A quaternionic treatment of inhomogeneous Cauchy–Riemann type systems in some traditional theories

Baruch Schneider

University of Ostrava

Motivation

Inhomogeneous Cauchy–Riemann equation in $\mathbb{C}$

$$\frac{\partial f}{\partial \bar{z}} = g,$$

general solution:

$$f = T[g] + h,$$

where $h \in Hol(\Omega, \mathbb{C})$ and

$$T : g \in C(\overline{\Omega}, \mathbb{C}) \to -\frac{1}{\pi} \int_{\Omega} \frac{g(\tau)}{\tau - z} dV_{\Omega}.$$

Motivation

$\Omega \subset \mathbb{R}^3$; $\Gamma = \partial\Omega$; $0 \leq \theta < 2\pi$.

$$\begin{cases} -\dfrac{\partial f_1}{\partial x_1} + \left(\dfrac{\partial f_2}{\partial x_2} - \dfrac{\partial f_3}{\partial x_3}\right)\sin\theta - \left(\dfrac{\partial f_3}{\partial x_2} + \dfrac{\partial f_2}{\partial x_3}\right)\cos\theta = g_0, \\ \left(\dfrac{\partial f_3}{\partial x_3} - \dfrac{\partial f_2}{\partial x_2}\right)\cos\theta - \left(\dfrac{\partial f_3}{\partial x_2} + \dfrac{\partial f_2}{\partial x_3}\right)\sin\theta = g_1, \\ -\dfrac{\partial f_3}{\partial x_1} + \dfrac{\partial f_1}{\partial x_3}\sin\theta + \dfrac{\partial f_1}{\partial x_2}\cos\theta = g_2, \\ \dfrac{\partial f_2}{\partial x_1} - \dfrac{\partial f_1}{\partial x_3}\cos\theta + \dfrac{\partial f_1}{\partial x_2}\sin\theta = g_3, \end{cases}$$

with unknown functions f_m and a known ones g_n, which belong to $C^1(\Omega, \mathbb{C}) \cap C^0(\Omega \cup \Gamma, \mathbb{C})$, $m = 1, 2, 3$ and $n = 0, 1, 2, 3$.

Motivation

inhomogeneous div-rot system: $\theta = 0$, $f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k} \leftrightarrow f_1\mathbf{i} + f_3\mathbf{j} + f_2\mathbf{k}$

$$\begin{cases} -\dfrac{\partial f_1}{\partial x_1} - \dfrac{\partial f_2}{\partial x_2} - \dfrac{\partial f_3}{\partial x_3} = g_0, \\ \dfrac{\partial f_2}{\partial x_3} - \dfrac{\partial f_3}{\partial x_2} = g_1, \quad \dfrac{\partial f_3}{\partial x_1} - \dfrac{\partial f_1}{\partial x_3} = g_2, \quad \dfrac{\partial f_1}{\partial x_2} - \dfrac{\partial f_2}{\partial x_1} = g_3, \end{cases}$$

Motivation

$$\begin{cases} \mathrm{div}\vec{f} = g_0, \\ \mathrm{rot}\vec{f} = \vec{g}, \end{cases}$$

where $\vec{f} := f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$

Motivation

particular case of the inhomogeneous Cimmino system: $\theta = \frac{\pi}{2}$

$$\begin{cases} \dfrac{\partial f_2}{\partial x_2} - \dfrac{\partial f_1}{\partial x_1} - \dfrac{\partial f_3}{\partial x_3} = g_0, \\ -\dfrac{\partial f_2}{\partial x_3} - \dfrac{\partial f_3}{\partial x_2} = g_1, \quad -\dfrac{\partial f_3}{\partial x_1} + \dfrac{\partial f_1}{\partial x_3} = g_2, \quad \dfrac{\partial f_1}{\partial x_2} + \dfrac{\partial f_2}{\partial x_1} = g_3, \end{cases}$$

Motivation

Riesz system: $\theta = \pi$; $f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k} \leftrightarrow f_0\mathbf{i} + f_2\mathbf{j} + f_1\mathbf{k}$ **as well as** $(x_1, x_2, x_3) \leftrightarrow (x_0, x_1, x_2)$.

$f : (x_0, x_1, x_2) \in \mathbb{R}^3 \to \mathrm{span}_{\mathbb{R}}\{1, \mathbf{i}, \mathbf{j}\}$

$$\begin{cases} \dfrac{\partial f_0}{\partial x_0} - \dfrac{\partial f_1}{\partial x_1} - \dfrac{\partial f_2}{\partial x_2} = 0, \\ \dfrac{\partial f_0}{\partial x_1} + \dfrac{\partial f_1}{\partial x_0} = 0, \quad \dfrac{\partial f_0}{\partial x_2} + \dfrac{\partial f_2}{\partial x_0} = 0, \quad \dfrac{\partial f_1}{\partial x_2} - \dfrac{\partial f_2}{\partial x_1} = 0, \end{cases}$$

Motivation

Riesz system

$$\begin{cases} \mathrm{div}\bar{f} = 0, \\ \mathrm{rot}\bar{f} = 0, \end{cases}$$

where $\bar{f} := f_0 - f_1\mathbf{i} - f_2\mathbf{j}$.

Motivation

time-harmonic relativistic Dirac bispinors theory: $\theta = \frac{3\pi}{2}$

$$\begin{cases} -\dfrac{\partial f_1}{\partial x_1} - \dfrac{\partial f_2}{\partial x_2} + \dfrac{\partial f_3}{\partial x_3} = g_0, \\ \dfrac{\partial f_2}{\partial x_3} + \dfrac{\partial f_3}{\partial x_2} = g_1, \quad -\dfrac{\partial f_3}{\partial x_1} - \dfrac{\partial f_1}{\partial x_3} = g_2, \quad -\dfrac{\partial f_1}{\partial x_2} + \dfrac{\partial f_2}{\partial x_1} = g_3. \end{cases}$$

Problems

- How to have general solution of inhomogeneous Cauchy-Riemann type systems in an appropriate quaternionic setting ?
- How to define Hilbert formulas ?

Notations

Let $\mathbb{H} := \mathbb{H}(\mathbb{R})$ and $\mathbb{H}(\mathbb{C})$ denote the sets of real and complex quaternions respectively.
An element in the $\mathbb{H}$ (or $\mathbb{H}(\mathbb{C})$) will be denoted by

$$a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$

where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and satisfy the following multiplication rules $\mathbf{i} \cdot \mathbf{j} = -\mathbf{j} \cdot \mathbf{i} = \mathbf{k}$; $\mathbf{j} \cdot \mathbf{k} = -\mathbf{k} \cdot \mathbf{j} = \mathbf{i}$; $\mathbf{k} \cdot \mathbf{i} = -\mathbf{i} \cdot \mathbf{k} = \mathbf{j}$.

The coefficients $\{a_k\} \subset \mathbb{R}$ if $a \in \mathbb{H}$ and $\{a_k\} \subset \mathbb{C}$ if $a \in \mathbb{H}(\mathbb{C})$.

Denote the complex imaginary unit in $\mathbb{C}$ by i as usual. By definition, i commutes with all the quaternionic imaginary units $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

For $a \in \mathbb{H}$ (or $a \in \mathbb{H}(\mathbb{C})$)

$$\vec{a} := a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

or we can write

$$a = a_0 + \vec{a},$$

$\mathrm{Sc}(a) := a_0$ will be called the scalar part and $\mathrm{Vec}(a) := \vec{a}$ the vector part of the quaternion. Also, if $a \in \mathbb{H}(\mathbb{C})$, then

$$a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = a^{(1)} + ia^{(2)} = a^{(1)} + a^{(2)}i,$$

where $\{a^{(1)}, a^{(2)}\} \subset \mathbb{H}$,

Elements of quaternionic analysis

For any $a, b \in \mathbb{H}(\mathbb{C})$:

$$a\,b := a_0\,b_0 - \langle \vec{a}, \vec{b} \rangle + a_0\,\vec{b} + b_0\,\vec{a} + [\vec{a}, \vec{b}],$$

where

$$\langle \vec{a}, \vec{b} \rangle := \sum_{k=1}^{3} a_k\,b_k, \quad [\vec{a}, \vec{b}] := \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

If $a_0 = b_0 = 0$ then $a\,b := -\langle \vec{a}, \vec{b} \rangle + [\vec{a}, \vec{b}]$.
Define:

$$\bar{a} := a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}.$$

If $\{a, b\} \subset \mathbb{H}$, then $\overline{a \cdot b} = \bar{b} \cdot \bar{a}$.

Elements of quaternionic analysis

$$a\bar{a} = \bar{a}a = \sum_{k=0}^{3} a_k^2 =: |a|^2 \in \mathbb{R}.$$

Note that for $a \in \mathbb{H}(\mathbb{C})$:

$$a\bar{a} = \bar{a}a = \sum_{k=0}^{3} a_k^2 = \left|a^{(1)}\right|^2 - \left|a^{(2)}\right|^2 + i\left(a^{(1)}\overline{a^{(2)}} + a^{(2)}\overline{a^{(1)}}\right)$$

$$= \left|a^{(1)}\right|^2 - \left|a^{(2)}\right|^2 + 2i\left\langle a^{(1)}, a^{(2)}\right\rangle_{\mathbb{R}^4} \in \mathbb{C},$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^4}$ is the usual scalar product.

Denote by $\mathfrak{S}$ the set of zero divisors from $\mathbb{H}(\mathbb{C})$ and by $G\mathbb{H}(\mathbb{C})$ the subset of invertible elements from $\mathbb{H}(\mathbb{C})$. If $a \notin \mathfrak{S} \cup \{0\}$ then $a^{-1} := \frac{\bar{a}}{(a\bar{a})}$ is the inverse of the complex quaternion a. Note that $G\mathbb{H}(\mathbb{C}) = \mathbb{H}(\mathbb{C}) \setminus (\mathfrak{S} \cup \{0\})$.

Elements of quaternionic analysis

$\Omega \subset \mathbb{R}^3; \Gamma = \partial\Omega$. We will consider $\mathbb{H}(\mathbb{C})$–valued functions defined on Ω:

$$f : \Omega \longrightarrow \mathbb{H}(\mathbb{C}).$$

for them we have the following representations:

$$i)\ f = f_0 + f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k},$$

where $f_k : \Omega \longrightarrow \mathbb{C}$;

$$ii)\ f = f^{(1)} + f^{(2)} i,$$

where $f^{(j)} : \Omega \longrightarrow \mathbb{H}$;

$$iii)\ f = f_0 + \vec{f},$$

where f_0 takes values on $\mathbb{C}$ and $\vec{f}$ is a $\mathbb{C}^3$–valued function.

Elements of quaternionic analysis

On $C^1(\Omega, \mathbb{H}(\mathbb{C}))$ an operator ${}^{\psi}D$ by the formula

$$ {}^{\psi}D := \frac{\partial}{\partial x_1}\psi^1 + \frac{\partial}{\partial x_2}\psi^2 + \frac{\partial}{\partial x_3}\psi^3, $$

where $\psi := \{\psi^1, \psi^2, \psi^3\}$, $\psi^k \in \mathbb{H}$. Denote by $\bar{\psi} := \{\bar{\psi}^1, \bar{\psi}^2, \bar{\psi}^3\}$.
On $C^2(\Omega, \mathbb{H}(\mathbb{C}))$ the following equalities

$$ {}^{\psi}D\,{}^{\bar{\psi}}D = {}^{\bar{\psi}}D\,{}^{\psi}D = \Delta, $$

hold if and only if

$$ \psi^n \bar{\psi}^m + \psi^m \bar{\psi}^n = 2\delta_{nm} $$

for any $n, m \in \mathbb{N}_3$; $\Delta[f] := \Delta_{\mathbb{R}^3}[f_0] + \Delta_{\mathbb{R}^3}[f_1] + \Delta_{\mathbb{R}^3}[f_2] + \Delta_{\mathbb{R}^3}[f_3]$.

ψ-Hyperholomorphic functions

Definition

Any solution to the homogeneous equation ${}^{\psi}D[f](\xi) = 0$ for each $\xi \in \Omega$ is called a (left) ψ-hyperholomorphic function in Ω.

Any solution to the homogeneous equation $[g]^{\psi}D(\xi) = 0$ for each $\xi \in \Omega$ is called a right ψ-hyperholomorphic function in Ω.

Notaition

$$ {}^{\psi}\mathcal{M}(\Omega, \mathbb{H}(\mathbb{C})) := \{f \in C^1(\Omega, \mathbb{H}(\mathbb{C})) : \ {}^{\psi}D[f] = 0\} $$
$$ \mathcal{M}^{\psi}(\Omega, \mathbb{H}(\mathbb{C})) := \{g \in C^1(\Omega, \mathbb{H}(\mathbb{C})) : \ [g]^{\psi}D = 0\}. $$

Definition

If the function $f := f_0 + \vec{f}$ is a ψ-hyperholomorphic function then the pair $(f_0, \vec{f})$ is called a pair of conjugate harmonic functions.

Quaternionic Stokes formula

$$\int_\Gamma g(\xi)\cdot n_\psi(\xi)\cdot f(\xi)\, d\Gamma_\xi = \int_\Omega ([g]^\psi D(\xi)\cdot f(\xi) + g(\xi)\cdot {}^\psi D[f](\xi))\, d\xi,$$

$n_\psi = n_1\psi^1 + n_2\psi^2 + n_3\psi^3$, $(n_1, n_2, n_3) \in \mathbb{R}^3$.

Definition of Cauchy kernel:

We will call the expression

$$\mathcal{K}_{\psi}(q) = -\frac{1}{4\pi|q|^3}\sum_{n=1}^{3} q_n \bar{\psi}^n.$$

the Cauchy kernel.

The convolution integral operator

$${}^{\psi}T[f](q) := \int_{\Omega} \mathcal{K}_{\psi}(q-\xi)\cdot f(\xi)\, dV_{\xi}, \;\; q \in \mathbb{R}^3,$$

such that

$${}^{\psi}D[{}^{\psi}T[f]](q) = f(q), \; q \in \Omega.$$

ψ-Hyperholomorphic functions

Cauchy type integral:

$$^{\psi}K[f](q) := -\int_{\Gamma} \mathcal{K}_{\psi}(\xi - q) \cdot n_{\psi}(\xi) \cdot f(\xi)\, d\Gamma_{\xi}, \quad q \in \mathbb{R}^3 \setminus \Gamma.$$

ψ-Hyperholomorphic functions

Borel-Pompeiu (= Cauchy-Green) formula

$$ {}^{\psi}K[f](q) + {}^{\psi}T\left[{}^{\psi}D[f]\right](q) = f(q), \quad q \in \Omega. $$

The Borel-Pompeiu formula solves the inhomogeneous equation

$$ {}^{\psi}D[f] = g, $$

in the standard way and the general solution is given by

$$ f = {}^{\psi}T[g] + h, $$

where $h \in {}^{\psi}\mathcal{M}(\Omega, \mathbb{H}(\mathbb{C}))$.

Let $\psi = \psi_\theta := \{\mathbf{i}, \mathbf{i}e^{\mathbf{i}\theta}\mathbf{j}, e^{\mathbf{i}\theta}\mathbf{j}\}$, for $0 \leq \theta < 2\pi$.

Then

$$^{\psi_\theta}D := \frac{\partial}{\partial x_1}\mathbf{i} + \frac{\partial}{\partial x_2}\mathbf{i}e^{\mathbf{i}\theta}\mathbf{j} + \frac{\partial}{\partial x_3}e^{\mathbf{i}\theta}\mathbf{j}.$$

Define the following partial differential operators for $f \in C^1(\Omega, \mathbb{H}(\mathbb{C}))$:

$$
\begin{aligned}
{}^{\psi_\theta}\mathrm{div}[\vec{f}] &:= \frac{\partial f_1}{\partial x_1} + \left(\frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3}\right)\mathbf{i}e^{\mathbf{i}\theta},\\
{}^{\psi_\theta}\mathrm{grad}[f_0] &:= \frac{\partial f_0}{\partial x_1}\mathbf{i} + \frac{\partial f_0}{\partial x_2}\mathbf{i}e^{\mathbf{i}\theta}\mathbf{j} + \frac{\partial f_0}{\partial x_3}e^{\mathbf{i}\theta}\mathbf{j},\\
{}^{\psi_\theta}\mathrm{rot}[\vec{f}] &:= \left(-\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right)e^{\mathbf{i}\theta} + \left(-\frac{\partial f_1}{\partial x_3}\mathbf{i}e^{\mathbf{i}\theta} - \frac{\partial f_3}{\partial x_1}\right)\mathbf{j} +\\
&\quad + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\mathbf{i}e^{\mathbf{i}\theta}\right)\mathbf{k}.
\end{aligned}
$$

The operator ${}^{\psi_\theta}D$

$$ {}^{\psi_\theta}D[f] = -{}^{\psi_\theta}\mathrm{div}[\vec{f}] + {}^{\psi_\theta}\mathrm{grad}[f_0] + {}^{\psi_\theta}\mathrm{rot}[\vec{f}], $$

The operator ${}^{\psi_\theta}D = 0$

$$ -{}^{\psi_\theta}\mathrm{div}[\vec{f}] + {}^{\psi_\theta}\mathrm{grad}[f_0] + {}^{\psi_\theta}\mathrm{rot}[\vec{f}] = 0. $$

In particular, if $\vec{f}$ is a solution to the homogeneous system

$$
\begin{cases}
-\dfrac{\partial f_1}{\partial x_1} + \left(\dfrac{\partial f_2}{\partial x_2} - \dfrac{\partial f_3}{\partial x_3}\right)\sin\theta - \left(\dfrac{\partial f_3}{\partial x_2} + \dfrac{\partial f_2}{\partial x_3}\right)\cos\theta = g_0, \\
\left(\dfrac{\partial f_3}{\partial x_3} - \dfrac{\partial f_2}{\partial x_2}\right)\cos\theta - \left(\dfrac{\partial f_3}{\partial x_2} + \dfrac{\partial f_2}{\partial x_3}\right)\sin\theta = g_1, \\
-\dfrac{\partial f_3}{\partial x_1} + \dfrac{\partial f_1}{\partial x_3}\sin\theta + \dfrac{\partial f_1}{\partial x_2}\cos\theta = g_2, \\
\dfrac{\partial f_2}{\partial x_1} - \dfrac{\partial f_1}{\partial x_3}\cos\theta + \dfrac{\partial f_1}{\partial x_2}\sin\theta = g_3,
\end{cases}
$$

$$
\begin{cases}
-\dfrac{\partial f_1}{\partial x_1}+\left(\dfrac{\partial f_2}{\partial x_2}-\dfrac{\partial f_3}{\partial x_3}\right)\sin\theta-\left(\dfrac{\partial f_3}{\partial x_2}+\dfrac{\partial f_2}{\partial x_3}\right)\cos\theta=0,\\
\left(\dfrac{\partial f_3}{\partial x_3}-\dfrac{\partial f_2}{\partial x_2}\right)\cos\theta-\left(\dfrac{\partial f_3}{\partial x_2}+\dfrac{\partial f_2}{\partial x_3}\right)\sin\theta=0,\\
-\dfrac{\partial f_3}{\partial x_1}+\dfrac{\partial f_1}{\partial x_3}\sin\theta+\dfrac{\partial f_1}{\partial x_2}\cos\theta=0,\\
\dfrac{\partial f_2}{\partial x_1}-\dfrac{\partial f_1}{\partial x_3}\cos\theta+\dfrac{\partial f_1}{\partial x_2}\sin\theta=0,
\end{cases}
$$

then $(0, \vec{f})$ is a pair of conjugate harmonic functions.

Integral Operators and Formulas: Analogue of the Cauchy-type integral

For a vector field $f = \vec{f}$, i.e., $f_0 \equiv 0$:

$$\mathrm{Sc}\left({}^{\psi_\theta}K[f](q)\right) = \frac{1}{4\pi}\int_\Gamma \left\{ \left\langle \left[{}^{\psi_\theta}\mathrm{grad}\frac{1}{|\xi - q|}, n_{\psi_\theta}(\xi) \right], \vec{f}(\xi) \right\rangle \right\} d\Gamma_\xi,$$

$$\mathrm{Vec}\left({}^{\psi_\theta}K[f](q)\right) = \frac{1}{4\pi}\int_\Gamma \left\{ \left\langle {}^{\psi_\theta}\mathrm{grad}\frac{1}{|\xi - q|}, n_{\psi_\theta}(\xi) \right\rangle \vec{f}(\xi) - \right.$$

$$\left. - \left[\left[{}^{\psi_\theta}\mathrm{grad}\frac{1}{|\xi - q|}, n_{\psi_\theta}(\xi) \right], \vec{f}(\xi) \right] \right\} d\Gamma_\xi.$$

Cauchy integral formula

If f is a ψ_θ-hyperholomorphic function, then the Cauchy integral formula holds:

$$f(q) = {}^{\psi_\theta}K[f](q) \quad \text{for } q \in \Omega.$$

If $f = \vec{f}$ and $f_0 \equiv 0$ is a solution to the homogeneous system, it implies that

$$0 = \frac{1}{4\pi} \int_\Gamma \left\{ \left\langle \left[{}^{\psi_\theta}\mathrm{grad} \frac{1}{|\xi - q|}, n_{\psi_\theta}(\xi) \right], \vec{f}(\xi) \right\rangle \right\} d\Gamma_\xi = \mathrm{Sc}\left({}^{\psi_\theta}K[f](q) \right),$$

in $\mathbb{R}^3 \setminus \Gamma$.

Cauchy integral in the theory of solutions to the homogeneous system

$$\vec{f} = \frac{1}{4\pi}\int_{\Gamma}\left\{\left\langle {}^{\psi_\theta}\mathrm{grad}\frac{1}{|\xi - q|}, n_{\psi_\theta}(\xi)\right\rangle \vec{f}(\xi) - \right.$$

$$\left. - \left[\left[{}^{\psi_\theta}\mathrm{grad}\frac{1}{|\xi - q|}, n_{\psi_\theta}(\xi)\right], \vec{f}(\xi)\right]\right\} d\Gamma_{\xi}.$$

Introduce a notion of the Cauchy-type integral (Cauchy transform) for the theory of solutions to the homogeneous systemas follows: if $\vec{f} \in C(\Gamma, \mathbb{C}^3)$ satisfies for $q \in \mathbb{R}^3 \setminus \Gamma$ the identity

$$\frac{1}{4\pi} \int_\Gamma \left\{ \left\langle \left[{}^{\psi_\theta}\mathrm{grad} \frac{1}{|\xi - q|}, n_{\psi_\theta}(\xi) \right], \vec{f}(\xi) \right\rangle \right\} d\Gamma_\xi = 0,$$

then for such vector fields the Cauchy-type integral is defined by

$${}^{\psi_\theta}\vec{K}[\vec{f}](q) := \frac{1}{4\pi} \int_\Gamma \left\{ \left\langle {}^{\psi_\theta}\mathrm{grad} \frac{1}{|\xi - q|}, n_{\psi_\theta}(\xi) \right\rangle \vec{f}(\xi) - \right.$$

$$\left. - \left[\left[{}^{\psi_\theta}\mathrm{grad} \frac{1}{|\xi - q|}, n_{\psi_\theta}(\xi) \right], \vec{f}(\xi) \right] \right\} d\Gamma_\xi .$$

Notation

Set

$$\mathfrak{W}(\Gamma, \mathbb{C}^3) :=$$

$$= \left\{ \vec{f} : \Gamma \to \mathbb{C}^3 : \frac{1}{4\pi} \int_\Gamma \left\{ \left\langle \left[{}^{\psi_\theta} \mathrm{grad} \frac{1}{|\xi - q|}, n_{\psi_\theta}(\xi) \right], \vec{f}(\xi) \right\rangle \right\} d\Gamma_\xi = 0, \ q \notin \Gamma \right\}.$$

Borel-Pompeiu formula

Let $f = \vec{f} \in \mathfrak{W}(\Gamma, \mathbb{C}^3)$ be a continuously differentiable vector field. Then

$$\frac{1}{4\pi} \int_\Gamma \left\{ \left\langle {}^{\psi_\theta}\mathrm{grad} \frac{1}{|\xi - q|}, n_{\psi_\theta}(\xi) \right\rangle \vec{f}(\xi) - \right.$$
$$\left. \left[\left[{}^{\psi_\theta}\mathrm{grad} \frac{1}{|\xi - q|}, n_{\psi_\theta}(\xi) \right], \vec{f}(\xi) \right] \right\} d\Gamma_\xi$$
$$+ \int_\Omega {}^{\psi_\theta}\mathrm{grad} \frac{1}{|\xi - q|} \left({}^{\psi_\theta}\mathrm{div}[\vec{f}](\xi) - {}^{\psi_\theta}\mathrm{rot}[\vec{f}](\xi) \right) dV_\xi$$
$$= \vec{f}(q), \quad q \in \Omega.$$

Cauchy integral formula

Let $f = \vec{f} \in \mathfrak{W}(\Gamma, \mathbb{C}^3)$ be a vector field solution to the homogeneous system. Then for $q \in \Omega$

$$\frac{1}{4\pi} \int_\Gamma \left\{ \left\langle {}^{\psi_\theta}\mathrm{grad} \frac{1}{|\xi - q|}, n_{\psi_\theta}(\xi) \right\rangle \vec{f}(\xi) - \right.$$
$$\left. - \left[\left[{}^{\psi_\theta}\mathrm{grad} \frac{1}{|\xi - q|}, n_{\psi_\theta}(\xi) \right], \vec{f}(\xi) \right] \right\} d\Gamma_\xi = \vec{f}(q).$$

Colombo, F., Luna-Elizarrarás, M. E., Sabadini, I., Shapiro, M. and Struppa, D. C. (2012): *A quaternionic treatment of the inhomogeneous div-rot system*. Moscow Math. J, 12(1), 37-48

${}^{\psi}T$ for the case $\psi = \psi_\theta$, can be explicitly described in terms of three integral operators:

$$
\begin{aligned}
{}^{\psi_\theta}T_{12}[\vec{g}](q) &:= -\int_\Omega \left\langle {}^{\psi_\theta}\mathrm{grad}\frac{1}{|\xi - q|}, \vec{g}(\xi) \right\rangle dV_\xi, \\
{}^{\psi_\theta}T_{21}[g_0](q) &:= \int_\Omega {}^{\psi_\theta}\mathrm{grad}\frac{1}{|\xi - q|} g_0(\xi) dV_\xi, \\
{}^{\psi_\theta}T_{22}[\vec{g}](q) &:= \int_\Omega \left[{}^{\psi_\theta}\mathrm{grad}\frac{1}{|\xi - q|}, \vec{g}(\xi) \right] dV_\xi,
\end{aligned}
$$

so that

$$
\begin{aligned}
{}^{\psi_\theta} T_{12} &: C(\Omega, \mathbb{C}^3) \to C^1(\Omega, \mathbb{C}), \\
{}^{\psi_\theta} T_{21} &: C(\Omega, \mathbb{C}) \to C^1(\Omega, \mathbb{C}^3), \\
{}^{\psi_\theta} T_{22} &: C(\Omega, \mathbb{C}^3) \to C^1(\Omega, \mathbb{C}^3).
\end{aligned}
$$

General solution of the inhomogeneous system

$$\begin{cases} -\dfrac{\partial f_1}{\partial x_1} + \left(\dfrac{\partial f_2}{\partial x_2} - \dfrac{\partial f_3}{\partial x_3}\right)\sin\theta - \left(\dfrac{\partial f_3}{\partial x_2} + \dfrac{\partial f_2}{\partial x_3}\right)\cos\theta = g_0, \\ \left(\dfrac{\partial f_3}{\partial x_3} - \dfrac{\partial f_2}{\partial x_2}\right)\cos\theta - \left(\dfrac{\partial f_3}{\partial x_2} + \dfrac{\partial f_2}{\partial x_3}\right)\sin\theta = g_1, \\ -\dfrac{\partial f_3}{\partial x_1} + \dfrac{\partial f_1}{\partial x_3}\sin\theta + \dfrac{\partial f_1}{\partial x_2}\cos\theta = g_2, \\ \dfrac{\partial f_2}{\partial x_1} - \dfrac{\partial f_1}{\partial x_3}\cos\theta + \dfrac{\partial f_1}{\partial x_2}\sin\theta = g_3, \end{cases} \tag{1}$$

Theorem

Let $g = g_0 + \vec{g} \in C(\Omega, \mathbb{H}(\mathbb{C}))$. The inhomogeneous system (1) has a solution if and only if for the scalar field

$$p_0 := {}^{\psi_\theta}T_{12}[\vec{g}], \tag{2}$$

General solution of the inhomogeneous system

Theorem Cont.

(A) either is identically zero;

(B) or has a hyper–conjugate harmonic function.

If it is true, then each solution f of (1) has the form:

(A*) either

$$\vec{f} = {}^{\psi_\theta} T_{21}[g_0] + {}^{\psi_\theta} T_{22}[\vec{g}] + \tilde{\vec{h}},$$

(B*) or

$$\vec{f} = {}^{\psi_\theta} T_{21}[g_0] + {}^{\psi_\theta} T_{22}[\vec{g}] + \vec{h}^T + \tilde{\vec{h}},$$

where $\vec{h}^T$ is a harmonic hyper–conjugate of $-p_0$, and $\tilde{\vec{h}}$ is an arbitrary hyperholomorphic vector field.

General solution of the inhomogeneous system

It is important to take into account the following remarks:

$$\begin{cases} -\dfrac{\partial f_1}{\partial x_1} - \dfrac{\partial f_2}{\partial x_2} - \dfrac{\partial f_3}{\partial x_3} = g_0, \\ \dfrac{\partial f_2}{\partial x_3} - \dfrac{\partial f_3}{\partial x_2} = g_1, \quad \dfrac{\partial f_3}{\partial x_1} - \dfrac{\partial f_1}{\partial x_3} = g_2, \quad \dfrac{\partial f_1}{\partial x_2} - \dfrac{\partial f_2}{\partial x_1} = g_3, \end{cases}$$

or

$$\begin{cases} \mathrm{div}\vec{f} = g_0, \\ \mathrm{rot}\vec{f} = \vec{g}, \end{cases}$$

is called sometimes the problem of reconstruction of the vector field by its divergence and rotation, or inverse problem of vector analysis.

General solution of the inhomogeneous system

Provide a necessary and sufficient condition for the solvability of the system and is described the general solution, they also show how the same problem could be studied using algebraic analysis and use it to obtain some additional results.

[Colombo, F., Luna-Elizarrarás, M. E., Sabadini, I., Shapiro, M. and Struppa, D. C.]

General solution of the inhomogeneous system

$$\begin{cases} \mathrm{div}\vec{f} = g_0, \\ \mathrm{rot}\vec{f} = \vec{g}, \end{cases}$$

can be obtained from

$$^{\psi_\theta}D[f] = -^{\psi_\theta}\mathrm{div}[\vec{f}] + {}^{\psi_\theta}\mathrm{grad}[f_0] + {}^{\psi_\theta}\mathrm{rot}[\vec{f}],$$

$\theta = 0$ with $f_0 \equiv 0$, and by assuming the correspondence on the values of the functions $f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k} \leftrightarrow f_1\mathbf{i} + f_3\mathbf{j} + f_2\mathbf{k}$. The results from

[Colombo, F., Luna-Elizarrarás, M. E., Sabadini, I., Shapiro, M. and Struppa, D. C.] can be directly achieved.

General solution of the inhomogeneous system

Cimmino system

Let Ω be a domain in $\mathbb{R}^4 \cong \mathbb{C}^2$ and f_l, $l = 0, 1, 2, 3$ be continuously differentiable $\mathbb{R}$-valued functions in Ω. The homogeneous Cimmino system may be written in the following way:

$$\begin{cases} \dfrac{\partial f_0}{\partial x_0} + \dfrac{\partial f_2}{\partial x_2} - \dfrac{\partial f_1}{\partial x_1} - \dfrac{\partial f_3}{\partial x_3} = 0, \\ \dfrac{\partial f_0}{\partial x_1} + \dfrac{\partial f_1}{\partial x_0} - \dfrac{\partial f_2}{\partial x_3} - \dfrac{\partial f_3}{\partial x_2} = 0, \\ \dfrac{\partial f_0}{\partial x_2} + \dfrac{\partial f_3}{\partial x_1} - \dfrac{\partial f_1}{\partial x_3} - \dfrac{\partial f_2}{\partial x_0} = 0, \\ \dfrac{\partial f_0}{\partial x_3} + \dfrac{\partial f_1}{\partial x_2} + \dfrac{\partial f_2}{\partial x_1} + \dfrac{\partial f_3}{\partial x_0} = 0. \end{cases}$$

G. Cimmino (12 March 1908 – 30 May 1989)

General solution of the inhomogeneous system

- G. Cimmino (1941): *Su alcuni sistemi lineari omegeni di equazioni alle derivate parziali del primo ordine*. Rend. Sem. Mat. Univ. Padova 12, 89–113 (Italian).
- S. Dragomir and E. Lanconelli (2006): *On first order linear PDE systems all of whose solutions are harmonic functions*. Tsukuba J. Math. 30(1), 149–170.

General solution of the inhomogeneous system

Cimmino system

Let us denote $z_1 = x_0 + \mathbf{i}x_1, z_2 = x_2 + \mathbf{i}x_3$, $u = f_0 + \mathbf{i}f_1$ and $v = f_2 + \mathbf{i}f_3$. In this way, CS may be rewritten as:

$$\begin{cases} \partial_{\bar{z}_1} u + \partial_{z_2} \bar{v} = 0, \\ \partial_{\bar{z}_2} u - \partial_{z_1} \bar{v} = 0, \end{cases} \tag{3}$$

where

$$\partial_{\bar{z}_1} := \frac{1}{2}\left(\frac{\partial}{\partial x_0} + \mathbf{i}\frac{\partial}{\partial x_1}\right), \partial_{\bar{z}_2} := \frac{1}{2}\left(\frac{\partial}{\partial x_2} + \mathbf{i}\frac{\partial}{\partial x_3}\right)$$

$$\partial_{z_1} := \frac{1}{2}\left(\frac{\partial}{\partial x_0} - \mathbf{i}\frac{\partial}{\partial x_1}\right), \partial_{z_2} := \frac{1}{2}\left(\frac{\partial}{\partial x_2} - \mathbf{i}\frac{\partial}{\partial x_3}\right).$$

If (u, v) of continuously differentiable $\mathbb{C}$-valued functions give a solution of (3) then

$$\Delta_{\mathbb{R}^4} u \cong \Delta_{\mathbb{C}^2} u = 2(\partial^2_{z_1 \bar{z}_1} + \partial^2_{z_2 \bar{z}_2})\, u = 0.$$

General solution of the inhomogeneous system

$$\begin{cases} \dfrac{\partial f_2}{\partial x_2} - \dfrac{\partial f_1}{\partial x_1} - \dfrac{\partial f_3}{\partial x_3} = g_0, \\ -\dfrac{\partial f_2}{\partial x_3} - \dfrac{\partial f_3}{\partial x_2} = g_1, \quad -\dfrac{\partial f_3}{\partial x_1} + \dfrac{\partial f_1}{\partial x_3} = g_2, \quad \dfrac{\partial f_1}{\partial x_2} + \dfrac{\partial f_2}{\partial x_1} = g_3, \end{cases}$$

can be considered as a particular case of the inhomogeneous Cimmino system when one looks for a solution (f_1, f_2, f_3), i.e. $f_0 \equiv 0$, where each $f_m,\ m = 1, 2, 3$ does not depend on x_0.

This system is obtained from

$$^{\psi_\theta}D[f] = -^{\psi_\theta}\mathrm{div}[\vec{f}] + {}^{\psi_\theta}\mathrm{grad}[f_0] + {}^{\psi_\theta}\mathrm{rot}[\vec{f}],$$

for $\theta = \frac{\pi}{2}$.

S. Dragomir, E. Lanconelli (Universida di Bologna): $\Omega \subset \mathbb{C}^2$:

$$\begin{aligned} \partial_{\bar{z}_1} u + \partial_{z_2} \bar{v} &= f, \\ \partial_{\bar{z}_2} u - \partial_{z_1} \bar{v} &= g, \text{ in } \Omega, \end{aligned}$$

$u = \phi$, $v = \psi$ on Γ where $F = (u, v) : \Omega \to \mathbb{C}^2$.

S. Dragomir, E. Lanconelli (Universida di Bologna): $\Omega \subset \mathbb{C}^2$:

$$\begin{aligned} \partial_{\bar{z}_1} u + \partial_{z_2} \bar{v} &= f, \\ \partial_{\bar{z}_2} u - \partial_{z_1} \bar{v} &= g, \text{ in } \Omega, \end{aligned}$$

$u = \phi$, $v = \psi$ on Γ where $F = (u, v) : \Omega \to \mathbb{C}^2$.

Only necessary solvability conditions are obtained!

Analogue of a Cauchy type integral in theory of the Cimmino system

$$^{\psi}K[u+vj](z_1,z_2) = \mathcal{K}_1[u,v](z_1,z_2)+\mathcal{K}_2[u,v](z_1,z_2)j,\ (z_1,z_2) \notin \Gamma,$$

$$^{\psi}S[u+vj](z_1,z_2) = \mathcal{S}_1[u,v](z_1,z_2)+\mathcal{S}_2[u,v](z_1,z_2)j,\ (z_1,z_2) \in \Gamma,$$

$$\mathcal{K}_1[u,v] = K_1[u] - K_2[\bar{v}],\ \mathcal{K}_2[u,v] = K_1[v] + K_2[\bar{u}],$$

$$\mathcal{S}_1[u,v] = S_1[u] - S_2[\bar{v}],\ \mathcal{S}_2[u,v] = S_1[v] + S_2[\bar{u}],$$

Analogue of a Cauchy type integral in theory of the Cimmino system

$$K_1[w](z_1, z_2) =$$

$$= \int\limits_{\Gamma} \frac{[(\bar{\zeta}_1 - \bar{z}_1)(n_0 + in_1) + (\bar{\zeta}_2 - \bar{z}_2)(n_2 + in_3)]}{2\pi^2(|\zeta_1 - z_1|^2 + |\zeta_2 - z_2|^2)^2} w(\zeta_1, \zeta_2) d\Gamma_{\zeta_1, \zeta_2},$$

$$K_2[w](z_1, z_2) =$$

$$= \int\limits_{\Gamma} \frac{[(\bar{\zeta}_2 - \bar{z}_2)\overline{(n_0 + in_1)} - (\bar{\zeta}_1 - \bar{z}_1)\overline{(n_2 + in_3)}]}{2\pi^2(|\zeta_1 - z_1|^2 + |\zeta_2 - z_2|^2)^2} w(\zeta_1, \zeta_2)\, d\Gamma_{\zeta_1, \zeta_2},$$

Analogue of a Cauchy type integral in theory of the Cimmino system

$$S_1[w](z_1, z_2) = 2K_1[w](z_1, z_2), \; S_2[w](z_1, z_2) = 2K_2[w](z_1, z_2).$$

Cauchy-Cimmino type integrals

$(\mathcal{K}_1, \mathcal{K}_2)$; for $(z_1, z_2) \in \mathbb{C}^2$.

Analogue of a Cauchy type integral in theory of the Cimmino system

$$S_1[w](z_1, z_2) = 2K_1[w](z_1, z_2), \; S_2[w](z_1, z_2) = 2K_2[w](z_1, z_2).$$

Cauchy-Cimmino type integrals

$(\mathcal{K}_1, \mathcal{K}_2)$; for $(z_1, z_2) \in \mathbb{C}^2$.

singular Cauchy-Cimmino integral operators

$(\mathcal{S}_1, \mathcal{S}_2)$.

The Teodorescu Cimmino type transforms

$(z_1, z_2) \in \mathbb{C}^2$

$$\mathcal{T}_1[u,v](z_1,z_2) = T_1[u](z_1,z_2) - T_2[\bar{v}](z_1,z_2)$$

$$\mathcal{T}_2[u,v](z_1,z_2) = T_1[v](z_1,z_2) + T_2[\bar{u}](z_1,z_2),$$

$$T_1[w](z_1,z_2) = \int_\Omega \frac{\bar{z}_1 - \bar{\zeta}_1}{2\pi^2(|\zeta_1 - z_1|^2 + |\zeta_2 - z_2|^2)^2} w(\zeta_1,\zeta_2)\, d\Omega_{\zeta_1,\zeta_2},$$

$$T_2[w](z_1,z_2) = \int_\Omega \frac{\bar{z}_2 - \bar{\zeta}_2}{2\pi^2(|\zeta_1 - z_1|^2 + |\zeta_2 - z_2|^2)^2} w(\zeta_1,\zeta_2)\, d\Omega_{\zeta_1,\zeta_2}.$$

Cimmino operators

Remark

$$ {}^{\psi}K[u+vj] = \mathcal{K}_1[u,v] + \mathcal{K}_1[v,-u]j, $$
$$ {}^{\psi}S[u+vj] = \mathcal{S}_1[u,v] + \mathcal{S}_1[v,-u]j, $$
$$ {}^{\psi}T[u+vj] = \mathcal{T}_1[u,v] + \mathcal{T}_1[v,-u]j. $$

Cimmino operators

Theorem (Stokes). $u, v, g_1, g_2 \in C^1(\Omega, \mathbb{C}) \cap C^0(\Omega \cup \Gamma, \mathbb{C})$:

$$\int_\Gamma \left\{ [g_1(n_0 + in_1) + g_2(n_2 + in_3)]u - [g_2\overline{(n_0 + in_1)} - g_1\overline{(n_2 + in_3)}]\bar{v} \right\} d\Gamma =$$

$$= 2\int_\Omega [(\partial_{\bar{z}_1} g_1 + \partial_{\bar{z}_2} g_2)u + (\partial_{z_2} g_1 - \partial_{z_1} g_2)\bar{v} + g_1(\partial_{\bar{z}_1} u + \partial_{z_2} \bar{v}) + g_2(\partial_{\bar{z}_2} u - \partial_{z_1} \bar{v})]\, d\Omega,$$

$$\int_\Gamma \left\{ [g_1(n_0 + in_1) + g_2(n_2 + in_3)]v + [g_2\overline{(n_0 + in_1)} - g_1\overline{(n_2 + in_3)}]\bar{u} \right\} d\Gamma =$$

$$= 2\int_\Omega [(\partial_{\bar{z}_1} g_1 + \partial_{\bar{z}_2} g_2)v - (\partial_{z_2} g_1 - \partial_{z_1} g_2)\bar{u} + g_2(\partial_{z_1} \bar{u} + \partial_{\bar{z}_2} v) - g_1(\partial_{z_2} \bar{u} - \partial_{\bar{z}_1} v)]\, d\Omega.$$

Stokes-Cimmino Theorem

$u, v \in C^1(\Omega, \mathbb{C}) \cap C^0(\Omega \cup \Gamma, \mathbb{C})$:

$$\int_\Gamma \left\{ [(n_0 + in_1) + (n_2 + in_3)]u - [\overline{(n_0 + in_1)} - \overline{(n_2 + in_3)}]\bar{v} \right\} d\Gamma =$$

$$= 2 \int_\Omega [(\partial_{\bar{z}_1} u + \partial_{z_2} \bar{v}) + (\partial_{\bar{z}_2} u - \partial_{z_1} \bar{v})] \, d\Omega.$$

Cimmino operators

Borel-Pompeiu Theorem

$u, v \in C^1(\Omega, \mathbb{C}) \cap C^0(\Omega \cup \Gamma, \mathbb{C})$; $(z_1, z_2) \in \Omega$:

$$u(z_1, z_2) = \mathcal{K}_1[u, v](z_1, z_2) + 2\mathcal{T}_1\left[\partial_{\bar{z}_1} u + \partial_{z_2} \bar{v}, -\overline{(\partial_{\bar{z}_2} u - \partial_{z_1} \bar{v})}\right](z_1, z_2),$$

$$v(z_1, z_2) = \mathcal{K}_2[u, v](z_1, z_2) + 2\mathcal{T}_2\left[\partial_{\bar{z}_1} u + \partial_{z_2} \bar{v}, -\overline{(\partial_{\bar{z}_2} u - \partial_{z_1} \bar{v})}\right](z_1, z_2).$$

Some properties of the Cauchy-Cimmino integrals

Theorem (Sokhotski-Plemelj formulas)

$(u, v) \in C^{0,\nu}(\Gamma, \mathbb{R}^2) \times C^{0,\nu}(\Gamma, \mathbb{R}^2)$:

$$\lim_{\Omega^{\pm} \ni (z_1, z_2) \to (\zeta_1, \zeta_2) \in \Gamma} (\mathcal{K}_1[u, v](z_1, z_2), \mathcal{K}_2[u, v](z_1, z_2)) =:$$

$$=: (\mathcal{K}_1^{\pm}[u, v](\zeta_1, \zeta_2), \mathcal{K}_2^{\pm}[u, v](\zeta_1, \zeta_2)),$$

$$(\mathcal{K}_1^{\pm}[u, v](\zeta_1, \zeta_2), \mathcal{K}_2^{\pm}[u, v](\zeta_1, \zeta_2)) =$$

$$= \frac{1}{2}[(\mathcal{S}_1[u, v](\zeta_1, \zeta_2), \mathcal{S}_2[u, v](\zeta_1, \zeta_2)) \pm (u(\zeta_1, \zeta_2), v(\zeta_1, \zeta_2))],$$

$\forall (\zeta_1, \zeta_2) \in \Gamma$.

Plemelj-Privalov type theorem for the Cimmino system

Theorem

$$(u, v) \in C^{0,\nu}(\Gamma, \mathbb{R}^2) \times C^{0,\nu}(\Gamma, \mathbb{R}^2) \Rightarrow$$

$$\Rightarrow (\mathcal{S}_1[u, v], \mathcal{S}_2[u, v]) \in C^{0,\nu}(\Gamma, \mathbb{R}^2) \times C^{0,\nu}(\Gamma, \mathbb{R}^2),$$

for $0 < \nu < 1$.

Extension of a given pair of complex-valued Hölder continuous function on Γ up to a solution of the Cimmino system

1. $(u, v) \in C^{0,\nu}(\Gamma, \mathbb{R}^2) \times C^{0,\nu}(\Gamma, \mathbb{R}^2)$ be a boundary value of a solution of Cimmino system (U, V) into $\Omega^+ \Longleftrightarrow$

$$(u(\zeta_1, \zeta_2), v(\zeta_1, \zeta_2)) = (\mathcal{S}_1[u, v](\zeta_1, \zeta_2), \mathcal{S}_2[u, v](\zeta_1, \zeta_2)),$$

$(\zeta_1, \zeta_2) \in \Gamma$.

2. $(u, v) \in C^{0,\nu}(\Gamma, \mathbb{R}^2) \times C^{0,\nu}(\Gamma, \mathbb{R}^2)$ be a boundary value of a solution of Cimmino system (U, V) into Ω^-, and vanishes at infinity $\Longleftrightarrow$

$$(u(\zeta_1, \zeta_2), v(\zeta_1, \zeta_2)) = (-\mathcal{S}_1[u, v](\zeta_1, \zeta_2), -\mathcal{S}_2[u, v](\zeta_1, \zeta_2)),$$

$(\zeta_1, \zeta_2) \in \Gamma$.

On the square of the singular Cauchy-Cimmino operators

Theorem

$(u, v) \in C^{0,\nu}(\Gamma, \mathbb{R}^2) \times C^{0,\nu}(\Gamma, \mathbb{R}^2)$:

$$\mathcal{S}_1^2[u, v] - \mathcal{S}_2^2[u, v] = u,$$

$$\mathcal{S}_1[u, v]\mathcal{S}_2[u, v] + \mathcal{S}_2[u, v]\mathcal{S}_1[u, v] = -v.$$

Boundary value problems

K. Gürlebeck, W. Sprössig (1997): over Sobolev-Slobodetzkij spaces the following properties hold:

- ${}^{\psi}T : L_2(\Omega, \mathbb{H}) \to W_2^1(\Omega, \mathbb{H})$,
- ${}^{\psi}K : W_2^{\frac{1}{2}}(\Gamma, \mathbb{H}) \to W_2^1(\Omega, \mathbb{H})$.

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Abbreviation:

- $im\mathcal{A}$: the range of operator $\mathcal{A}$.

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Abbreviation:

- $im\mathcal{A}$: the range of operator $\mathcal{A}$.
- $\mathrm{tr}_\Gamma f$: restriction of a $\mathbb{H}$-valued function f of a Sobolev space in Ω to one the boundary Γ.

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Abbreviation:

- $im\mathcal{A}$: the range of operator $\mathcal{A}$.
- $\mathrm{tr}_\Gamma f$: restriction of a $\mathbb{H}$-valued function f of a Sobolev space in Ω to one the boundary Γ.

Introduce the projection operators:

$$ {}^{\psi}P[f] := \frac{1}{2}(f + {}^{\psi}S[f]), \quad {}^{\psi}Q[f] := \frac{1}{2}(f - {}^{\psi}S[f]). $$

Remark

$$\mathrm{tr}_{\Gamma}{}^{\psi}\, T[f] \in im^{\psi} Q.$$

BVP in the quaternionic framework

Theorem. $F \in L_2(\Omega, \mathbb{H})$, $G \in L_2(\Gamma, \mathbb{H})$.

$$^{\psi}D[f] = F, \text{ in } \Omega, \tag{4}$$

$$f = G, \text{ on } \Gamma, \tag{5}$$

has solution $f \in C^1(\Omega, \mathbb{H}) \cap C^0(\Omega \cup \Gamma, \mathbb{H})$, then F, G satisfy the relation

$$\int_{\Gamma} g(\xi) n_{\psi}(\xi) G(\xi)\, d\Gamma_{\xi} - \int_{\Omega} g(\xi) F(\xi)\, d\Omega_{\xi} = 0,$$

for any function $g \in \mathcal{M}^{\psi}(\Omega, \mathbb{H}) \cap C^0(\Omega \cup \Gamma, \mathbb{H})$.

Boundary value problems

Theorem. Let $F \in L_2(\Omega, \mathbb{H})$, $G \in W_2^{\frac{1}{2}}(\Gamma, \mathbb{H})$. Then the bvp (4)-(5) has solution if and only if

$$^{\psi}Q[G] = \mathrm{tr}_{\Gamma}\, {}^{\psi}T[F]$$

is fulfilled. The only solution $f \in W_2^1(\Omega, \mathbb{H})$ admits the representation

$$f = {}^{\psi}K[G] + {}^{\psi}T[F].$$

Boundary value problems for Cimmino system

We are looking for the following boundary value problem associated to the Cimmino system:

$$\partial_{\bar{z}_1} u + \partial_{z_2} \bar{v} = F_1, \;\; \partial_{\bar{z}_2} u - \partial_{z_1} \bar{v} = F_2, \text{ in } \Omega \tag{6}$$

$$u = G_1, \;\; v = G_2 \text{ on } \Gamma, \tag{7}$$

where $F_1, F_2 \in L_2(\Omega, \mathbb{C})$, $G_1, G_2 \in W_2^{\frac{1}{2}}(\Gamma, \mathbb{C})$.

Boundary value problems for Cimmino system

Theorem. Let $F_1, F_2 \in L_2(\Omega, \mathbb{C})$ and $G_1, G_2 \in L_2(\Gamma, \mathbb{C})$. If there exist $u, v \in C^1(\Omega, \mathbb{C}) \cap C^0(\Omega \cup \Gamma, \mathbb{C})$ solutions of (6)-(7), then the following relations

$$2 \int_\Omega (g_1 F_1 + g_2 F_2)\, d\Omega =$$

$$\int_\Gamma \left\{ [g_1(n_0 + in_1) + g_2(n_2 + in_3)]G_1 - \left[g_2\overline{(n_0 + in_1)} - g_1\overline{(n_2 + in_3)} \right] \overline{G}_2 \right\} d\Gamma,$$

$$2 \int_\Omega (g_2 \overline{F}_1 - g_1 \overline{F}_2)\, d\Omega =$$

$$\int_\Gamma \left\{ [g_1(n_0 + in_1) + g_2(n_2 + in_3)]G_2 + \left[g_2\overline{(n_0 + in_1)} - g_1\overline{(n_2 + in_3)} \right] \overline{G}_1 \right\} d\Gamma,$$

holds for every $g_1, g_2 \in C^1(\Omega, \mathbb{C}) \cap C^0(\Omega \cup \Gamma, \mathbb{C})$ solutions of the system

$$\partial_{\bar{z}_1} g_1 + \partial_{\bar{z}_2} g_2 = 0, \quad \partial_{\bar{z}_2} \bar{g}_1 - \partial_{\bar{z}_1} \bar{g}_2 = 0, \text{ in } \Omega. \tag{8}$$

Theorem. Let $F_1, F_2 \in L_2(\Omega, \mathbb{C})$ and $G_1, G_2 \in W_2^{\frac{1}{2}}(\Gamma, \mathbb{C})$. The boundary value problem (6)-(7) has solutions $u, v \in W_2^1(\Omega, \mathbb{C})$ if and only if F_1, F_2, G_1, G_2 satisfy the following relations

$$2\mathcal{T}_1\left[F_1, -\overline{F}_2\right] = \frac{1}{2}G_1 - \mathcal{K}_1[G_1, G_2] \text{ for a.e. on } \Gamma,$$

$$2\mathcal{T}_2\left[F_1, -\overline{F}_2\right] = \frac{1}{2}G_2 - \mathcal{K}_2[G_1, G_2] \text{ for a.e. on } \Gamma.$$

Moreover,

$$u = \mathcal{K}_1[G_1, G_2] + 2\mathcal{T}_1\left[F_1, -\overline{F}_2\right] \text{ in } \Omega,$$

$$v = \mathcal{K}_2[G_1, G_2] + 2\mathcal{T}_2\left[F_1, -\overline{F}_2\right] \text{ in } \Omega.$$

General solution of the inhomogeneous system

The Riesz system

For $f : (x_0, x_1, x_2) \in \mathbb{R}^3 \to \mathrm{span}_{\mathbb{R}}\{1, \mathbf{i}, \mathbf{j}\}$

$$\begin{cases} \dfrac{\partial f_0}{\partial x_0} - \dfrac{\partial f_1}{\partial x_1} - \dfrac{\partial f_2}{\partial x_2} = 0, \\ \dfrac{\partial f_0}{\partial x_1} + \dfrac{\partial f_1}{\partial x_0} = 0, \quad \dfrac{\partial f_0}{\partial x_2} + \dfrac{\partial f_2}{\partial x_0} = 0, \quad \dfrac{\partial f_1}{\partial x_2} - \dfrac{\partial f_2}{\partial x_1} = 0, \end{cases}$$

which is equivalent to the system

$$\begin{cases} \mathrm{div}\overline{f} = 0, \\ \mathrm{rot}\overline{f} = 0. \end{cases}$$

General solution of the inhomogeneous system

The Riesz system

This system can be obtained from

$$ {}^{\psi_\theta}D[f] = -{}^{\psi_\theta}\mathrm{div}[\vec{f}] + {}^{\psi_\theta}\mathrm{grad}[f_0] + {}^{\psi_\theta}\mathrm{rot}[\vec{f}], $$

for $\theta = \pi$ with $f_0 \equiv 0$, by assuming the correspondence on the values of the functions $f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k} \leftrightarrow f_0\mathbf{i} + f_2\mathbf{j} + f_1\mathbf{k}$ as well as on the variables $(x_1, x_2, x_3) \leftrightarrow (x_0, x_1, x_2)$.

General solution of the inhomogeneous system

Dirac

$$\begin{cases} -\dfrac{\partial f_1}{\partial x_1} - \dfrac{\partial f_2}{\partial x_2} + \dfrac{\partial f_3}{\partial x_3} = 0, \\ \dfrac{\partial f_2}{\partial x_3} + \dfrac{\partial f_3}{\partial x_2} = 0, \quad -\dfrac{\partial f_3}{\partial x_1} - \dfrac{\partial f_1}{\partial x_3} = 0, \quad -\dfrac{\partial f_1}{\partial x_2} + \dfrac{\partial f_2}{\partial x_1} = 0, \end{cases}$$

is a particular case of the version of the α-hyperholomorphic function theory that is in a deep relation with the time-harmonic solutions of the relativistic Dirac equation.

The system (Dirac) can be obtained from

$$^{\psi_\theta}D[f] = -{}^{\psi_\theta}\mathrm{div}[\vec{f}] + {}^{\psi_\theta}\mathrm{grad}[f_0] + {}^{\psi_\theta}\mathrm{rot}[\vec{f}],$$

for $\theta = \frac{3\pi}{2}$ with $f_0 \equiv 0$.

Now, let $\mathbb{S}^2 = \mathbb{S}^2(0;1)$ be the unit sphere in $\mathbb{R}^3$ which is the boundary of the unit ball $\mathbb{B}^2 = \mathbb{B}^2(0;1)$, the following formulas define linear bounded operators on both spaces of our interest which are $C^{0,\epsilon}(\mathbb{S}^2, \mathbb{C}^3)$, $0 < \epsilon \leq 1$, and $L_p(\mathbb{S}^2, \mathbb{C}^3)$, $1 < p < \infty$:

$$M^0_{\mathbb{S}^2}[f](x) := \frac{1}{2\pi} \int_{\mathbb{S}^2} \frac{1}{2|x-\xi|} f(\xi) d\Gamma_\xi,$$

$$\mathcal{H}^1_{\mathbb{S}^2}[f](x) := \frac{1}{2\pi} \int_{\mathbb{S}^2} \frac{(x_3 y_2 - x_2 y_3)}{|x-\xi|^3} f(\xi) d\Gamma_\xi,$$

$$\mathcal{H}^2_{\mathbb{S}^2}[f](x) := \frac{1}{2\pi} \int_{\mathbb{S}^2} \frac{(x_2 y_1 - x_1 y_2)\cos\theta + (x_3 y_1 - x_1 y_3)\sin\theta}{|x-\xi|^3} f(\xi) d\Gamma_\xi,$$

$$\mathcal{H}^3_{\mathbb{S}^2}[f](x) := \frac{1}{2\pi} \int_{\mathbb{S}^2} \frac{(x_1 y_3 - x_3 y_1)\cos\theta + (x_2 y_1 - x_1 y_2)\sin\theta}{|x-\xi|^3} f(\xi) d\Gamma_\xi.$$

$$0 = \frac{1}{4\pi} \int_{\Gamma} \left\{ \left\langle \left[{}^{\psi_\theta}\mathrm{grad} \frac{1}{|\xi - x|}, n_{\psi_\theta}(\xi) \right], \vec{f}(\xi) \right\rangle \right\} d\Gamma_{\xi}, \quad x \in \Gamma, \quad (9)$$

where the integral is understood in the sense of Cauchy's principal value. Also, one can consider $\vec{f} \in L_p(\Gamma, \mathbb{C}^3)$ with $1 < p < \infty$.
Consider the following real linear spaces:

$$\hat{C}^{0,\epsilon}(\Gamma, \mathbb{C}^3) := \{\vec{f} \in C^{0,\epsilon}(\Gamma, \mathbb{C}^3),\ 0 < \epsilon \leq 1;\ (9) \text{ is valid}\}.$$

$$\hat{L}_p(\Gamma, \mathbb{C}^3) := \{\vec{f} \in L_p(\Gamma, \mathbb{C}^3),\ 1 < p < \infty;\ (9) \text{ is valid}\}.$$

Definition

$\vec{\mathfrak{U}}(\mathbb{B}^2(0;1);\hat{C}^{0,\epsilon}(\mathbb{S}^2,\mathbb{C}^3))$, $0<\epsilon\leq 1$, denotes the class of vector fields $\vec{F}$ such that

1) $\vec{F}$ is a solution to the homogeneous system in $\mathbb{B}^2(0;1)$;
2) there exists everywhere on $\mathbb{S}^2$ the limit $\lim_{\mathbb{B}^2(0;1)\ni q\to x\in\mathbb{S}^2}\vec{F}(q)=:\vec{f}(x)$ generating the vector field $\vec{f}$ in $\hat{C}^{0,\epsilon}(\mathbb{S}^2,\mathbb{C}^3)$.

Integral Hilbert formulas

A vector field $\vec{f} = \mathbf{i} f_1 + \mathbf{j} f_2 + \mathbf{k} f_3$ is the limit function of $\vec{F} \in \vec{\mathfrak{U}}(\mathbb{B}^2(0;1); \hat{C}^{0,\epsilon}(\mathbb{S}^2, \mathbb{C}^3))$ or $\vec{F} \in \vec{\mathfrak{U}}(\mathbb{B}^2(0;1); \hat{L}_p(\mathbb{S}^2, \mathbb{C}^3))$ if and only if the following relations between its components hold:

$$\begin{aligned} f_1 &= M^0_{\mathbb{S}^2}[f_1] + \mathcal{H}^2_{\mathbb{S}^2}[f_3] - \mathcal{H}^3_{\mathbb{S}^2}[f_2], \\ f_2 &= M^0_{\mathbb{S}^2}[f_2] + \mathcal{H}^3_{\mathbb{S}^2}[f_1] - \mathcal{H}^1_{\mathbb{S}^2}[f_3], \\ f_3 &= M^0_{\mathbb{S}^2}[f_3] + \mathcal{H}^1_{\mathbb{S}^2}[f_2] - \mathcal{H}^2_{\mathbb{S}^2}[f_1]. \end{aligned} \tag{10}$$

On $\ker M^0_{\mathbb{S}^2}$ formulas (10) become

$$\begin{aligned} f_1 &= \mathcal{H}^2_{\mathbb{S}^2}[f_3] - \mathcal{H}^3_{\mathbb{S}^2}[f_2], \\ f_2 &= \mathcal{H}^3_{\mathbb{S}^2}[f_1] - \mathcal{H}^1_{\mathbb{S}^2}[f_3], \\ f_3 &= \mathcal{H}^1_{\mathbb{S}^2}[f_2] - \mathcal{H}^2_{\mathbb{S}^2}[f_1]. \end{aligned}$$

Thank you for listening!