpha_r} u^{\alpha_1}_{H_1} u^{\alpha_2}_{H_2} \cdots u^{\alpha_r}_{H_r}$

In the case $r = 2$ there is a constructive proof

Determinants (3)

Consider homogeneous polynomials $A^{H_1 H_2 \cdots H_r}_{\alpha_1 \alpha_2 \cdots \alpha_r} u^{\alpha_1}_{H_1} u^{\alpha_2}_{H_2} \cdots u^{\alpha_r}_{H_r}$

In the case $r = 2$ there is a constructive proof

Partition the quadratic terms by $H_1 + H_2 = H$ and put

$$\psi_H = \sum_{H_1+H_2=H} A^{H_1 H_2}_{\alpha_1 \alpha_2} u^{\alpha_1}_{H_1} u^{\alpha_2}_{H_2}$$

Choose a term $A^{K_1 K_2}_{\alpha_1 \alpha_2} u^{\alpha_1}_{K_1} u^{\alpha_2}_{K_2}$ arbitrarily, so from E–L we have

$$A^{K_1 K_2}_{\alpha_1 \alpha_2} = \sum_{H_1+H_2=H,(H_1,H_2)\neq(K_1,K_2)} -A^{H_1 H_2}_{\alpha_1 \alpha_2}$$

and so

$$\psi_H = \sum_{H_1+H_2=H} A^{H_1 H_2}_{\alpha_1 \alpha_2} (u^{\alpha_1}_{H_1} u^{\alpha_2}_{H_2} - u^{\alpha_1}_{K_1} u^{\alpha_2}_{K_2})$$

Determinants (4)

For cubic and higher terms, there is no obvious algorithm to give an explicit construction

(although ad-hoc methods work for all examples investigated)

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(although ad-hoc methods work for all examples investigated)

A possible approach would use an abstract dimension argument:

The number of variables u_I^α, $|I| = k$, is known,
and so the dimension of the space of homogeneous polynomials of degree r is also known

The number of E–L constraints for quadratic polynomials is known, so the number of constraints for degree r polynomials can in principle be calculated

The theorem will be proved if there are enough independent $r \times r$ determinants of the correct type