THE HENRYK NIEWODNICZAŃSKI
INSTITUTE OF NUCLEAR PHYSICS
POLISH ACADEMY OF SCIENCES

Subordination and memory depend kinetics

Ostrava, May 10, 2022

Katarzyna Górska

Institute of Nuclear Physics, Polish Academy of Sciences

Diffusion (Fick, Einstein, Smoluchowski)

DIFFUSION - the process of spontaneous spreading out particles or energy in a given medium (e.g., gas, liquid, or solid) resulting from collisions of diffusing substance particles among themselves or with particles of the surrounding medium.

The diffusion process is described by the Fokker-Planck (FP) equation in which the FP operator $\mathscr{A}$ is equal to ∂_x^2. This equation reads

$$\partial_t N(x,t) = D\partial_x^2 N(x,t)$$

whose fundamental solution (it means that we use $N(x,0) = \delta(x)$) is the Gaussian:

$$N(x,t) = \frac{e^{-x^2/(4Dt)}}{\sqrt{4\pi D t}}$$

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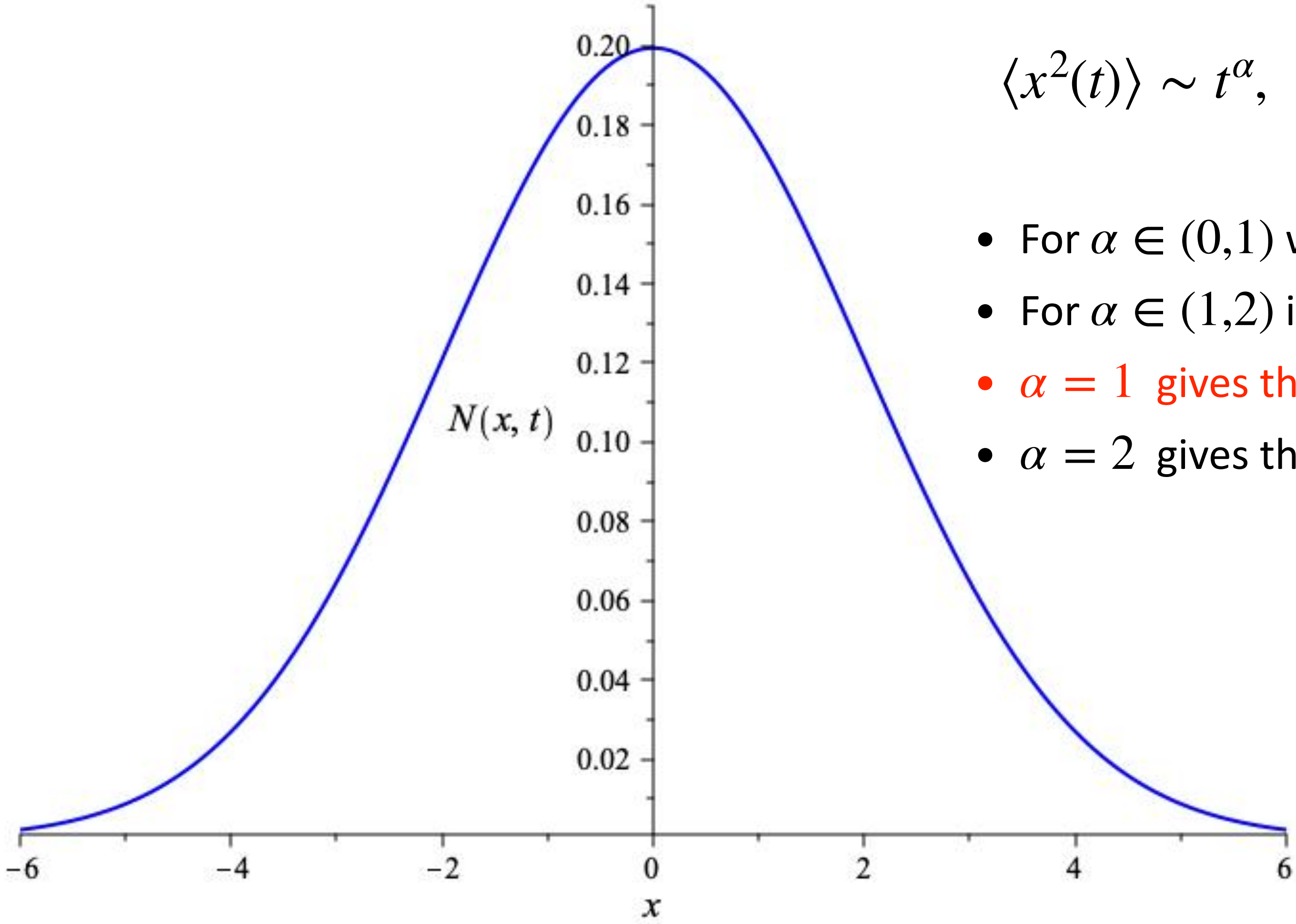
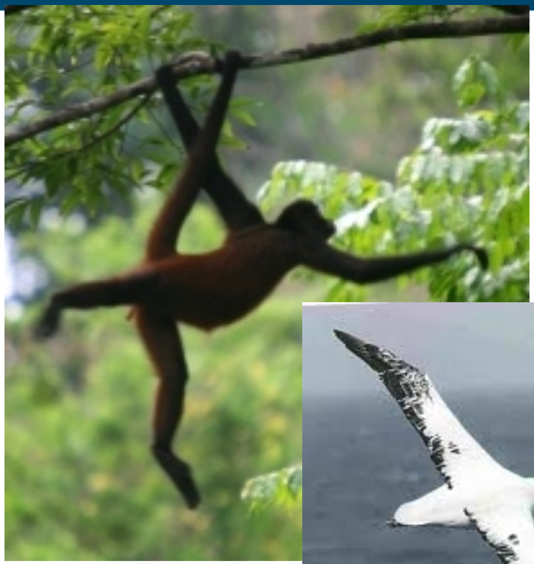
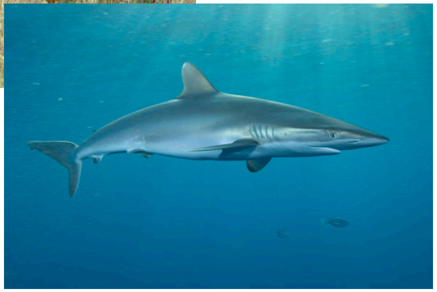


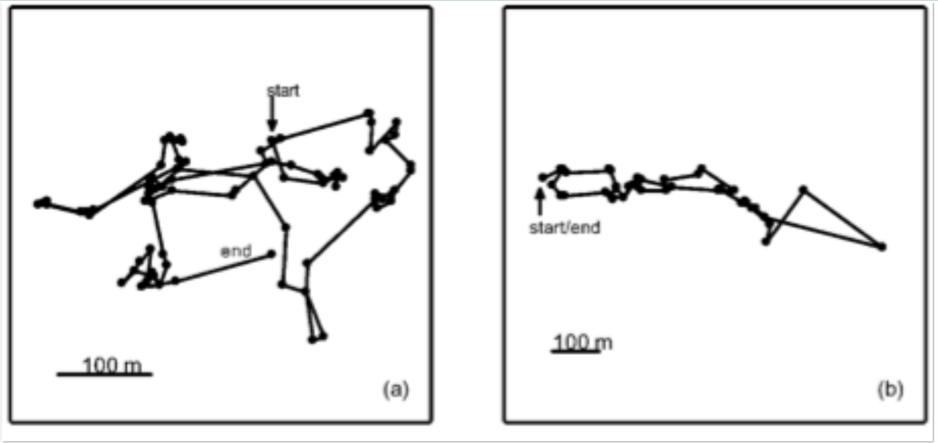
$$\langle x^2(t)\rangle \sim t^{\alpha}, \qquad \alpha > 0$$

- For $\alpha \in (0,1)$ we have sub-diffusion
- For $\alpha \in (1,2)$ it is super-diffusion
- $\alpha = 1$ gives the normal diffusion
- $\alpha = 2$ gives the ballistic motion

Anomalous diffusion







Gabriel Ramos-Fernández · José L. Mateos · Octavio Miramontes · Germinal Cocho · Hernán Larralde · Bárbara Ayala-Orozco

Lévy walk patterns in the foraging movements of spider monkeys (*Ateles geoffroyi*)

letters to nature

Nature **381**, 413 - 415 (30 May 1996); doi:10.1038/381413a0

Lévy flight search patterns of wandering albatrosses

G. M. VISWANATHAN*, V. AFANASYEV†, S. V. BULDYREV*, E. J. MURPHY†, P. A. PRINCE† & H. E. STANLEY*

OIKOS 98: 134–140, 2002

Scale-free dynamics in the movement patterns of jackals

R. P. D. Atkinson, C. J. Rhodes, D. W. Macdonald and R. M. Anderson

We consider the generalized Fokker-Planck (FP) equation which figures out

$$\partial_t p(x,t) = \mathscr{O}\mathscr{A}p(x,t) \quad \Longrightarrow \partial_t p(x,t) = \int_0^t O(t-\xi)\mathscr{A}p(x,\xi)\,\mathrm{d}\xi$$

where $x \in \mathbb{R}, t \in \mathbb{R}_+$, the linear operator $\mathscr{O}$ acts on the time variable only (it carries responsibility for the memory effects). The time independent FP operator $\mathscr{A}$ influences only the x-dependence of solutions.

The above equation is usually presented in two forms, namely

(1)

$$\partial_t p(x,t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t M(t-\xi)\mathscr{A}p(x,\xi)\,\mathrm{d}\xi$$

$$p(x,t) = p(x,0) + \int_0^t M(t-\xi)\,\mathscr{A}p(x,\xi)\,\mathrm{d}\xi$$

or

(2)

$$\int_0^t k(t-\xi)\partial_\xi p(x,\xi)\,\mathrm{d}\xi = \mathscr{A}p(x,t)$$

(Mainardi, Sokolov, Chechkin, Kochubei, Tomovski, Sandev, and so on)

Remark 1. These two equations can be rewritten to each other for the memory kernels $M(t)$ and $k(t)$ satisfied the Sonine equation

$$\int_0^t M(t-t')k(t')\mathrm{d}t' = \int_0^t M(t')k(t-t')\mathrm{d}t' = 1$$

which in the Laplace space reads $\hat{M}(z)\hat{k}(z) = \dfrac{1}{z}$.

(Kochubei, Hanyga, Górska, Horzela)

Remark 2. In the Laplace space we have $\hat{O}(z) = z\hat{M}(z) = [\hat{k}(z)]^{-1}$.

Remark 3. In the description of relaxation phenomena we restrict the kinetic problem under consideration to be depended on the time only. That means that the action of the FP operator reduces to multiplication by a constant factor B.

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… Sonine equation

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$$\hat{M}(z)\hat{k}(z) = \frac{1}{z}\,.$$

(Kochubei, Hanyga, Górska, Horzela)

The Laplace transform reads

$$\hat{f}(z) = \mathscr{L}[f(t); z] = \int_0^\infty f(t)\, \mathrm{e}^{-zt} \mathrm{d}t$$

where the inverse Laplace transform is given by the Bromwich integral

$$f(t) = \mathscr{L}^{-1}[\hat{f}(z); t] = \int_L \hat{f}(z)\, \mathrm{e}^{zt}\, \frac{\mathrm{d}z}{2\pi i}$$

with L being the Bromwich contour

Remark 2. … …we have $\hat{O}(z) = z\hat{M}(z) = [\hat{k}(z)]^{-1}\,.$

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Remark 3. In the description of relaxation phenomena we restrict the kinetic problem under consideration to be depended on the time only. That means that the action of the FP operator reduces to multiplication by a constant factor B.

Bochner, 1955; Sokolov 2002; Chechkin, Sokolov, 2021

The solution is presented as an integral decomposition

$$p(x,t) = \int_0^\infty h(x,\xi) f(\xi,t) \mathrm{d}\xi$$

in which functional forms of $h(x,\xi)$ and $f(\xi,t)$ are to be determined from the primary equation.

Remark 4. In the integral decomposition the functions $h(x,\xi)$ and $f(\xi,t)$ are not determined uniquely, so we can keep one factor and another change. For us ξ is only the integral variable.

Remark 5. If we impose a probabilistic interpretation then the integral decomposition can represent the joint probability in which $h(x,\xi)$ and $f(\xi,t)$ are independent probability density (pdf) of the parent and leading processes. ξ is interpreted as the operational (internal) time.

We do not want to use the probabilistic interpretation !!!

We assume only that $h(x, \xi)$ and $f(\xi, t)$ are

- normalizable in the first argument, i.e $\int_{\mathbb{R}} h(x, \xi)\mathrm{d}x = 1$ and $\int_{\mathbb{R}_+} f(\xi, t)\mathrm{d}\xi = 1$,
- and non-negative which is satisfied, e.g., by the special classes of functions; i.e. completely monotonicity function, (completely) Bernstein function

According to **the Bernstein theorem** we can connect in a unique way the completely monotonicity (CM) function and non-negative functions: $s \in [0,\infty) \to G(s) \in CM$ iff

$$G(s) = \int_0^\infty \exp(-st)g(t)\mathrm{d}t \qquad \text{(Laplace integral)}$$

and if $g(t) \geq 0$ for all $t \in [0,\infty)$.

We have $G(s)$ which we would like to expand to $\hat{f}(z)$.

Theorem. The Laplace transform $\hat{f}(z)$ of a function $f(t)$ that is locally integrable on $\mathbb{R}_+$ and completely monotonic, has the following properties:

(a) $\hat{f}(z)$ an analytical extension to the region $\mathbb{C}\backslash\mathbb{R}_-$

(b) $\hat{f}(s) = \hat{f}^\star(s)$ for $s \in (0,\infty)$;

(c) $\lim\limits_{s\to\infty} \hat{f}(s) = 0$;

(d) $\mathrm{Im}[\hat{f}(z)] < 0$ for $\mathrm{Im}(z) > 0$;

(e) $\mathrm{Im}[z\hat{f}(z)] \geq 0$ for $\mathrm{Im}(z) > 0$ and $\hat{f}(s) \geq 0$ for $s \in (0,\infty)$

Conversely, every function $\hat{f}(z)$ that satisfies (a)–(c) together with (d) or (e), is the Laplace transform of a function $f(t)$, which is locally integrable on $\mathbb{R}_+$ and completely monotonic on $(0,\infty)$.

Definition. A real function $G(s)$ with domain $(0,\infty)$ is said to be a ***complete monotone (CM) function***, if it posses derivatives $G^{(n)}(s)$ for all $n = 0,1,2,3,\ldots$ and if $(-1)^n G^{(n)}(s) \geq 0$ for all $s > 0$.

Properties:

- the product of two CM functions is also a CM function
- the composition of a CM function and a Bernstein function is another CM function

Definition. A real function $H(s)$ is a ***Bernstein (B) function***, if it $(-1)^{n-1} H^{(n)}(s) \geq 0$ for all $s > 0$ and all $n = 1,2,3,\ldots$

Definition. A real function $F(s)$ is a ***complete Bernstein (CB) function***, if it a Bernstein function and $F(s)/s$ is the Laplace transform of CM function restricted to the positive semiaxis, or, equivalently, in the same way restricted Stieltjes transform of a positive function named also the Stieltjes function (SF).

CB functions form a subclass of the Bernstein functions

The Laplace-Fourier (FL) transform of the generalized FP equations (1) and (2) for $\mathscr{A} = D\partial_x^2$, $D > 0$, is equal to

$$\hat{\tilde{p}}(\kappa, z) = \frac{1}{z\hat{M}(z)} \frac{1}{\frac{1}{\hat{M}(z)} + D\kappa^2} = \frac{\hat{k}(z)}{z\hat{k}(z) + D\kappa^2}\,.$$

We want to calculate $\tilde{p}(\kappa, t) = \mathscr{L}^{-1}[\hat{\tilde{p}}(\kappa, z); t]$.

I recall that the generalized FP equations reads

(1)

$$p(x,t) = p(x,0) + \int_0^t M(t-\xi)\, D\partial_x^2 p(x,\xi)\,\mathrm{d}\xi$$

or

(2)

$$\int_0^t k(t-\xi)\partial_\xi p(x,\xi)\,\mathrm{d}\xi = D\partial_x^2 p(x,t)$$

The Laplace-Fourier (FL) transform of the generalized FP equations (1) and (2) for $\mathscr{A} = D\partial_x^2$, $D > 0$, is equal to

The Fourier transform has the form $\tilde{f}(\kappa) = \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-i\kappa x}\,\mathrm{d}x$ while its inverse reads

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\kappa)\,\mathrm{e}^{i\kappa x}\,\mathrm{d}\kappa\,.$$

We want to calculate

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$$p(x, t) = p(x,0) + \int_0^t M(t-\xi)\, D\partial_x^2 p(x, \xi)\, \mathrm{d}\xi$$

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$$\int_0^t k(t-\xi)\partial_\xi p(x, \xi)\, \mathrm{d}\xi = D\partial_x^2 p(x, t)$$

Efross 1935; Włodarski 1952; Graf 2004; Górska & Penson 2012; Górska 2021

The **Efross theorem** generalizes the convolution (Borel) theorem for the Laplace transform. It states as follows:

Theorem. If $\hat{G}(z)$ and $\hat{q}(z)$ are analytic functions, and $\mathscr{L}[h(x,\xi);z] = \hat{h}(x,z)$

(Laplace pair $\xi \div z$)

as well as $\mathscr{L}[f(\xi,t);z] = \hat{G}(z)\,\mathrm{e}^{-\xi\hat{q}(z)}$ (Laplace pair $t \div z$)

then

$$\hat{G}(z)\,\hat{h}(x,\hat{q}(z)) = \mathscr{L}\left[\int_0^\infty h(x,\xi)\,f(\xi,t)\,\mathrm{d}\xi;\; z\right].$$

(Laplace pair $t \div z$)

From Efross theorem appears that

$$\mathscr{L}^{-1}\left[\hat{G}(z)\hat{h}(x,\hat{q}(z);t\right] = \int_0^\infty \mathscr{L}^{-1}[\hat{h}(x,z),\xi]\,\mathscr{L}^{-1}[\hat{f}(\xi,z),t]\,\mathrm{d}\xi\,.$$

$$\tilde{p}(\kappa, t) = \mathscr{L}^{-1}\left[\frac{\hat{k}(z)}{z\hat{k}(z) + D\kappa^2}; t\right]$$

Examples:

(A) To have the <u>anomalous diffusion equation</u> we take $\hat{q}(z) = z\hat{k}(z)$ and $\hat{G}(z) = \hat{k}(z)$.

The Efross theorem allows us to express $\tilde{p}(\kappa, t)$ as

$$\tilde{p}(\kappa, t) = \int_0^\infty \boxed{\mathscr{L}^{-1}\left[\frac{1}{z + D\kappa^2}; \xi\right]} \mathscr{L}^{-1}[\hat{k}(z)\, e^{-\xi z \hat{k}(z)}; t]\, d\xi$$

For the fundamental initial condition, i.e $h(x,0) = \delta(x)$,

the inverse Laplace transform of $(z + D\kappa^2)^{-1}$ gives

$$\partial_\xi h(x, \xi) = D\, \partial_x^2 h(x, \xi)$$

whose basic solution is the Gaussian $h(x, \xi) \equiv N(x, \xi)$.

In the integral decomposition $p(x, t) = \int_0^\infty N(x, \xi) f_N(\xi, t)\, d\xi$ at first we kept $N(x, \xi)$.

$$\tilde{p}(\kappa, t) = \mathscr{L}^{-1}\left[\frac{\hat{k}(z)}{z\hat{k}(z) + D\kappa^2}; t\right]$$

(B) Let us take $\hat{q}(z) = z\hat{k}(z)$ and $\hat{G}(z) = \hat{k}(z)$, where $\hat{k}(z) = \tau z[\hat{\gamma}(z)]^2 + \hat{\gamma}(z)$.

The choice $\hat{q}(z) = z\hat{k}(z)$ does not change $h(x, \xi)$ but changes $f(\xi, t)$.

The Efross theorem gives

$$p(x, t) = \int_0^\infty N(x, \xi)\, f_{\mathrm{CV1}}(\tau; \xi, t)\, \mathrm{d}\xi$$ where

$$f_{\mathrm{CV1}}(\tau; \xi, t) = \mathscr{L}^{-1}\left\{\left[\tau z \hat{\gamma}^2(z) + \hat{\gamma}(z)\right] \mathrm{e}^{-\xi\,[\tau z^2 \hat{\gamma}^2(z) + z\hat{\gamma}(z)]};\ t\right\}$$

Remain that $f_{\mathrm{CV1}}(\tau; \xi, t)$ should be given by a non-negative function.

Example. For $\hat{\gamma}(z) = z^{\alpha-1}$, $\alpha \in (0,1)$ we have

$\tau s^{2\alpha-2} + s^{\alpha-1}$ is a completely monotonic (CM) function for $\alpha \in (0,1/2]$

It is not CM function for $\alpha \in (1/2, 1)$

(Tomovski)

$$\tilde{p}(\kappa, t) = \mathscr{L}^{-1}\left[\frac{\hat{k}(z)}{z\hat{k}(z) + D\kappa^2};\ t\right]$$

For $\hat{k}(z) = \tau z[\hat{\gamma}(z)]^2 + \hat{\gamma}(z)$ we express $\tilde{\hat{p}}(\kappa, z)$ in the form

$$\tilde{\hat{p}}(\kappa, z) = \frac{\tau z\hat{\gamma}^2(z) + \hat{\gamma}(z)}{\tau z^2\hat{\gamma}^2(z) + z\hat{\gamma}(z) + D\kappa^2}$$

which for the localized initial conditions

($p(x,0) = \delta(x)$ and $\dot{p}(x,t)\,|_{t=0} = 0$) is obtained from the integro-differential equation called the generalized Cattaneo-Vernotte equation:

$$\tau\int_0^t \eta(t-\xi)\,\partial_\xi^2 p(x,\xi)\,\mathrm{d}\xi + \int_0^t \gamma(t-\xi)\,\partial_\xi p(x,\xi)\,\mathrm{d}\xi = D\,\partial_x^2 p(x,t)\,.$$

The memory kernel $\gamma(t) = \mathscr{L}^{-1}[\hat{\gamma}(z); t]$ is responsible for the time smearing of the first time derivative and

$$\eta(t) = \int_0^t \gamma(u)\gamma(t-u)\mathrm{d}u, \quad \hat{\eta}(z) = \hat{\gamma}^2(z)$$

Represent the time smearing of the second time derivative (smearing of smeared derivative)

$$\tilde{p}(\kappa, t) = \mathscr{L}^{-1}\left[\frac{\tau z\,\hat{\gamma}^2(z) + \hat{\gamma}(z)}{\tau\, z^2\,\hat{\gamma}^2(z) + z\,\hat{\gamma}(z) + D\kappa^2}; t\right]$$

Now, in the Efross theorem we take $\hat{q}(z)$ and $\hat{G}(z)$ like in the diffusion case, i.e. $\hat{q}(z) = z\hat{\gamma}(z)$ and $\hat{G}(z) = \hat{\gamma(z)} = \hat{q}(z)/z$. Thus, we change $h(x, \xi)$ and keep $f(\xi, t)$:

$$p(x, t) = \int_0^\infty p_{\rm CV}(\tau; x, \xi)\, f_N(\xi, t)\, \mathrm{d}\xi \qquad \text{where } f_N(\xi, t) = \mathscr{L}^{-1}[\hat{\gamma}(z)\mathrm{e}^{-\xi z\hat{\gamma}(z)}; t].$$

(Górska)

The Laplace-Fourier transform of $p_{\rm CV}(\tau; x, \xi)$ reads

$$\tilde{\hat{p}}_{\rm CV}(\tau; \kappa, z) = \frac{\tau z + 1}{\tau z^2 + z + D\kappa^2}$$

Together with the initial conditions, i.e. $p_{\rm CV}(\tau; x,0) = \delta(x)$ and $\dot{p}_{\rm CV}(\tau; x, t)\,|_{t=0} = 0$, it gives the Cattaneo-Vernotte equation

$$\tau\partial_t^2 p_{\rm CV}(\tau; x, t) + \partial_t p_{\rm CV}(\tau; x, t) = D\partial_x^2 p_{\rm CV}(\tau; x, t)$$

also used for description of diffusion phenomena. The relaxation time is denoted as τ whereas the diffusion coefficient by $D > 0$.

The Cattaneo-Vernotte equation

Weymann, 1967

$|x| \ll a\,t$

$a = \sqrt{D/\tau}$

$\tau = 0$

Górska, 2021

Gaussian $N(x,t)$

$$N(x,t) = \frac{e^{-x^2/(4Dt)}}{\sqrt{4\pi D t}}$$

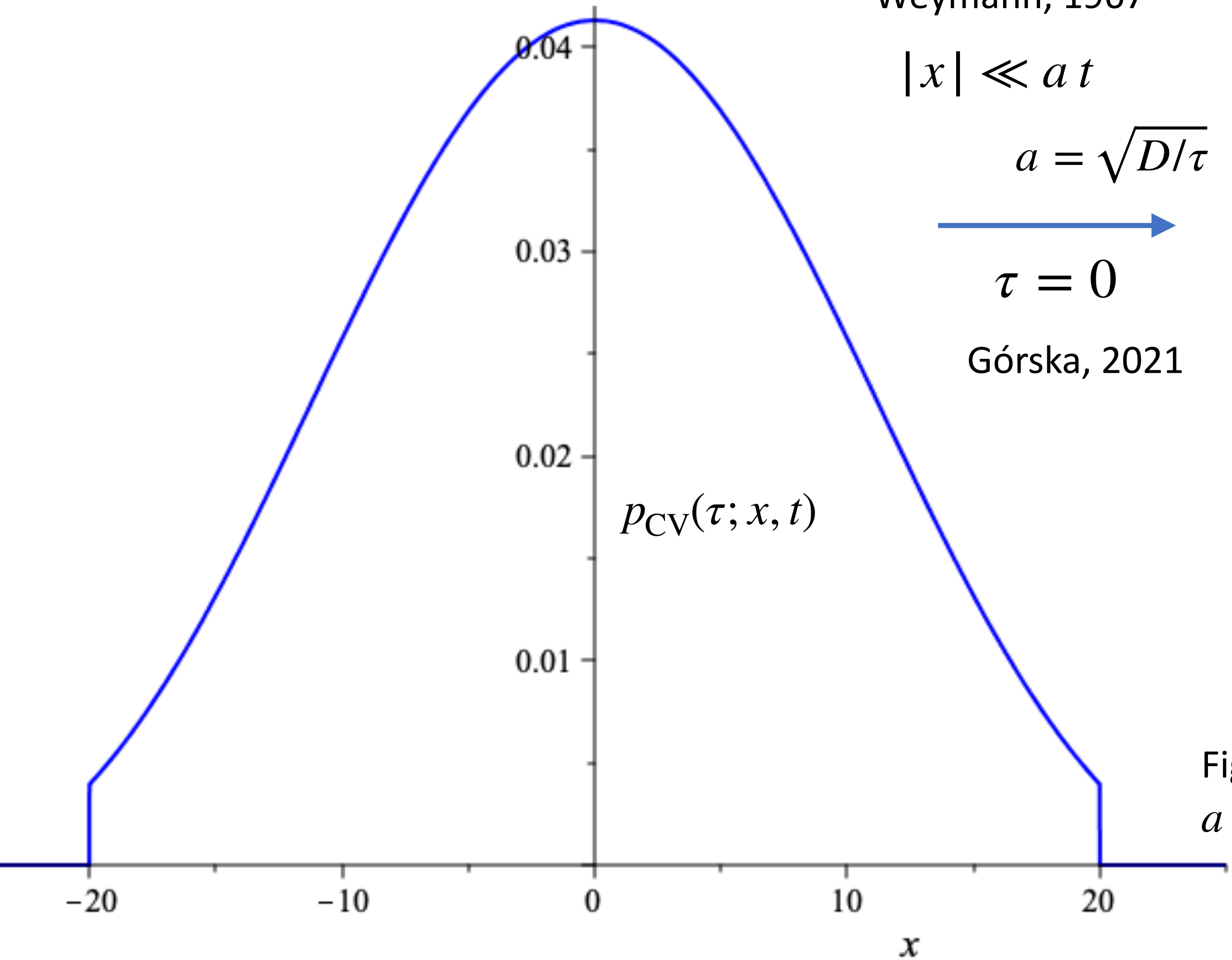


Figure. $p_{CV}(\tau; x, t)$ for $\tau = 0.22$, $a = 10$, and $t = 2$.

The generalized Cattaneo-Vernotte equation

The solution is equal to $p(x,t) = \int_0^\infty p_{\mathrm{CV}}(\tau; x, \xi)\, f_N(\xi, t)\ \mathrm{d}\xi\,.$

where $f_N(\xi, t) = \mathscr{L}^{-1}[\hat{\gamma}(z) \mathrm{e}^{-\xi z \hat{\gamma}(z)}; t]$.

Examples: (a) for $\hat{\eta}(z) = \hat{\gamma}(z) = 1 \quad \Rightarrow \quad \eta(t) = \gamma(t) = \delta(t)$ we get the Cattaneo-Vernotte equation.

$$f_N(\xi, t) = \mathscr{L}^{-1}[\hat{\gamma}(z) e^{-\xi z \hat{\gamma}(z)}; t].$$

Górska 2021

$$\hat{\gamma}(z) = z^{\alpha-1} \quad \Rightarrow \quad \gamma(t) = t^{-\alpha}/\Gamma(1-\alpha), \quad \text{for } \alpha \in (0,1]$$

$$\hat{\eta}(z) = z^{2(\alpha-1)} \quad \Rightarrow \quad \eta(t) = t^{1-2\alpha}/\Gamma(2-2\alpha)$$

Because of $f(\alpha; \xi, t) = \frac{t}{\alpha \xi^{1+1/\alpha}} \, g_\alpha\left(\frac{t}{\xi^{1/\alpha}}\right)$

and $g_\alpha(u)$, $u > 0$, is one-sided Levy stable distribution then $p(\alpha, \tau; x, t)$ for fixed t vanishes only at infinity.

Górska & Penson 2010;

$) = t^{-\alpha}/\Gamma(1-\alpha)$, for $\alpha \in (0,1]$

$= t^{1-2\alpha}/\Gamma(2-2\alpha)$

$= \frac{t}{\alpha \xi^{1+1/\alpha}}\, g_\alpha\left(\frac{t}{\xi^{1/\alpha}}\right)$

ded Levy stable

$x, t)$ for fixed t

$(\tau; x, t) \equiv p(\alpha, \tau; x, t)$

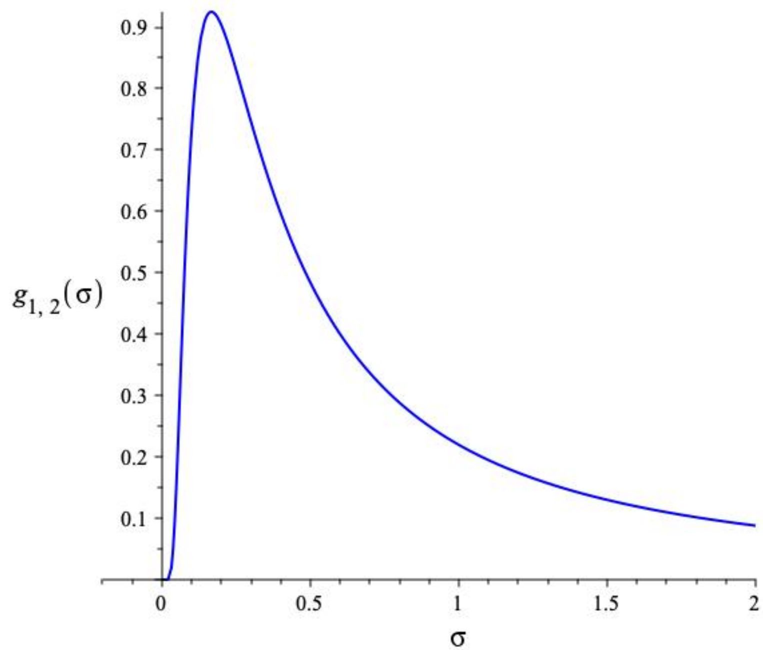
$, \tau = 0.22, a = 10$, and

The Levy stable distribution is defined as follows:

(b)

$$g_\alpha(\sigma) = \begin{cases} \frac{\sqrt{kl}}{(2\pi)^{(k-l)/2}} \frac{1}{\sigma} G^{k,0}_{l,k}\left(\frac{l^l}{k^k \sigma^l} \middle| \begin{matrix} \Delta(l,0) \\ \Delta(k,0) \end{matrix}\right), & \text{for} \quad \sigma > 0 \\ 0, & \text{for} \quad \sigma \le 0 \end{cases}$$

The example of one-sided Levy stable distribution is Levy-Smirnov distribution:

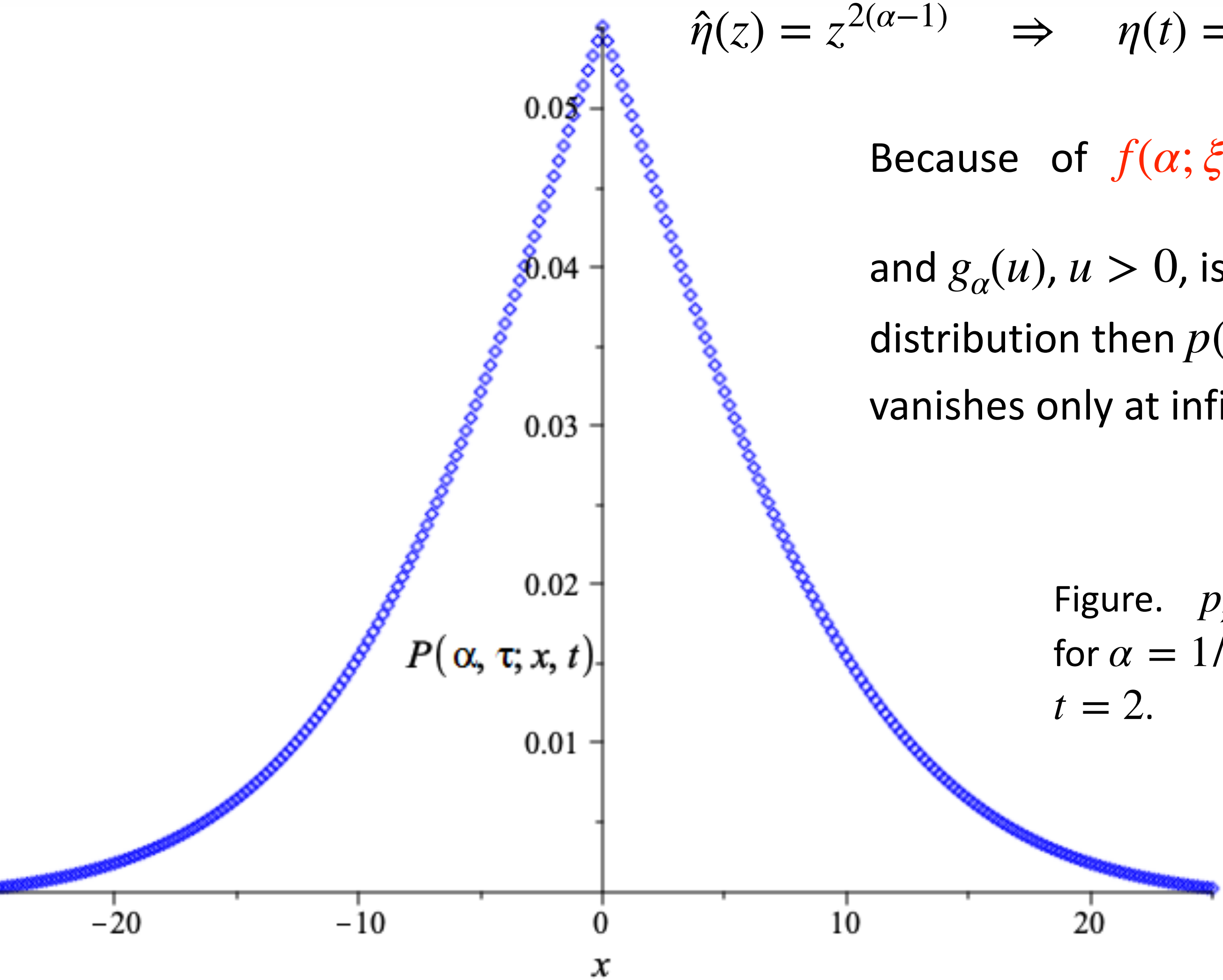


$$f_N(\xi, t) = \mathscr{L}^{-1}[\hat{\gamma}(z)\mathrm{e}^{-\xi z\hat{\gamma}(z)}; t].$$

Górska 2021

$\hat{\gamma}(z) = z^{\alpha-1} \quad \Rightarrow \quad \gamma(t) = t^{-\alpha}/\Gamma(1-\alpha)$, for $\alpha \in (0,1]$

$\hat{\eta}(z) = z^{2(\alpha-1)} \quad \Rightarrow \quad \eta(t) = t^{1-2\alpha}/\Gamma(2-2\alpha)$



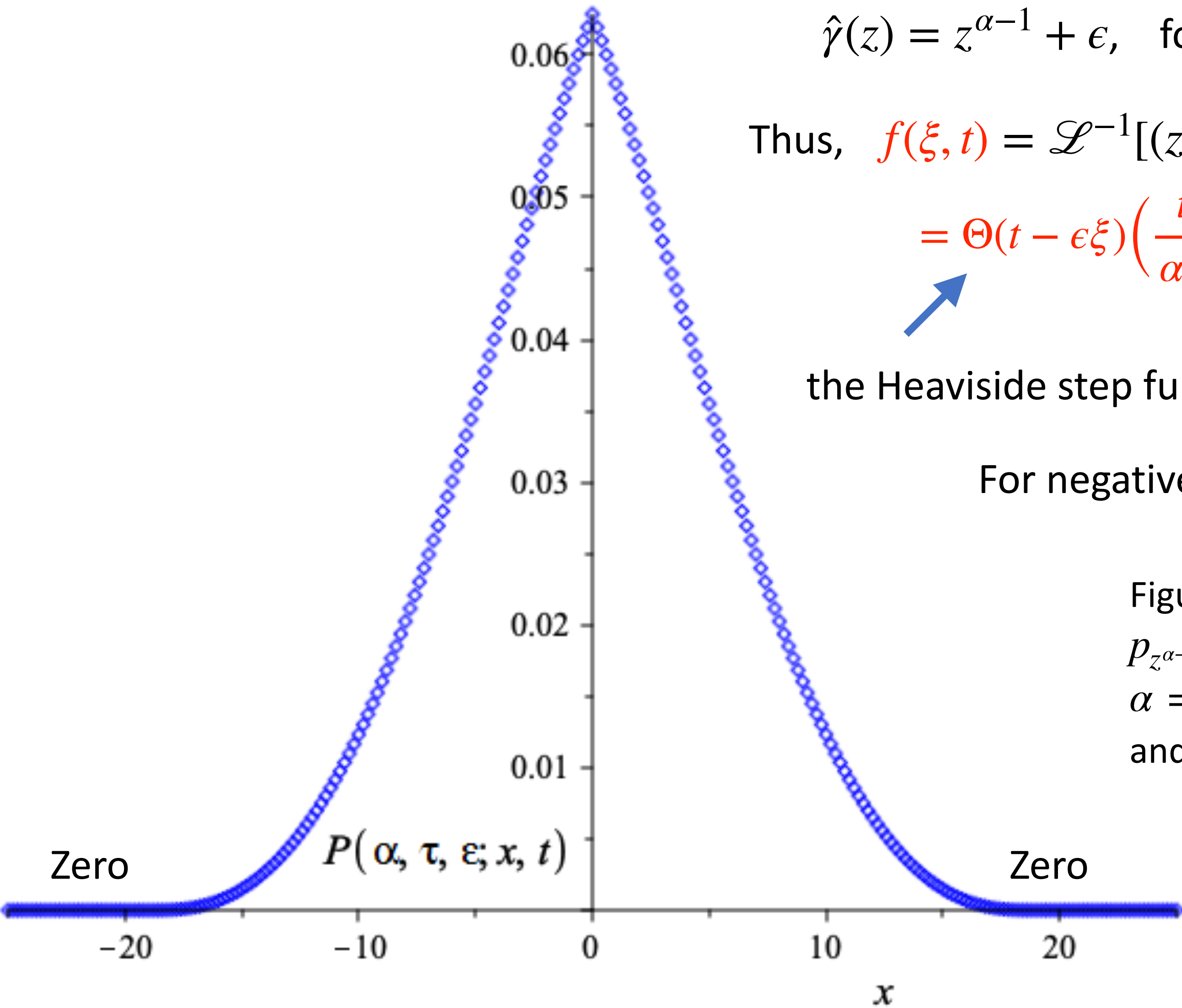
Because of $f(\alpha; \xi, t) = \dfrac{t}{\alpha\xi^{1+1/\alpha}}\, g_\alpha\!\left(\dfrac{t}{\xi^{1/\alpha}}\right)$

and $g_\alpha(u)$, $u > 0$, is one-sided Levy stable distribution then $p(\alpha, \tau; x, t)$ for fixed t vanishes only at infinity.

Figure. $p_{z^{\alpha-1}}(\tau; x, t) \equiv p(\alpha, \tau; x, t)$ for $\alpha = 1/2$, $\tau = 0.22$, $a = 10$, and $t = 2$.

$$f_N(\xi, t) = \mathscr{L}^{-1}[\hat{\gamma}(z)\mathrm{e}^{-\xi z \hat{\gamma}(z)}; t].$$

Górska 2021



$\hat{\gamma}(z) = z^{\alpha-1} + \epsilon$, for $\alpha \in (0,1]$ and $\epsilon \geq 0$.

Thus, $f(\xi, t) = \mathscr{L}^{-1}[(z^{\alpha-1} + \epsilon)\,\mathrm{e}^{-\xi\,(z^{\alpha}+\epsilon z)}; t]$

$$= \Theta(t - \epsilon\xi)\left(\frac{t}{\alpha\xi} - \epsilon\frac{1-\alpha}{\alpha}\right)\frac{1}{\xi^{1/\alpha}} g_\alpha\left(\frac{t-\epsilon\xi}{\xi^\alpha}\right)$$

the Heaviside step function

For negative argument it is equal to zero !

Figure.
$p_{z^{\alpha-1}+\epsilon}(\tau; x, t) \equiv p(\alpha, \tau, \epsilon; x, t)$ for $\alpha = 1/2, \epsilon = 1, \tau = 0.22, a = 10,$ and $t = 2$.

Conclusions

In the presented examples the role of $h(x, \xi)$ is played by the non-negative functions $N(x, \xi)$ and $p_{\mathrm{CV}}(\tau; x, \xi)$.

We also make the natural choice of $\hat{q}(z)$ and $\hat{G}(z)$ such that for given $\hat{q}(z)$ we take $\hat{G}(z) = \dfrac{\hat{q}(z)}{z}$. It means the in the Laplace space we have the so-called direct subordination

$$\hat{f}(\xi, z) = \frac{\hat{q}(s)}{z}\, \mathrm{e}^{-\xi\, \hat{q}(z)} = -\, z^{-1} \frac{\mathrm{d}}{\mathrm{d}\xi}\, \hat{g}(\xi, z),$$

where $\hat{g}(\xi, z) = \exp[-\xi\, \hat{q}(z)]$ is named the parametric subordinator.

(Chechkin, Sokolov)

- The function $f(\xi, t)$ is normalized in the first argument.
- The non-negativity is ensured by the Bernstein theorem and a condition saying that $\hat{q}(s)$ is a completely Bernstein function which also ensured the infinitely divisibility of $\hat{g}(\xi, s)$.

Conclusions

- Mathematically it is quite clear, also physically it is well justified - physical processes looking the same at first glance in fact may be rooted in different primary laws. In below tables we recall dualities of integral decompositions found. Note that separating out the parent and leading processes we restricted ourselves to simple models dependent on a single memory function
- We are convinced that the methods based on Efross theorem will work also for much complicated models involving larger number of memory functions or governed by " nested" processes which seems to be typical situation for various transport and relaxation phenomena taking place in complex systems.

$p(x,t)$	$h(x,\xi)$	$f(\xi,t)$
anomalous diffusion	$N(x,\xi)$	$f_N(\xi,t)$
generalized Cattaneo-Vernotte. equation	$N(x,\xi)$	$f_{CV1}(\tau;\xi,t)$
	$p_{CV}(\tau;x,\xi)$	$f_N(\xi,t)$

$n(t)$	$h(\xi)$	$f(\xi,t)$
Cole-Cole relaxation	$n_D(\tau;\xi) = e^{-t/\tau}$	$f_N(\xi,t)$
Havriliak-Negami relaxation	$n_D(\tau;t)$	$f_{HN}(\tau;\xi,t)$
	$n_{CD}(\tau;\xi)$	$f_N(\xi,t)$

Thank you for attention

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$h(x,\xi)$ is a non-negative function

In the presented examples the role of $h(x,\xi)$ is played by $N(x,\xi)$, $p_{\rm CV}(\tau; x, \xi)$ for diffusion process, and $n_{\rm D}(\xi)$, $n_{\rm CD}(\xi)$ for the Havriliak-Negami relaxation phenomena. These functions are non-negative and infinitely divisible.

(DIFFUSION)

- $N(x,\xi)$ is also called the normal distribution and satisfied the central limit theorem which ensures its infinitely divisibility.
- $p_{\rm CV}(\tau; x, \xi)$ can be derived from the central limit theorem if we consider its behavior for $|x|$ larger than $\mathcal{O}(\sqrt{t})$ (Keller, PNAS, 2004)

(RELAXATION PHENOMENA)

- $n_{\rm D}(t) = \exp(-t/\tau)$ is non-negative and its Laplace transform, i.e., its characteristic function, is also infinitely divisible with respect to s which results from applying [Schilling, Song, Vondracek, see Lemma 5.8]
- $n_{\rm CD}(\beta, t) = \Gamma(\beta, t/\tau)/\Gamma(\beta)$ where for $\mathrm{Re}\beta > 0, \Gamma(a,z) = \int_z^\infty \xi^{a-1} \mathrm{e}^{-\xi} \mathrm{d}\xi$ is the incomplete gamma function.