Spectral Convergence of Neumann Laplacian on Non-Compact Quasi-One-Dimensional Spaces and Some Geometric Domains

LY HONG HAI

Department of Mathematics
Faculty of Science, University of Ostrava
Czech Republic
p18180@student.osu.cz

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Outline

INTRODUCTION

- Non-compact quasi-one-dimentional spaces can be approximated by underlying metric graph. A *metric* or *quantum graph* is a graph considered as one-dimentional space where each edge is assigned a length.
- A *quasi-one-dimentional space* consists of a family of *graph-like* manifolds, i.e., a family of manifolds X_ε shrinking to the underlying metric graph X_0.
- The family of graph-like manifolds is constructed of building blocks $U_{\varepsilon,v}$ and $U_{\varepsilon,e}$ for each vertex $v \in V$ and $e \in E$ of the graph.

Introduction

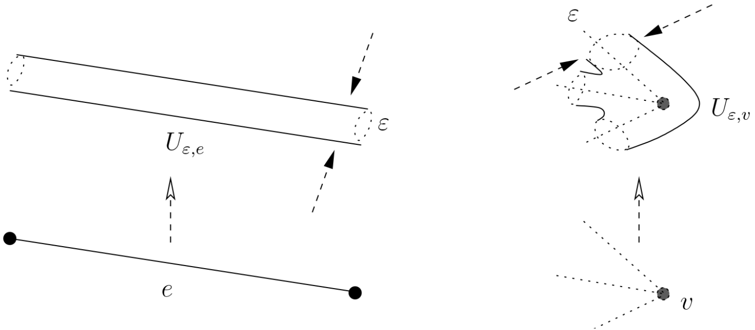

Figure: The associated edge and vertex neighbourhoods with $F_\varepsilon = \mathbb{S}^1_\varepsilon$, i.e., $U_{\varepsilon,e}$ and $U_{\varepsilon,v}$ are 2-dimentional manifolds with boundary.

Introduction

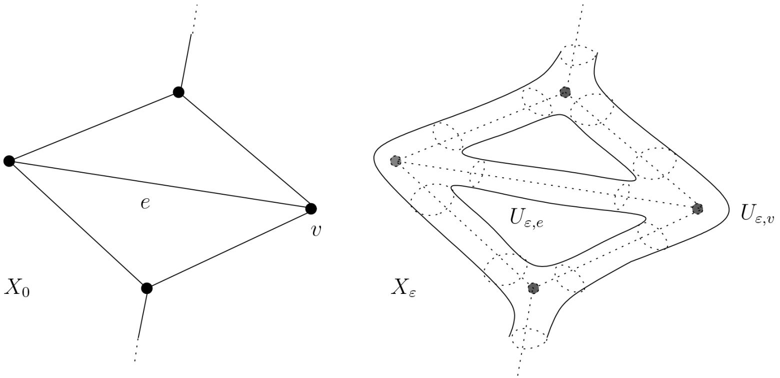

Figure: On the left, we have the graph X_0, on the right, the associated family of graph-like manifolds. Here $F_\varepsilon = \mathbb{S}^1_\varepsilon$ is the tranversal section of radius ε and X_ε is a 2-dimentional manifold.

INTRODUCTION

- On the graph-like manifold X_ε, we consider the Laplacian $\widetilde{H} := \Delta_{X_\varepsilon} \geq 0$ acting in the Hilbert space $\widetilde{H} := L^2(X_\varepsilon)$.
- If X_ε has a boundary, we impose *Neumann* boundary conditions.
- On the graph X_0, we choose $H := \Delta_{X_0}$ be the generalised Neumann (Kirchhoff) Laplacian acting on the each edge as a one-dimensional weighted Laplacian.
- Δ_{X_0} acts on $\mathcal{H} := \oplus_e L^2(e)$ where e is identified with $(0, l_e)$ $(0 < l_e < \infty)$ - in contrast to the *discrete* graph Laplacian acting as difference operator on the space of vertices, $l_2(V)$.

Main Theorem.

Suppose X_ε is a family of (non-compact) graph-like manifolds associated to a metric graph X_0. if X_ε and X_0 satisfy some natural uniformity conditions, then the resolvent of Δ_{X_ε} converges in norm to the resolvent of Δ_{X_0} (with suitable identification operators) as $\varepsilon \to 0$. In particular, the corresponding essential and discrete spectra converge uniformly in any bounded interval. Furthermore, the eigenfunctions converge as well.

1. Scale of Hilbert spaces associated with a non-negative operator.

- To a Hilbert space $\mathcal{H}$ with inner product $\langle ., . \rangle$ and norm $\|.\|$ together with a non-negative, unbounded, operator H, we associate the scale of Hilbert spaces

$$\mathcal{H}_k := dom(H+1)^{k/2}, \quad \|u\|_k := \|(H+1)^{/k2}\|, \quad k \geq 0. \tag{1}$$

- For negative exponents, define

$$\mathcal{H}_{-k} := \mathcal{H}_k^*. \tag{2}$$

- Note that $\mathcal{H} = \mathcal{H}_0$ embeds naturally into $\mathcal{H}_{-k}$ via $u \mapsto \langle u, . \rangle$ since

$$\|\langle u, . \rangle\|_{-k} = \sup_{v \in \mathcal{H}_k} \frac{|\langle u, v \rangle|}{\|v\|_k} = \sup_{v \in \mathcal{H}_0} \frac{\left|\left\langle R^{k/2} u, w \right\rangle\right|}{\|w\|_0} = \left\| R^{k/2} u \right\|_0,$$

where

$$R := (H + 1)^{-1} \tag{3}$$

- Denotes the resolvent of $H \geq 0$. The last equality used the natural identification $\mathcal{H} \cong \mathcal{H}^*$ via $u \mapsto \langle u, . \rangle$. Therefore, we can interprete $\mathcal{H}_{-k}$ as the completion of $\mathcal{H}$ in the norm $\|.\|_{-k}$. With this identification, we have

$$\|u\|_{-k} = \sup_{v \in \mathcal{H}_k} \frac{|\langle u, v \rangle|}{\|v\|_k} \text{ for all } k \in \mathbb{R}. \tag{4}$$

- For a second Hilbert space $\widetilde{\mathcal{H}}$ with inner product $\langle ., . \rangle$ and norm$\|.\|$ together with a non-negative, unbounded, operator $\widetilde{A}$, we define in the same way a scale of Hilbert spaces $\widetilde{\mathcal{H}}_k$ with norms $\|.\|_k$.
- Given by the classical application $A = -\Delta_X$ in $\mathcal{H} = L^2(X)$ for a complete manifold X, we call k the *regularity order*. In this case, $\mathcal{H}_k$ corresponds to the $k - th$ Sobolev space $H^k(X)$.

2. Operators on scales.

- Suppose we have two scales of Hilbert spaces $\mathcal{H}_k$, $\widetilde{\mathcal{H}}_k$ associated to the non-negative operators $H, \widetilde{H}$ with resolvents $R := (H+1)^{-1}$, $\widetilde{R} := (\widetilde{H}+1)^{-1}$, respectively. The norm of an operator $A : \mathcal{H}_k \to \widetilde{\mathcal{H}}_{-k}$ is

$$\|A\|_{k \to -\widetilde{k}} := \sup_{u \in H_k} \frac{\|Au\|_{-\widetilde{k}}}{\|u\|_k} = \|\widetilde{R}^{\widetilde{k}/2} A R^{k/2}\|_{0 \to 0}. \qquad (5)$$

- The norm of the adjoint $A^* : \widetilde{\mathcal{H}}_{\widetilde{k}} \to \mathcal{H}_{-k}$ is given by

$$\|A^*\|_{\widetilde{k} \to -k} = \|A\|_{k \to -\widetilde{k}}. \qquad (6)$$

Furthermore,

$$\|A\|_{k\to -\widetilde{k}} \leq \|A\|_{m\to -\widetilde{m}} \quad \textit{provided} \quad k \geq m, \widetilde{k} \geq \widetilde{m} \qquad (7)$$

since

$$\begin{aligned} \|A\|_{k\to -\widetilde{k}} &= \|\widetilde{R}^{\widetilde{k}/2} A R^{k/2}\|_{0\to 0} \\ &= \|\widetilde{R}^{(\widetilde{k}-\widetilde{m})/2} \widetilde{R}^{\widetilde{m}/2} A R^{m/2} R^{(k-m)/2}\|_{0\to 0} \\ &\leq \|A\|_{m\to -\widetilde{m}} \end{aligned}$$

and $\|R\|, \|\widetilde{R}\| \leq 1$.

3. Closeness assumption.

Let us explain the following concept of *quasi-unitary* opertators in the case of unitary operators: Suppose we have a unitary operator $J : \mathcal{H} \to \widetilde{\mathcal{H}}$ with inverse $J' = J^* : \widetilde{\mathcal{H}} \to \mathcal{H}$ respecting the quadratic form domains, i.e. $J_1 := J|_{\mathcal{H}_1} : \mathcal{H}_1 \to \widetilde{\mathcal{H}}_1$ and $J'_1 := J^*|_{\widetilde{\mathcal{H}}_1} : \widetilde{\mathcal{H}}_1 \to \mathcal{H}_1$. If

$$J_1'^* H = \widetilde{H} J_1$$

then H and $\widetilde{H}$ are unitarily equivalent and have therefore the same spectral properties. Note that J respects the quadratic form domain and therefore, $J_1'^* : \mathcal{H}_{-1} \to \widetilde{\mathcal{H}}_{-1}$ is an extention of $J : \mathcal{H} \to \widetilde{\mathcal{H}}$.

Suppose H and $\widetilde{H}$ are self-adjoint non-negative operators acting in the Hilbert spaces $\mathcal{H}$ and $\widetilde{\mathcal{H}}$. h and $\widetilde{h}$ denote the sesquilinear closed forms associated to H and $\widetilde{H}$. We have linear operators

$$J : \mathcal{H} \to \widetilde{\mathcal{H}}, J' : \widetilde{\mathcal{H}} \to \mathcal{H}, \qquad J_1 : \mathcal{H}_1 \to \widetilde{\mathcal{H}}_1, J_1' : \widetilde{\mathcal{H}}_1 \to \mathcal{H}_1. \quad (8)$$

Let $\delta > 0$ and $k \geq 1$. We say that $(H, \mathcal{H})$ and $(\widetilde{H}, \widetilde{\mathcal{H}})$ are $\delta -$ *close* with respect to the *quasi-unitary maps* (J, J_1) and (J', J_1') of *order* k iff the following conditions are fullfilled:

$$\|Jf - J_1 f\|_0 \leq \delta \|f\|_1, \|J'u - J_1' u\|_0 \leq \delta \|f\|_1 \quad (9)$$

$$|\langle Jf, u\rangle - \langle f, J'u\rangle| \leq \delta \|f\|_0 \|u\|_0 \quad (10)$$

$$|\widetilde{h}(J_1 f, u) - h(f, J_1' u)| \leq \delta \|f\|_k \|u\|_1 \quad (11)$$

$$\|f - J'Jf\|_0 \leq \delta \|f\|_1, \|u - JJ'u\|_0 \leq \delta \|u\|_1 \quad (12)$$

$$\|Jf\|_0 \leq 2, \|J'u\|_0 \leq 2 \quad (13)$$

Definition 1. (Definition of δ-closeness)

for all f, u in the appropiate spaces.
Here $\|f\|_0 = \|f\|_{\mathcal{H}}, \quad \|f\|_1 := \|f\|_{\mathcal{H}_1} = h[f, f] + \|f\|_0$.
And $h(f, g) = \langle H^{1/2}, H^{1/2} \rangle$ for $f, g \in \operatorname{dom} h = \mathcal{H}_1$ and similarly for $\widetilde{h}$.

Examples

Suppose that $\widetilde{\mathcal{H}} = \mathcal{H}$, $J = J' = \mathbf{1}$, $J_1 = J_1' = \mathbf{1}$,$k = 1$ and $\delta = \delta_n \to 0$ as $n \to \infty$. Assume in addition that the quadractic form domains of A and $\widetilde{H} = H_n$ agree. Now the only non-trivial assumption in Definition 1 is Equation (11), which is equivalent to

$$\|H_n - H\|_{1 \to -1} = \|R_n^{1/2}(H_n - H)R^{1/2}\|_{0 \to 0} \to 0$$

whereas $H_n \to H$ in norm resolvent convergence means

$$\|R_n - R\|_{0\to 0} = \|R_n(H_n - H)R\|_{0\to 0} = \|H_n - H\|_{2\to -2} \to 0$$

as $n \to \infty$. Therefore, we see that our assumption (11) implies the norm resolvent convergence but not vice versa.

Remark

We have expressed the closeness of certain quantities in dependence on the initial closeness data $\delta > 0$. Although, in our applications, $(\widetilde{H}, \widetilde{\mathcal{H}})$ will depend on some parameter $\varepsilon > 0$ with $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ we prefer to express the dependece only in terms of δ. In particular, an assertion like $\|JR - \widetilde{R}J\| \le 4\delta$ means that it is true for all $(H, \mathcal{H})$ and $(\widetilde{H}, \widetilde{\mathcal{H}})$ should be considered as "variables" being close to each other.

Lemma.

Suppose that Assumption (10), (12) and (13) are fulfilled, then

$$\|f\|_0 - \delta' \|Jf\|_1 \leq \|Jf\|_0 \leq \|f\|_0 + \delta' \|f\|_1 \quad \textit{with} \quad \delta' := \sqrt{3\delta} \quad (14)$$

and similarly for J'.

4. Resolvent convergence and functional calculus.

To prove the result on resolvent convergence, we estimate the errors in terms of δ. All the results below are valid for pairs of non-negative operators and Hilbert spaces $(H, \mathcal{H})$ and $(\widetilde{H}, \widetilde{\mathcal{H}})$ which are δ-close of order k. We set

$$m := max\{0, k-2\} \tag{15}$$

as regularity order for the resolvent difference. Note that $m = 0$ if $k = 1$ or $k = 2$.

Theorem 1.

Suppose (9), (10) and (11), then

$$\|\widetilde{R}J - JR\|_{0\to 0} = \|JH - \widetilde{H}J\|_{2\to -2} \leq 4\delta, \tag{16}$$

$$\|\widetilde{R}^j J - JR^j\|_{0\to 0} \leq 4j\delta \tag{17}$$

$\forall j \in \mathbb{N}$.

The authors want to extend their results to more general functions $\varphi(H)$ of the operator H and similarly for $\widetilde{H}$. They start with continuous functions on $\mathbb{R}_+ := [0, \infty)$ such that $lim_{\lambda \to \infty} \varphi(\lambda)$ exist, i.e., with functions continuous on $\overline{\mathbb{R}}_+ := [0, \infty]$. We denote this space by $C(\overline{\mathbb{R}}_+)$.

Theorem 2.

Suppose that (9), (10), (11) and (13) are fulfilled, then

$$\|\varphi(\widetilde{H})J - J\varphi(H)\|_{m\to 0} \leq \eta_\varphi(\delta) \tag{18}$$

for all $\varphi \in C(\overline{\mathbb{R}}_+)$ where $\eta_\varphi(\delta) \to 0$ as $\delta \to 0$.

In a second step authors extend the previous result to certain bounded measurable functions $\psi : \overline{\mathbb{R}}_+ \to \mathbb{C}$.

Theorem 3.

Suppose that $U \in \overline{\mathbb{R}}_+$ and that $\psi : \overline{\mathbb{R}}_+ \to \mathbb{C}$ is measurable, bounded function, continuous on U such that $\lim_{\lambda\to\infty} \psi(\lambda)$ exists. Then

$$\|\psi(\widetilde{H})J - J\psi(H)\|_{m\to 0} \leq \eta_\psi(\delta) \tag{19}$$

for the pairs of non-negative operators and Hilbert spaces $(H, \mathcal{H})$ and $(\widetilde{H}, \widetilde{\mathcal{H}})$ which are δ-close provided

$$\sigma(H) \subset U \quad or \quad \sigma(\widetilde{H}) \subset U.$$

Furthermore, $\eta_\psi(\delta) \to 0$ as $\delta \to 0$.

Examples

Consider $\psi = \mathbf{1}_I$ with interval I such that $\partial I \cap \sigma(H) = \emptyset$ or $\partial I \cap \sigma(\widetilde{H}) = \emptyset$ then the spectral projections satisfy

$$\|\mathbf{1}_I(\widetilde{H})J - J\mathbf{1}_I(H)\|_{m\to 0} \le \eta_{\mathbf{1}_I}(\delta). \tag{20}$$

Theorem 4.

Suppose that (10), (12), (13) and $\|\varphi(\widetilde{H})J - J\varphi(H)\|_{m\to 0} \le \eta$ for some function φ and some constant $\eta > 0$. Then we have

$$\|\varphi(H)J' - J'\varphi(\widetilde{H})\|_{0\to -m} \le 2\|\varphi\|_\infty \delta + \eta \tag{21}$$

$$\|\varphi(H) - J'\varphi(\widetilde{H})J\|_{m\to 0} \le C\delta + 2\eta \tag{22}$$

$$\|\varphi(\widetilde{H}) - J'\varphi(H)J\|_{0\to 0} \le 5C\delta + 2\eta \tag{23}$$

provided $m = 0$ for the last estimate. Here $C := \|\varphi\|_\infty$ if $m \ge 1$ and $C > 0$ is a constant satisfying $|\varphi(\lambda)| \le C(\lambda + 1)^{-1/2}$ for all λ if $m = 0$.

5. Spectral convergence.

The authors proved some convergence results for spectral projections and (parts) of spectrum.

Theorem 5.

Let I be a measurable and bounded subset of $\mathbb{R}$. Then there exists $\delta_0 = \delta_0(I, k) > 0$ such that for all $\delta > 0$ we have

$$\dim P = \dim \widetilde{P}$$

for all pairs of non-negative operators and Hilbert spaces $(H, \mathcal{H})$ and $(\widetilde{H}, \widetilde{\mathcal{H}})$ are δ-close of order k provided

$$\partial I \cap \sigma(H) = \emptyset \quad or \quad \partial I \cap \sigma(\widetilde{H}) = \emptyset.$$

Here, $P := 1_I(H)$ and $\dim P := \dim \mathrm{P}(\mathcal{H})$, similarly for $\widetilde{H}$.

In case of 1-dimentional projections we can even show the convergence of the corresponding eigenvectors. Note that generically, the eigenvalues are simple :

Theorem 6.

Suppose that φ is a normalised eigenvector of H with eigenvalue λ and that dim $\mathbf{1}_I(H) = 1$ for some open, bounded interval $\mathrm{I} \subset [0,\infty)$ containing λ. Then there exists $\delta_0 = \delta(I,k) > 0$ such that $\widetilde{H}$ has only one eigenvalue $\tilde{\lambda}$ of multiplicity 1 in I for all $(\widetilde{H}, \widetilde{\mathcal{H}})$ being δ-close of order k to $(H, \mathcal{H})$ and all $0 < \delta < \delta_0$.

In addition, there exist a unique eigenvector $\tilde{\varphi}$ (up to a unitary scalar factor close to 1) and functions $\eta_{1,2}(\delta) \to 0$ as $\delta \to 0$ depending only on λ and k such that

$$\|J\varphi - \widetilde{\varphi}\| \leq \eta_1(\delta), \quad \|J'\widetilde{\varphi} - \varphi\| \leq \eta_2(\delta).$$

Theorem 7.

There exists $\eta(\delta) > 0$ with $\eta(\delta) \to 0$ as $\delta \to 0$ such that

$$\overline{d}(\sigma_\bullet(H), \sigma_\bullet(\widetilde{H})) \leq \eta(\delta)$$

for the pairs of non-negative operators and Hilbert spaces $(H, \mathcal{H})$ and $(\widetilde{H}, \widetilde{\mathcal{H}})$ which are δ-close. Here $\sigma_\bullet(H)$ denotes either the entire spectrum, the essential or the discrete spectrum of H.

Furthermore, the multiplicity of the discrete spectrum is preserved, i.e., if $\lambda \in \sigma_{disc}(H)$ has multiplicity $m > 0$ then dim $\mathbf{1}_I(\widetilde{H}) = m$ for $I := (\lambda - \eta(\delta), \lambda + \eta(\delta))$ provided δ is small enough.

We have the following consequences when $\sigma_{disc}(H) = \emptyset$ resp.$\sigma_{ess}(H) = \emptyset$:

Corollary 8.

Suppose that H has purely essential spectrum. Then for each $\lambda \in \sigma_{ess}(H)$ there is essential spectrum close to λ for $\widetilde{H}$ being δ-close to H. Either $\widetilde{H}$ has no discrete spectrum or the discrete spectrum merges into the essential spectrum as $\delta \to 0$.

Corollary 9.

Suppose that H has purely discrete spectrum denoted by λ_k (repeated according to multiplicity). Then the infimum of the essential spectrum of $\widetilde{H}$ tends to infinity (if there where any) and there exists $\eta_k(\delta) > 0$ with $\eta_k(\delta) \to 0$ as $\delta \to 0$ such that $|\lambda_k - \tilde{\lambda}_k| \leq \eta_k(\delta)$ for all $(\widetilde{H}, \widetilde{\mathcal{H}})$ being δ-close. Here, $\tilde{\lambda}_k$ denotes the discrete spectrum of $\widetilde{H}$ (below the essential spectrum) repeated according to multiplicity.

3.1. Metric graphs.

Metric graph $X_0 = (V, E, \partial, l)$ is a countable, connected metric graph, i.e., V denotes the set of vertices, E the set of edges and $\partial : E \to V \times V, \partial e = (\partial_+ e, \partial_- e)$ denotes the pair of the end point and the starting point of the edge e. For each vertex $v \in V$ we denote by

$$E_v^{\pm} := \{e \in E | \partial_{\pm e} = v\}$$

the edges starting $(-)$ ending $(+)$ at v. Let $E_v := E_v^+ \cup E_v^-$ be *disjoint* union of all edges emanating at v. The *degree* of a vertex v is the number of vertices emanating at v, i.e.,

$$\deg v := |E_v| = |E_v^+| + |E_v^-|.$$

We assume that X_0 is locally finite, i.e., $\deg v \in \mathbb{N}$. Note that we allow loops, i.e., edges e with $\partial_+ e = \partial_- e = v$. A loop e will be counted twice in *degv* and occurs twice in E_v due to the disjoint union.

In addition, we assume that ∂e always consists of two elements, even if $\partial_- e = \partial_+ e = v$ for a loop e. We also allow multiple edges, i.e., edges $e_1 \neq e_2$ having the same starting and end points. Finally, $l : E \to (0, \infty]$ assigns a length l_e to each edge $e \in E$ making the graph V, E, ∂ a *metric or quantum* graph.

Remark.

- A *finite* metric graph is a graph with finitely many vertices and edges.
- A *compact* graph must in addition have finite edge length for each edge.
- A compact metric graph is finite but not vice versa (star-shaped metric graph with on vertex and a finite number of leads attached to the vertex).

We also assign a *density* p_e to each edge $e \in E$,i.e., a measurable function $p_e : e \to (0, \infty)$. For simplicity, we asume that p_e is smooth in order to obtain a smooth metric in the graph-like manifold. The data (V, E, ∂, l, p), $p = (p_e)_e$ discrible a *weighted* metric graph.

The Hilbert space associated to such a graph is

$$\mathcal{H} := L^2(X_0) = \oplus L^2(e)$$

which consists of all functions f with finite norm

$$\|f\|^2 = \|\|^2_{X_0} = \sum_{e\in E} \|f_e\|^2_e = \sum_{e\in E} \int_e |f_e(x)|^2 p_e(x)dx.$$

We define the limit operator H via the quadratic form

$$h(f) := \sum_{e \in E} \int_e |f_e'(x)|^2 p_e(x) dx$$

for functions f in

$$\mathcal{H}_1 := H^1(X_0) := C(X_0) \cap \oplus H^1(e).$$

Note that a weakly differentiable function on interval e, i.e., $f_e \in H^1(e)$, is automatically continuous. Therefore, the continuity is only a condition at each vertex. **h** is closed form, i.e.,$\mathcal{H}_1$ together with the norm

$$\|f\|_1^2 = \|f\|_{1,X_0}^2 := \|f\|_{X_0}^2 + h(f)$$

is complete.

The associated self-adjoint, non-negative operator $H = \Delta_{X_0}$ is given by

$$(\Delta_{X_0} f)_e = -\frac{1}{p_e}(p_e f_e')'$$

on each edge e. If $l_e > l_0$ for all $e \in E$ then the domain $\mathcal{H}_2$ of $H = \Delta_{X_0}$ consists of all functions $f \in L_2(X_0)$ such that $\Delta_{X_0} f \in L_2(X_0)$. f satisfied the so-called *(generalised) Neumann boundary condition (Kirchhoff)* at each vertex v,i.e., f is continuous at v and

$$\sum_{v \in E_v} p_e(v) f_e'(v) = 0$$

for all $v \in V$. We set $f_e'(v) := f_e'(0)$ if $v = \partial_{-e}$ and $f_e'(v) := f_e'(l_e)$ if $v = \partial_{+e}$ (considering f_e as function on the interval $(0, l_e)$). We call Δ_{X_0} the *(generalised) weighted Neumann Laplacian* on X_0.

3.2. Graph-like manifolds.

Let X_0 be a weighted mertic graph. The corresponding family of graph-like manifolds X_ε is given as follows: For each $0 < \varepsilon < \varepsilon_0$ we associate with the graph X_0 a connected Riemannian manifold X_ε of dimension $d \geq 2$ with or without boundary equipped with a metric g_ε. The boundary of X_ε need not to be smooth; we allow singularities on the boundary of the vertex neighborhood $U_{\varepsilon,v}$. X_ε is the union of the closure of open subsets $U_{\varepsilon,e}$ and $U_{\varepsilon,v}$ such that the $U_{\varepsilon,e}$ and $U_{\varepsilon,v}$ are mutually disjoint for all possible combinations of $e \in E$ and $v \in V$, i.e.,

$$X_\varepsilon = \bigcup_{e \in E} \overline{U}_{\varepsilon,e} \cup \bigcup_{v \in V} \overline{U}_{\varepsilon,v}.$$

- We assume that $U_{\varepsilon,e}$ and $U_{\varepsilon,v}$ are independent of ε as manifolds, i.e., only their metric g_ε depend on ε.
- $U_{\varepsilon,e}$ is diffeomorphic to $U_e := e \times F$ for all $0 < \varepsilon \le \varepsilon_0$ where F denotes a compact and connected manifold (with or without a boundary) of dimension $m := d - 1$. We fix a metric h on F and assume for simplicity that $volF = 1$.
- $U_{\varepsilon,v}$ is diffeomorphic to an ε-independent manifold U_v for $0 < \varepsilon \le \varepsilon_0$.
  Therefore, $U_{\varepsilon,e} \cong (U_e, g_{\varepsilon,e})$ and $U_{\varepsilon,v} = (U_v, g_{\varepsilon,v})$.
- The corresponding Hilbert space is then

$$\widetilde{\mathcal{H}} := L^2(X_\varepsilon) = \bigoplus_{e\in E} L^2(U_{\varepsilon,e}) \oplus \bigoplus_{v\in V} L^2(U_{\varepsilon,v})$$

which consists of all functions u with finite norm

$$\begin{aligned} \|u\|^2 = \|u\|^2_{X_\varepsilon} &= \sum_{e\in E} \|u_e\|^2_{U_{\varepsilon,e}} + \sum_{v\in V} \|u_v\|^2_{U_{\varepsilon,v}} \\ &= \sum_{e\in E} \int_{e\times F} |u_e|^2 detg^{1/2}_{\varepsilon,e} dxdy + \sum_{v\in V} \int_{U_v} |u_v|^2 detg^{1/2}_{\varepsilon,e} dz \end{aligned}$$

where y and z represent coordinates of F and U_v.

- The operator $\widetilde{H}$ is the Laplacian on X_ε, i.e., $\widetilde{H} = \Delta_{X_\varepsilon}$. We assume Neumann boundary conditions on the boundary part coming from ∂F.
- We define Δ_{X_ε} via its quadractic form $\widetilde{h}$ given by

$$\widetilde{h} = \int_{X_\varepsilon} |du|^2_{g_\varepsilon} dX_\varepsilon$$

for functions $u \in \widetilde{\mathcal{H}}_1 = H^1(X_\varepsilon)$ with the norm

$$\|u\|^2_1 = \|u\|^2_{1,X_\varepsilon} := \|u\|^2_{X_\varepsilon} + \widetilde{h}(u).$$

3.3 Quasi-unitary operators.

We define the operator $J : \mathcal{H} \to \widetilde{\mathcal{H}}$ by

$$Jf(z) = \begin{cases} \varepsilon^{-m/2} f_e(x) & \text{if} \quad z = (x, y) \in U_e, \\ 0 & \text{if} \quad z \in U_v \end{cases}$$

and the operator $J_1 : \mathcal{H}_1 \to \widetilde{\mathcal{H}_1}$ by

$$J_1 f(z) = \begin{cases} \varepsilon^{-m/2} f_e(x) & \text{if} \quad z = (x, y) \in U_e, \\ 0 & \text{if} \quad z \in U_v \end{cases}$$

We introduce the following averaging operators

$$(N_e u) := \langle \varphi_{F,1}, u_e(x, .) \rangle_F = \int_F u_e(x, y) dF(y),$$

$$C_v u : = \langle \varphi_{U_v,1}, u_v \rangle_{U_v} = \frac{1}{vol U_v} \int_{U_v} u dU_v$$

for $u \in \widetilde{\mathcal{H}} = L^2(X_\varepsilon)$ giving the coefficient corresponding to the first eigenfunction φ_1 on U_e resp. U_v. Note that these eigenfunctions are constant and vol $F = 1$.
We define $J' : \widetilde{\mathcal{H}} \to \mathcal{H}$ by

$$(J'u)_e(x) := \varepsilon^{m/2}(N_e u)(x), \quad x \in e$$

and the operator $J'_1 : \widetilde{\mathcal{H}}_1 \to \mathcal{H}_1$ by

$$(J'_1 u)_e(x) := \varepsilon^{m/2} [N_e u(x) \\ + p_e^+(x)[C_{\partial_{+e}} u - N_e u(\partial_{+e})] + p_e^-(x)[C_{\partial_{-e}} u - N_e u(\partial_{-e})]]$$

for $x \in e$. Here, $\rho_e^\pm : \mathbb{R} \to [0, 1]$ are the continuous, piecewise affine functions given by $p_e^+(\partial_{+e}) = 1$ and $p_e^+(x) = 0$ for all $\text{dist}(x, \partial_{+e}) \geq min 1, l_e/2$ and similarly for p_e^- and ∂_{-e}. Note that $(J'_1 u)_e(v) = C_v u$ for $v = \partial_{\pm e}$. In particular, $J'_1 u$ is a continuous function on X_0. Again, the operator J'_1 is only defined on $\widetilde{\mathcal{H}}_1 = H^1(X_\varepsilon)$.

The closeness assumption as follows:

$$\|Jf - J_1 f\|^2 = \sum_{v\in V} \varepsilon^{-m} volU_{\varepsilon,v}|f(v)|^2$$

$$\|J'u - J_1' u\|^2 = \sum_{e\in E}\sum_{v\in\partial e} \varepsilon^{m}\|p_e^{\pm}\|_e^2|C_v u - N_e u(v)|^2$$

$$\begin{aligned}&|\langle Jf, u\rangle - \langle f, J'u\rangle| \\ &= |\sum_{e\in E}\int_{e\times F} \overline{f}(x)u(x,y)\varepsilon^{-m/2}[dU_{\varepsilon,e}(x,y) - \varepsilon^m dF(y)p_e(x)dx]|\end{aligned}$$

$$\begin{aligned}&|\widetilde{h}(J_1 f, u) - h(f, J_1' u)| \\ &= |\sum_{e\in E}\int_{e\times F} \overline{f'}(x)\partial_x u(x,y)\varepsilon^{-m/2}[g^{xx}_{\varepsilon,e}dU_{\varepsilon,e}(x,y) - \varepsilon^m dF(y)p_e(x)dx] \\ &\qquad - \sum_{e\in E}\sum_{v\in\partial e}\varepsilon^{-m/2}(C_v u - N_e u(v))\left\langle f_e', (\rho_e^{\pm})'\right\rangle_e |\end{aligned}$$

$$\|JJ'u - u\|^2 = \sum_{e \in E} \|N_e u - u\|^2_{U_{\varepsilon,e}} + \sum_{v \in V} \|u\|^2_{U_{\varepsilon,v}}$$

$$\|Jf\|^2 = \sum_{e \in E} \int_{e \times F} |f(x)|^2 \varepsilon^{-m} dU_{\varepsilon,e}(x,y)$$

$$\|J'u\|^2 \leq \int_{e \times F} |u(x,y)|^2 \varepsilon^m dF(y) p_e(x) dx$$

Here, the sign in $\rho_e^{\pm}$ is used according to $v = \partial_{\pm} e$. Note that $J'Jf = f$, vol $U_{\varepsilon,e} = o(\varepsilon^m)$, $g_{\varepsilon,e}$ must be close to a product metric on $U_e = e \times F$.

3.4. Assumption on the graph.

For the graph data we require that the degree is uniformly bounded,i.e., that there exists $d_0 \in \mathbb{N}$ such that

$$\deg v \leq d_0, \quad v \in V. \tag{G1}$$

There is a uniform lower bound on the set of length, i.e., there exists $l_0 > 0$ (without loss of generality $l_0 \leq 1$) such that

$$l_e \geq l_0 \quad \text{for all } e \in E. \tag{G2}$$

We assume that the density function p_e is uniformly bounded, i.e., there exist constants $p_\pm > 0$ such that

$$\begin{aligned} p_{\leq} p_e(x), \quad \operatorname{dist}(x, \partial_\pm e) \leq min 1, l_e/2, \quad e \in E, \\ p_e(x) \leq p_+, \quad x \in e, \quad e \in E. \end{aligned} \tag{G3}$$

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Definition.

A *uniform* weighted metric graph is a weighted metric graph $X_0 = (V, E, \partial, l, p)$ satisfying (G1)-(G3). We conlude the following estimates:

Lemma.

We have

$$\sum_{v \in V} |f(v)|^2 \leq \frac{4}{l_0 p_-} \|f\|_1^2$$

for all $f \in \mathcal{H}_1 = H^1(X_0)$.

Lemma.

The estimate

$$\|p_e^{\pm}\|_e^2 \leq p_+ \quad \textit{and} \quad \|(\rho_e^{\pm})'\|_e^2 \leq \frac{2p_+}{l_0}$$

holds for all $e \in E$.

Assumptions on the manifold.

We assume that the metric $g_{\varepsilon,e}$ on the edge neighborhood $U_e = e \times F$ is given as a perturbation of the product metric

$$\overline{g}_{\varepsilon,e} := dx^2 + \varepsilon^2 r_e^2(x) h(y), \quad (x,y) \in U_e = e \times F$$

with $r_e(x) := (p_e(x))^{1/m}$ where h is the fixed metric on F, $m = dimF = d - 1$ and p_e is the density function of the metric graph on the edge e.

We denote by $G_{\varepsilon,e}$ and $\overline{G}_{\varepsilon,e}$ the $d \times d$-matrices associated to the metrics $g_{\varepsilon,e}$ and $\overline{g}_{\varepsilon,e}$ with respect to the coordinates (x, y) and assume that the two metrics coincide up to an error term as $\varepsilon \to 0$, more specifically

$$G_{\varepsilon,e} = \overline{G}_{\varepsilon,e} + \begin{pmatrix} o(1) & o(\varepsilon) r_e \\ o(\varepsilon) r_e & o(\varepsilon^2) r_e^2 \end{pmatrix} = \begin{pmatrix} 1 + o(1) & o(\varepsilon) r_e \\ o(\varepsilon) r_e & (\varepsilon^2 + o(\varepsilon^2)) r_e^2 \end{pmatrix} \tag{G4}$$

uniformly on U_e.

We can show the following estimates

$$dU_{\varepsilon,e}(x,y) = (1+o_1(1))\varepsilon^m dF(y)p_e(x)dx \quad (24)$$

$$g^{xx}_{\varepsilon,e} := (G^{-1}_{\varepsilon,e})_{xx} = 1 + 0_2(1) \quad (25)$$

$$|dxu|^2 \leq O_3(1)|du|^2_{g_{\varepsilon,e}} \quad (26)$$

$$|d_F u|^2_h \leq o_4(\varepsilon)|du|^2_{g_{\varepsilon,e}} \quad (27)$$

On the vertex neighborhood U_v we assume that the metric $g_{\varepsilon,v}$ satisfies

$$c_-\varepsilon^2 g_v(z)(w,w) \leq g_{\varepsilon,v} \leq c_+\varepsilon^{2\alpha} g_v \quad \text{(G5)}$$

The number α in the exponent is assumed to satisfy the inequalities

$$\frac{d-1}{d} < \alpha \leq 1 \quad \text{(G6)}$$

$$C_{\text{vol}} := \sup_{v\in V} \text{vol} U_v < \infty \text{ and } \lambda_2 := \inf \lambda_2^N(U_v) > 0 \quad \text{(G7)}$$

where $\lambda_2^N(U_v)$ denotes the second (i.e., the first non-zero) Neumann eigenvalue of Δ_{U_v}.

Definition.

A family of graph-like manifolds X_ε with respect to the uniform metric graph X_0 will be called *uniform* if (G4)-(G7) are satisfied.

We are now able to estimate the RHS of the closeness assumptions as following:

$$\|Jf - J_1 f\|^2 \leq \frac{4c_+^{d/2} C_{vol}}{l_0 p_-} \varepsilon^{\alpha d - m} \|f\|_1^2.$$

Next we have

$$\|J'u - J_1'u\|^2 \leq p_+ \widetilde{c}_{tr} \varepsilon^{2\alpha - 1} \sum_{e \in E} \sum_{v \in \partial e} \|du\|^2_{U_\varepsilon, v} \leq d_0 p_+ \widetilde{c}_{tr} \varepsilon^{2\alpha - 1} \widetilde{h}(u).$$

We have the estimate

$$|\widetilde{h}(J_1 f, u) - h(f, J_1' u)| \leq \left(o(1) + \left[\frac{2 d_0 p_+ \widetilde{c}_{tr}}{l_0} \right]^{1/2} \right) h(f)^{1/2} \widetilde{h}(u)^{1/2}$$

and

$$\begin{aligned} \|JJ'u - u\|^2 &= \sum_{e \in E} \|N_e u - u\|^2_{U_{\varepsilon,e}} + \sum_{v \in V} \|u\|^2_{U_{\varepsilon,v}} \\ &\leq c_{ed} o_4(\varepsilon) \sum_{e \in E} \|du\|^2_{U_{\varepsilon,e}} + c_{vx} \varepsilon^{\alpha d - m} \sum_{v \in V} \|u\|^2_{1,\widehat{U}_{\varepsilon,v}} \\ &\leq 3 \left(c_{ed} o_4(\varepsilon) + c_{vx} \varepsilon^{\alpha d - m} \right) \|u\|_1^2. \end{aligned}$$

$$\|Jf\|^2 \leq (1 + o_1(1)) \|f\|^2 \text{ and } \|u\|^2 \leq \frac{1}{1 - o_1(1)} \|u\|^2.$$

We therefore have proven.

Theorem.

Suppose that the metric graph X_0 and the family of graph-like manifolds X_ε is given as below and satisfy the uniformity condition (G1)-(G7). Then the generalised weighted Neumann Laplacian on the graph $(\Delta_{X_0}, L^2(X_0))$ and the (Neumann) Laplacian on the manifold $(\Delta_{X_\varepsilon}, L^2(X_\varepsilon))$ are δ-close of order 1 where $\delta = o(1)$ as $\varepsilon \to 0$. In particular, all the results of Appendix A are true, e.g., the convergence of eigenfunctions stated in Theorem 6 or the spectral convergence in Theorem 7.

EXAMPLES AND APPLICATIONS OF SPECTRAL CONVERGENCE

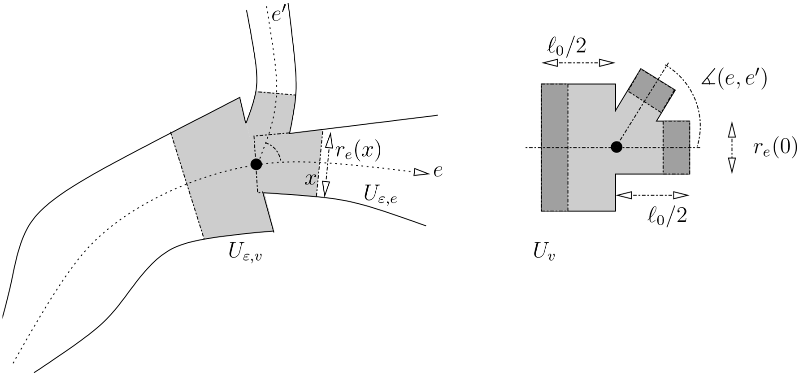

Figure: Decomposition of the weighted neighborhood X_ε and the unscaled vertex neighborhood U_v.

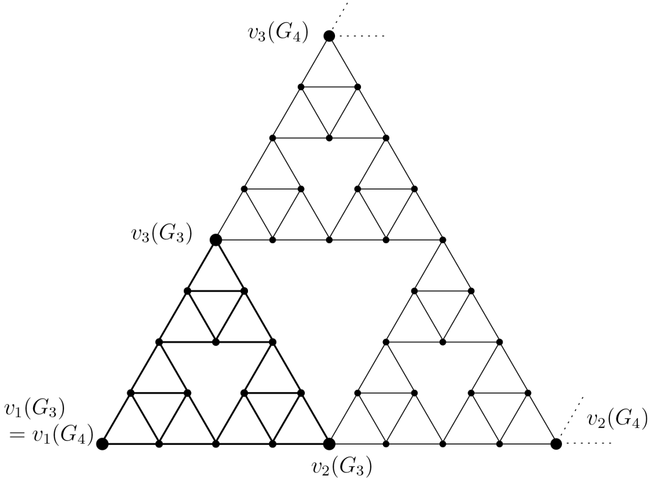

Figure: The first four generations G_4 of infinite Sierpiński graph, each edge having unit length. The graph G_3 is denoted with thick edges and is naturally embedded into G_4.

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Thank you very much for your attention!

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