Selected applications of multivalued algebras in the algebraic theory of quadratic forms

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Hypergroups, hyperrings and hyperfields.

Hypergroups.

A **hypergroup** is an object just like a group, but with binary operation allowed to take multiple values:

$(H,+,-,0)$, $+\colon H\times H\to 2^H$, $-\colon H\to H$, $0\in H$.

i. $(a+b)+c=a+(b+c)$, $a,b,c\in H$,

ii. $a+0=0+a=\{a\}$, $a\in H$,

iii. $a+b=b+a$, $a,b\in H$,

iv. $a\in b+c\Rightarrow b\in a+(-c)$, $a,b,c\in H$.

1. F. Marty, *Sur une generalisation de la notion de group*, in: 8th Congress Math. Scandinaves, Stockholm, 1934, pp. 45–49.

2. M. Krasner, *Approximation des corps values complets de caracteristique $p\neq 0$ par ceux de caracteristique 0*, in: Colloque d'Algebre Superieure, Brussels, 1956, pp. 129–206.

Some examples.

Example 1. Any abelian group $(G,+)$ becomes a hypergroup with $+: G\times G\to 2^G$ defined by

$$a+b=\{a+b\}.$$

Example 2. Let $Q_2=\{-1,0,1\}$. Define $+: Q_2\times Q_2\to 2^{Q_2}$ by taking 0 to be the neutral element and

$$(-1)+(-1)=\{-1\},\ \ 1+1=\{1\},\ \ 1+(-1)=(-1)+1=\{-1,0,1\}.$$

Think of "1" as "positive reals", "-1" as "negative reals" and "$+$" as the possible outcome of addition.

This is a hypergroup.

Category of hypergroups.

A **morphism** of hypergroups $H_1 \xrightarrow{f} H_2$ is a function $f\colon H_1 \to H_2$ such that

i. $f(a+b) \subseteq f(a) + f(b)$, $a, b \in H_1$,

ii. $f(-a) = -f(a)$, $a \in H_1$,

iii. $f(0) = 0$.

There is a fair amount of controversy on how to define morphisms. For what we are doing here the above definition is just fine, but, for example, the category of hypergroups with morphisms where the axiom i. has "$\subseteq$" instead of "$=$" fails to have equalizers.

Hyperrings and hyperfields.

A **hyperring** is a hypergroup $(H, +, -, 0)$ equipped with a binary operation $\cdot : H \times H \to H$ with the neutral element $1 \in H$ such that $(H, \cdot, 1)$ is a commutative monoid and

i. $a \cdot 0 = 0$, $a \in H$,

ii. $a \cdot (b + c) \subseteq a \cdot b + a \cdot c$,

iii. $1 \neq 0$.

If, on top of that, every nonzero element $a \in H$ has a multiplicative inverse, we call H a **hyperfield**.

A **morphism** $H_1 \xrightarrow{f} H_2$ of hyperrings is just a morphism of underlying hypergroups such that

i. $f(a \cdot b) = f(a) \cdot f(b)$, $a, b \in H_1$,

ii. $f(1) = 1$.

Note that axioms of hyperfields slightly differ from axioms of fields: here, for example, the axiom i. is not a consequence of other axioms.

Modern applications.

A) Number theory, incidence geometry, and geometry in characteristic one.

1. A. Connes, C. Consani, *From monoids to hyperstructures: in search of an absolute arithmetic*, in: G. van Dijk (ed.) et al., Casimir force, Casimir operators and Riemann hypothesis. Mathematics for innovation in industry and science. Proceedings of the conference, Fukuoka, Japan, November 9–13, 2009, 147-198, de Gruyter, Berlin, 2010.
2. A. Connes, C. Consani, *The hyperring of adele classes,* J. Number Theory 131 (2011), 159–194.
3. A. Connes, C. Consani, *Universal Thickening of the Field of Real Numbers,* in: A. Alaca (ed.) et al., Advances in the The- ory of Numbers. Proceedings of the Thirteenth Conference of the Canadian Number Theory Association, 11–74, Fields Institute Communications 77, Springer, New York, 2015.

B) Tropical geometry.

1. O. Viro, *Hyperfields for tropical geometry i. hyperfields and dequantization,* arXiv:1006.3034, 2010.
2. O. Viro, *On basic concepts of tropical geometry,* Proc. Steklov Inst. Math. 273 (2011), 252–282.

C) Supertropical algebras.

1. Z. Izhakian, M. Knebusch, L. Rowen, *Layered tropical mathematics,* J. Algebra 416 (2014), 200–273.
2. Z. Izhakian, L. Rowen, *Supertropical algebra,* Adv. Math. 225 (2010), 2222-2286.

D) Algebraic geometry over hyperrings.

1. J. Jun, *Algebraic geometry over hyperrings,* Adv. Math. 323 (2018), 142-192.
2. J. Jun, *Hyperstructures of affine group schemes,* J. Number Theory 167 (2016), 336-352.

Applications to quadratic forms.

A) Witt equivalence.

1. P.G., M. Marshall, *Witt equivalence of function fields over global fields,* Trans. Amer. Math. Soc. 369 (2017), 7861-7881.
2. P.G., M. Marshall, *Witt equivalence of function fields of curves over local fields,* Comm. Algebra 45 (2017), 5002-5013.
3. P.G., M. Marshall, *Witt equivalence of function fields of conics*, Algebra & Disc. Math. 30 (2020), 63-78.
4. P.G., *Witt equivalence of fields: a survery with a special emphasis on applications of hyperfields,* in Ordered Algebraic Structures and Related Topic, 169-185, Contemp. Math. 697, Amer. Math. Soc., Providence, RI, 2017.

B) Orderings of higher level and root selections.

1. P.G., *Orderings of higher level in multirings and multifields,* Ann. Math. Silesianae 24 (2010), 15-25.
2. P.G., M. Marshall, *Orderings and signatures of higher level on multirings and hyperfields*, J. K-Theory 10 (2012), 489-518.
3. P.G., *n-th roots and orderings of level n*, Ann. Math. Silesianae, 33 (2019), 106-120.
4. P.G., *Root selections and 2^p-th root selections in hyperfields,* Discuss. Math., Gen. Algebra Appl. 39 (2019), 43-53.

C) Axiomatic theory of quadratic forms.

1. P.G., K. Worytkiewicz, *Witt rings of quadratically presentable fields,* Categ. Gen. Algebr. Struct. Appl. 12 (2020), 1-23.
2. P.G., K. Worytkiewicz, *Ordered monoids with exchange as generalised hyperalgebras*, submitted.

Witt equivalence.

Witt ring.

Similarity classes of nonsingular quadratic forms over F, $\operatorname{char} F \neq 2$, with addition and multiplication induced by orthogonal sum and tensor product form a commutative ring called the **Witt ring** of the field F and denoted by $W(F)$.

The Witt ring of a field F encodes, more or less, all information relevant to the orthogonal geometry over F.

1. E. Witt, *Theorie der quadratischen Formen in beliebigen Korpern*. J. Reine Ang. Math. 176 (1937) 31–44.

Witt equivalence.

We say two fields K and L are **Witt equivalent**, denoted $K \sim L$, if $W(K)$ and $W(L)$ are isomorphic as rings.

We shall explain in some detail what are the implications of Witt equivalence.

Two fields K and L of characteristic $\neq 2$ are said to be **equivalent with respect to quadratic forms**, if there exists a pair of bijections $t\colon K^\times / K^{\times 2} \to L^\times / L^{\times 2}$ and $T\colon \mathrm{Cl}(K) \to \mathrm{Cl}(L)$, where $\mathrm{Cl}(\cdot)$ denotes the set of equivalence classes of nonsingular quadratic forms over a given field, such that the following four conditions are satisfied:

i. $T(\langle a_1, \ldots, a_n \rangle) = \langle t(a_1), \ldots, t(a_n) \rangle$, $a_1, \ldots, a_n \in K^\times / K^{\times 2}$,

ii. $\det T(q) = t(\det q)$, for every nonsingular quadratic form q over K,

iii. $D_L(T(q)) = t(D_K(q))$, for every nonsingular quadratic form q over K,

iv. $t(1) = 1$, $t(-1) = -1$.

Theorem 3. (Harrison-Cordes criterion) *For two fields K and L of characteristic $\neq 2$ the following conditions are equivalent:*

i. K *and* L *are equivalent with respect to quadratic forms,*

ii. *there exists a group isomorphism* $t\colon K^\times/K^{\times 2} \to L^\times/L^{\times 2}$ *such that* $t(-1) = -1$*, and, for all* $a, b \in K^\times/K^{\times 2}$*:*

$$1 \in D_K(\langle a, b\rangle) \Leftrightarrow 1 \in D_L(\langle t(a), t(b)\rangle),$$

iii. $K \sim L$*,*

iv. $W(K)/I^3(K) \cong W(L)/I^3(L)$*,* $I(\cdot)$ *denoting the fundamental ideal of a given Witt ring.*

So, basically, for two fields to be Witt equivalent means to have same orthogonal geometries.

1. D.K. Harrison, *Witt rings.* University of Kentucky Notes, Lexington, Kentucky (1970).
2. C. Cordes, *The Witt group and equivalence of fields with respect to quadratic forms,* J. Algebra 26 (1973), 400–421.

Quotient hyperfields.

Let $(H, +, -, \cdot, 0, 1)$ be a hyperfield, let T be a subgroup of the multiplicative group $H^{\times}$.

Denote by $H/_m T$ the set of equivalence classes with respect to the equivalence relation $\sim$ on H defined by

$$a \sim b \text{ if and only if } as = bt \text{ for some } s, t \in T.$$

Denoting by $\bar{a}$ the equivalence class of a set $\bar{a}\bar{b} = \overline{ab}$, $-\bar{a} = \overline{-a}$, $0 = \bar{0}$, $1 = \bar{1}$ and

$$\bar{a} \in \bar{b} + \bar{c} \text{ if and only if } as \in bt + cu \text{ for some } s, t, u \in T.$$

$(H/_m T, +, \cdot, -, 0, 1)$ is then a hyperfield that we shall refer to as **quotient hyperfield**.

Quadratic hyperfields.

Let K be a field, $\operatorname{char} K \neq 2$, $K \neq \mathbb{F}_3, \mathbb{F}_5$.

Observe that, for $z, a, b \in K$ the following equivalence holds true:

$$z = ax^2 + by^2 \text{ for some } x, y \in K \text{ if and only if } \bar{z} \in \bar{a} + \bar{b} \text{ in } K /_m K^{\times 2}.$$

The hyperfield $K /_m K^{\times 2}$ will be called the **quadratic hyperfield** of K and denoted by $Q(K)$.

Harrison-Cordes criterion revisited.

Theorem 4. *Let K and L be any fields. Then $K \sim L$ if and only if $Q(K)$ and $Q(L)$ are isomorphis as hyperfields.*

What is known then?

A) Trivial examples: quadratically closed fields, real closed fields.

B) Less trivial, but still easy: finite fields.

C) Somewhat non-trivial, but elementary: local fields.

D) Global fields.

1. R. Perlis, K. Szymiczek, P.E. Conner, R. Litherland, Matching Witts with global fields. Contemp. Math. 155 (1994) 365–378.
2. K. Szymiczek, Matching Witts locally and globally. Math. Slovaca 41 (1991) 315–330.
3. K. Szymiczek, Hilbert-symbol equivalence of number fields, Tatra Mount. Math. Publ. 11 (1997), 7–16.

E) Function fields over algebraically closed fields and real closed fields.

1. P. Koprowski, Witt equivalence of algebraic function fields over real closed fields. Math. Z. 242 (2002) 323–345.
2. N. Grenier-Boley, D.W. Hoffmann, Isomorphism criteria for Witt rings of real fields. With appendix by Claus Scheiderer. Forum Math. 25 (2013) 1–18.

F) Function fields over global and local fields.

1. P.G., M. Marshall, *Witt equivalence of function fields over global fields,* Trans. Amer. Math. Soc. 369 (2017), 7861-7881.
2. P.G., M. Marshall, *Witt equivalence of function fields of curves over local fields,* Comm. Algebra 45 (2017), 5002-5013.
3. P.G., M. Marshall, *Witt equivalence of function fields of conics*, Algebra & Disc. Math. 30 (2020), 63-78.

Witt equivalence of function fields over global fields.

Theorem 5. *If function fields F and E over global fields are Witt equivalent, then the corresponding isomorphism of quadratic hyperfields $Q(F)$ and $Q(E)$ induces, in a canonical way, a bijection between the Abhyankar valuations of F and E, whose residue fields are neither finite, nor of characteristic 2.*

Recall that if F is a function field over k and v is a valuation on K, the Abhyankar inequality asserts that

$$\operatorname{trdeg}(F\colon k) \geq \operatorname{rk}_{\mathbb{Q}}(\Gamma_v/\Gamma_{v|k}) + \operatorname{trdeg}(F_v\colon k_{v|k})$$

where $v|k$ denotes the restriction of v to k.

For any abelian group Γ, $\operatorname{rk}_{\mathbb{Q}}(\Gamma) := \dim_{\mathbb{Q}}(\Gamma \otimes_{\mathbb{Z}} \mathbb{Q})$.

We will say the valuation v is *Abhyankar* (relative to k) if $\geq$ in the Abhyankar inequality is replaced with $=$.

In this case it is well known that $\Gamma_v/\Gamma_{v|k} \cong \mathbb{Z} \times \ldots \times \mathbb{Z}$ (with $\operatorname{rk}_{\mathbb{Q}}(\Gamma_v/\Gamma_{v|k})$ factors) and F_v is a function field over $k_{v|k}$.

Moreover, if v is Abhyankar (relative to k) then $\Gamma_v \cong \mathbb{Z} \times \ldots \times \mathbb{Z}$ (with $\operatorname{rk}_{\mathbb{Q}}(\Gamma_v)$ factors) and F_v is either a function field over a global field or a finite field.

For any field F, we define the *nominal transcendence degree* of F by

$$\operatorname{ntd}(F) = \begin{cases} \operatorname{trdeg}(F:\mathbb{Q}), & \text{if } \operatorname{char} F = 0, \\ \operatorname{trdeg}(F:\mathbb{F}_p) - 1, & \text{if } \operatorname{char} F = p. \end{cases}$$

Let F be a function field in n variables over a global field. For $0 \leq i \leq n$ denote by $\nu_{F,i}$ the set of Abyankar valuations v on F with $\operatorname{ntd}(F_v) = i$. Observe that

$$\nu_{F,i} = \nu_{F,i,0} \,\dot{\cup}\, \nu_{F,i,1} \,\dot{\cup}\, \nu_{F,i,2},$$

where

1. $\nu_{F,i,0}$ is the set of valuations of $\nu_{F,i}$ such that $\operatorname{char} F_v = 0$,
2. $\nu_{F,i,1}$ is the set of valuations of $\nu_{F,i}$ such that $\operatorname{char} F_v \neq 0, 2$,
3. $\nu_{F,i,0}$ is the set of valuations of $\nu_{F,i}$ such that $\operatorname{char} F_v = 2$.

Of course, some of the sets $\nu_{F,i,j}$ may be empty. Specifically, if $\operatorname{char}(F) = p$ for some odd prime p then $\nu_{F,i,j} = \emptyset$ for $j \in \{0, 2\}$, and if $\operatorname{char}(F) = 2$ then $\nu_{F,i,j} = \emptyset$ for $j \in \{0, 1\}$

The correspondence of Theorem 5 preserves the sets $\nu_{F,i,j}$. To be more specific, one has the following:

Theorem 6. *Suppose F, E are function fields in n variables over global fields which are Witt equivalent via a hyperfield isomorphism $\alpha\colon Q(F) \to Q(E)$. Then for each $i \in \{0, 1, ..., n\}$ and each $j \in \{0, 1, 2\}$ there is a uniquely defined bijection between $\nu_{F,i,j}$ and $\nu_{E,i,j}$ such that, if $v \leftrightarrow w$ under this bijection, then α maps $(1 + M_v)\, F^{\times 2}/F^{\times 2}$ onto $(1 + M_w)\, E^{\times 2}/E^{\times 2}$ and $U_v\, F^{\times 2}/F^{\times 2}$ onto $U_w\, E^{\times 2}/E^{\times 2}$.*

In particular, considering the bijection between $\nu_{F,0,0}$ and $\nu_{E,0,0}$ yields the following result:

Theorem 7. *Let $F \sim E$ be function fields over number fields, with fields of constants k and ℓ respectively. If there exists $v \in \nu_{F,0,0}$ with $F_v = k$ and $w \in \nu_{E,0,0}$ with $E_w = \ell$ then $k \sim \ell$.*

The correspondence of Theorem 5 also yields some interesting quantitive results.

If k is a number field, every ordering of k is archimedean, i.e., corresponds to a real embedding $k \hookrightarrow \mathbb{R}$.

Let r_1 be the number of real embeddings of k, and r_2 the number of conjugate pairs of complex embeddings of k. Thus $[k\colon \mathbb{Q}] = r_1 + 2\,r_2$. Let

$$V_k = \{r \in k^\times | \, (r) = \mathfrak{a}^2 \text{ for some fractional ideal } \mathfrak{a} \text{ of } k\}.$$

Clearly V_k is a subgroup of $k^\times$ and $k^{\times 2} \subseteq V_k$. In this case the local-global principle for function fields over global fields can be improved in the following sense:

Theorem 8. *Suppose $F = k(x_1, \ldots, x_n)$ and $E = \ell(x_1, \ldots, x_n)$ where $n \geq 1$ and k and ℓ are number fields, and $\alpha\colon Q(E) \to Q(F)$ is a hyperfield isomorphism. Then*

(1) $r \in k^\times / k^{\times 2}$ iff $\alpha(r) \in \ell^\times / \ell^{\times 2}$.

(2) The map $r \mapsto \alpha(r)$ defines a hyperfield isomorphism between $Q(k)$ and $Q(\ell)$.

(3) α maps $V_k / k^{\times 2}$ to $V_\ell / \ell^{\times 2}$.

(4)The 2-ranks of the ideal class groups of k and ℓ are equal.

If ℓ is a number field, $[\ell\colon\mathbb{Q}]$ even, and $\ell \neq \mathbb{Q}(\sqrt{-1})$, then, for each integer $t \geq 1$, there exists a number field k such that $k \sim \ell$ and the 2-rank of the class group of k is $\geq t$.

Combining this with Theorem yields the following:

Corollary 9. *For a fixed field $n \geq 1$ and a fixed number field ℓ, $[\ell\colon\mathbb{Q}]$ even, $\ell \neq \mathbb{Q}(\sqrt{-1})$, there are infinitely many Witt inequivalent fields of the form $k(x_1, \ldots, x_n)$, k a number field with $k \sim \ell$.*

The case when $[\ell\colon\mathbb{Q}]$ is odd remains open.

Likewise, it is not known, if, for arbitrary fields F and E, $F(x) \sim E(x)$ implies $F \sim E$, or if the assumption in Theorem that F is purely transcendental over k is really necessary.

End of part I

Orderings of higher level on multirings and hyperfields.

Preorderings and orderings of higher level in fields and rings.

Let F be a field.

A **preordering of level** n is a subset T of F such that:

$$T + T \subseteq T, TT \subseteq T, \text{and } a^{2^n} \in T \text{ for all } a \in F,$$

and an **ordering of level** n is a subset P of F such that

$$P + P \subseteq P, P^{\times} \text{ is a subgroup of } F^{\times}, P \cup -P = F, \text{ and } F^{\times}/P^{\times} \text{is cyclic with } |F^{\times}/P^{\times}| \,|\, 2^n.$$

An $\boldsymbol{n}$**-formally real** field is one where -1 is not a sum of 2^n-th powers.

The definitions of a preordering of level n for rings and n-formally real rings coincide with the ones for fields, whereas an **ordering of level** n in a ring A is a subset $P \subseteq A$ such that

i. $P+P \subseteq P, PP \subseteq P, \text{and } a^{2^n} \in P \text{ for all } a \in A$,

ii. $P \cap -P = \mathfrak{p}$ is a prime ideal of A,

iii. if $ab^{2^n} \in P$, then $a \in P$ or $b \in P$,

iv. the set

$$\overline{P} = \left\{ \sum_{i=1}^{k} a_i^{2^n} \overline{p}_i \,|\, a_1, \ldots, a_k \in k(\mathfrak{p}), p_1, \ldots, p_k \in P, k \in \mathbb{N} \right\}$$

is an ordering of level n in the field of fractions $k(\mathfrak{p})$ of the ring $A/\mathfrak{p}$. Here $\overline{p}_i = p_i + \mathfrak{p} \in A/\mathfrak{p}$, $i \in \{1, \ldots, k\}$.

Basic properties.

For a fixed n, the fundamental facts of the classical theory of n-ordered fields can be summarized as follows:

1. if T is a proper preordering of level n, $a \notin T$, and P is a preordering of level n maximal subject to the conditions that $T \subseteq P$ and $a \notin P$, then P is an ordering of level n; the set of all orderings of level n containing a preordering T will be denoted by X_T, and the set of all orderings of level n of F will be denoted by X_F;
2. for every proper preordering of level n T, one has $T = \bigcap_{P \in X_T} P$;
3. a field F is formally n-real $\Leftrightarrow$ F admits a proper preordering of level n $\Leftrightarrow$ F admits an ordering of level n.

Playing a similar game for multirings and hyperfields...?

If H is a hyperfield, a **preordering of level** n is a subset T of H such that

$$T + T \subseteq T, TT \subseteq T, \text{and } a^{2^n} \in T \text{ for all } a \in H,$$

which is *proper* if $-1 \notin T$, an **ordering of level** n is a subset P of H such that

$P + P \subseteq P, P^\times$ is a subgroup of $H^\times, P \cup -P = H,$ and $H^\times/P^\times$ is cyclic with $|H^\times/P^\times| \mid 2^n$,

which is of **exact level** n if $|F^\times/P^\times| = 2^n$, and a hyperfield is n-**formally real** when -1 is not in a sum of 2^n-th powers.

Theorem 10. *Let H be a hyperfield. The following conditions are equivalent:*

1. *H is formally $n-$ real,*
2. *H admits an ordering of level n,*
3. *H admits a proper preordering of level n.*

Theorem 11. *Let H be a hyperfield, $T \subset H$ a preordering of level n. If T is proper, then $T = \bigcap_{P \in X_T} P$.*

1. P.G., *Orderings of higher level in multirings and multifields,* Ann. Math. Silesianae 24 (2010), 15-25.

It becomes much more complicated for multirings without the extra assumption that $A = T - T$.

Theorem 12.

1. *Let H be a hyperfield, $\operatorname{char} H = 0$, let $n \geq 0$. Then $H = \sum H^{2^n} - \sum H^{2^n}$.*
2. *Let A be a multiring such that for each maximal ideal $\mathfrak{m}$ of A and each $s \in A \setminus \mathfrak{m}$*

$$\left(\bigcup_{k \geq 2} \underbrace{s + \ldots + s}_{k} \right) \cap \mathfrak{m} = \emptyset,$$

let $n \geq 0$. Then $A = \sum A^{2^n} - \sum A^{2^n}$.

1. P.G., M. Marshall, *Orderings and signatures of higher level on multirings and hyperfields*, J. K-Theory 10 (2012), 489-518.

n-real reduced multirings and hyperfields.

n-real reduced multirings are non-zero multirings satisfying the following additional axioms:

1. $a^{2^n+1} = a$,
2. $a + a\,b^{2^n} = \{a\}$,
3. $a^{2^n} + b^{2^n}$ contains a unique element.

1-real reduced hyperfields correspond to spaces of orderings, so it is natural to wonder if n-real reduced hyperfields correspond to the spaces of signatures introduced.

In fact, this is not the case: the following symmetry property:

for all odd integers $1 \leq k \leq 2^n$, $a \in b + c \Rightarrow a^k \in b^k + c^k$,

is satisfied by spaces of signatures but is not true for general n-real reduced hyperfields.

Axiomatic theory of quadratic forms.

Realization problem.

The following question is at least 40 years old:

Question 13. *Every field F gives rise to a hyperfield H (namely $H = Q(F)$) such that*

i. $a^2 = 1$*, for all* $a \in H^\times$*, and*

ii. if $a \neq -1$*, then* $1 + a$ *is a subgroup of* $H^\times$*.*

Is it true that for every hyperfield H satisfying i. and ii. there is a field F such that $H \cong Q(F)$?

Everybody believes the answer to be negative, but no examples are known so far.

Cohomological invariants.

The structre of $W(F)$ for a field F, $\operatorname{char} F \neq 2$, is closely tied with the filtration

$$W(F) \supseteq I(F) \supseteq I^2(F) \supseteq I^3(F) \supseteq \dots$$

and quotients $I^n(F)/I^{n+1}(F)$ can be determined by homomorphisms with values in the n-th Galois cohomology groups of F. This is, roughly speaking, Milnor conjecture.

Can it be done in the axiomatic setting?

One would need a different cohomological theory...

Presentable posets.

Let A be a poset.

We shall write $\bigsqcup X$ for the supremum of $X \subseteq A$.

Let $\mathcal{S}_A$ be the set of A's minimal elements.

We shall write $\mathcal{S}_a$ for the set of all minimal elements below $a \in A$, and $\mathcal{S}_X \stackrel{\text{def.}}{=} \bigcup_{x \in X} \mathcal{S}_x$ for the set of minimal elements below $X \subseteq A$.

A poset $(A, \leqslant)$ is **presentable** if

i. every non-empty subset $X \subseteq A$ admits a supremum;

ii. $\mathcal{S}_a$ is non-empty and $a = \bigsqcup \mathcal{S}_a$ for all $a \in A$;

iii. every minimal element $s \in \mathcal{S}_A$ is **compact** in the following sense: if $Y \subseteq A$ is a nonempty subset and $s \leqslant \bigsqcup Y$, then there is an element $y \in Y$ such that $s \leq y$.

The minimal elements of a presentable poset are called **supercompacts.**

A **presentable monoid** $(M, \leq, 0, +)$ is a pointed presentable poset $(M, \leq, 0)$ with a distinguished supercompact 0 and a suprema-preserving binary addition $+ \colon M \times M \to M$ such that

i. $a + (b + c) = (a + b) + c$ for all $a, b, c \in M$;

ii. $a + 0 = 0 + a = a$ for all $a \in M$;

iii. $a + b = b + a$ for all $a, b \in M$.

A **presentable group** G is a presentable monoid equipped with a suprema preserving involutive homomorphism $- \colon G \to G$ called **inversion**, verifying

$$(s \leq t + u) \Rightarrow (t \leq s + (-u))$$

for all $s, t, u \in \mathcal{S}_G$.

A **presentable ring** R is a presentable group $(R, \leq, 0, +, -)$ consisting of at least two elements as well as a commutative monoid $(R, \cdot, 1)$, such that the element $1 \in R$ is a supercompact, $\cdot$ is compatible with $\leq$ (i.e. $a \leq$ b implies $a \cdot c \leq b \cdot c$, for all $a, b, c \in R$) and $-$(i.e. $a \cdot (-b) = -(a \cdot b)$, for all $a, b \in R$), distributative with respect to $+$, that $0 \cdot a = 0$, for all $a \in R$, and that $\cdot$ verifies

$$\mathcal{S}_{a \cdot b} \;=\; \{s \cdot t | s \in \mathcal{S}_a, t \in \mathcal{S}_b\}$$

for all $a, b \in R$. A presentable ring R such that $\mathcal{S}_R^* = \mathcal{S}_R \setminus \{0\}$ is a multiplicative group will be called a **presentable field**.

Quadratically presentable fields.

A presentable field R is **pre-quadratically presentable**, if the following conditions hold

i. $a \leq a + b$ for all $a \in \mathcal{S}_R^*, b \in \mathcal{S}_R$;

ii. $(a \leq 1 - b) \wedge (a \leq 1 - c) \Rightarrow (a \leq 1 - b\,c)$ for all $a, b, c \in \mathcal{S}_R$;

iii. $a^2 = 1$ for all $a \in \mathcal{S}_R \setminus \{0\}$.

A **form** over a pre-quadratically presentable field R is an n-tuple $\langle a_1, \ldots, a_n \rangle$ of elements of $\mathcal{S}_R^*$. The relation $\cong$ of **isometry** of forms of the same dimension is given by induction:

- $\langle a \rangle \cong \langle b \rangle$ if and only if $a = b$;
- $\langle a_1, a_2 \rangle \cong \langle b_1, b_2 \rangle$ if and only if $a_1\, a_2 = b_1\, b_2$ and $b_1 \leq a_1 + a_2$;
- $\langle a_1, \ldots, a_n \rangle \cong \langle b_1, \ldots, b_n \rangle$ if and only if there exist $x, y, c_3, \ldots, c_n \in \mathcal{S}_R^*$ such that

  i. $\langle a_1, x \rangle \cong \langle b_1, y \rangle$;

  ii. $\langle a_2, \ldots, a_n \rangle \cong \langle x, c_3, \ldots, c_n \rangle$;

  iii. $\langle b_2, \ldots, b_n \rangle \cong \langle y, c_3 \ldots, c_n \rangle$.

If this relation is an equivalence, then R is called a **quadratically presentable** field.

Witt rings of quadratically presentable fields can be defined:

Theorem 14. *For a field F, $W(\mathcal{P}^*(Q(F)))$ is just the usual Witt ring $W(F)$ of non-degenerate symmetric bilinear forms of F.*

1. P.G., K. Worytkiewicz, *Witt rings of quadratically presentable fields,* Categ. Gen. Algebr. Struct. Appl., to appear.

Thank you!