# Neumann Laplacian in a perturbed domain

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# Overview

# Domains with holes

- What kind of spectral convergence can we expect for the Laplace operator under perturbations of the domain such as removing small holes?
- It is a common expectation that small perturbations of the physical situation lead only to a small change of the spectrum.
- Domain perturbations is largely true for Dirichlet boundary conditions while the Neumann case is more delicate.
- Such questions received already quite a lot of answers starting from the seminal work of Rauch and Taylor concerning the spectrum of the Laplace operator of domains with holes.
- In Neumann case even small perturbations may cause abrupt change of the spectrum. For example, such an effect is observed when the hole has a "split-ring" geometry.

# Domains with holes

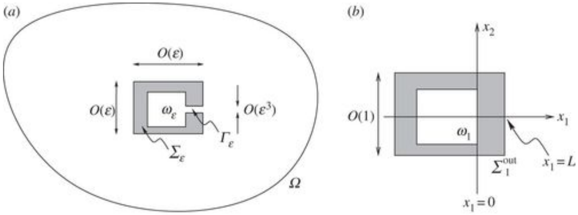



Figure: The geometry of the Helmholtz resonator in the two- dimensional.

# Domains with holes

- Maz'ya, Nazarov and Plamenewski, see (9), have considered Laplace operator on domain with obstacles, imposing the Dirichlet condition on their boundary and have proved the validity of a complete asymptotic expansion for the eigenvalues.
- In (3), (6), (7), (8) authors have considered the Dirichlet Laplacians on Euclidean domains or manifolds with holes and studied the problems of the resolvent convergence.
- The problems with Neumann obstacles having more general geometry (7), the authors required the hole to satisfy the so called "uniform extension property" which means that $H^1$- functions on the domain with a hole can be extended to $H^1$ function on the unperturbed domain and the norm of this extension operator does not depend on the hole diameter. In this case the authors established the spectral convergence.

# Motivation

- Another related paper, which makes use imposing Neumann boundary conditions on the boundary of hole which has satisfies the suitable geometrical assumptions to consider the upper and lower estimates for the ground state eigenvalue, have been extensively studied by Hempel [10].
- But, there is one case not considered. These are holes with zero Lebesgue measure (e.g. an interval or a piece of a curve). Evidently, for the holes with Lebesgue measure zero such an extension required in the work of A. Colette and O.Post [7] is not possible.
- We will be interested in a two-dimensional bounded domain with a single hole $K_\varepsilon$ (for a fixed parameter $\varepsilon$) having zero Lebesgue measure. The main purpose our work is to prove the spectral convergence of the Neumann Laplacian on $\Omega_{K_\varepsilon}$ as $\varepsilon \to 0$ in terms of the Hausdorff distance under some additional

# Spectrum of Operator

## Definition

Let $U \neq \{0\}$ be a complex normed space and $T : \mathrm{Dom}(T) \to U$ a linear operator with domain $\mathrm{Dom}(T) \subset U$. With $T$ we associate the operator

$$T_\lambda = T - \lambda \mathbb{I}$$

where $\lambda$ is a complex number and $\mathbb{I}$ is the identity operator on $Dom(T)$. If$T_\lambda$ has an inverse, we denote it by $R_\lambda(T) = T_\lambda^{-1} = (T - \lambda \mathbb{I})^{-1}$ and call it the resolvent operator of $T$ or, simply, the resolvent of $T$.

## Remark

$R_\lambda$ is a linear operator.

## Definition

Let $U \neq \{0\}$ be a complex normed space and $T : \mathrm{Dom}(T) \to U$ a linear operator with domain $\mathrm{Dom}(T) \subset U$ . A regular value $\lambda$ of $T$ is a complex number such that

- $R_\lambda(T)$ exists
- $R_\lambda(T)$ is bounded
- $R_\lambda(T)$ is defined whole $U$.

## Definition

The resolvent set $\rho(T)$ of $T$ is the set of all regular values $\lambda$ of $T$. Its complement $\sigma(T) = \mathbb{C} \backslash \rho(T)$ in the complex plane $\mathbb{C}$ is called the spectrum of $T$, and a $\lambda \in \sigma(T)$ is called a spectral value of $T$.

Furthermore, the spectrum $\sigma(T)$ is partitioned into three disjoint sets as follows:

### definiton

The **point spectrum** $\sigma_p(T)$ is the set such that $R_\lambda(T)$ does not exist. A $\lambda \in \sigma_p(T)$ is called an eigenvalue of T. The **continuous spectrum** $\sigma_c(T)$ is the set such that $R_\lambda(T)$ exists, defined on the dense set in $U$ but is unbounded. The **residual spectrum** $\sigma_r(T)$ is the set such that $R_\lambda(T)$ exists (and may be bounded or not) but the domain of $R_\lambda(T)$ is not dense in $U$.

# Scale of Hilbert spaces associated with a non-negative operator

- To a Hilbert space $H$ with inner product $\langle ., . \rangle$ and norm $\|.\|$ together with a non-negative, unbounded, operator $A$, we associate the scale of Hilbert spaces

$$\mathcal{H}_k := \mathrm{Dom}((A+I)^{k/2}), \quad \|u\|_k := \|(A+I)^{k/2}u\|, \quad k \geq 0,$$

- For negative exponents, define

$$\mathcal{H}_{-k} := \mathcal{H}_k^*$$

where $I$ is the identity operator.

We think of $(H', A')$ being some perturbation of $(H, A)$ and want to lessen the assumption such that the spectral properties are not the same but still are close.

# Definition

Suppose we have linear operators

$$
\begin{aligned}
J &: H \to H', \qquad & J_1 &: H_1 \to H_1' \\
J' &: H' \to H, \qquad & J_1' &: H_1' \to H_1.
\end{aligned}
$$

Let $\delta > 0$ and $k \geq 1$. We say that $(H, A)$ and $(H', A')$ are $\delta$-close of order $k$ iff the following conditions are fulfilled:

$$
\begin{aligned}
\|Jf - J_1 f\|_0 &\leq \delta \|f\|_1, && (3.1)\\
|(Jf, u) - (f, J'u)| &\leq \delta \|f\|_0 \|u\|_0, && (3.2)\\
\|u - JJ'u\|_0 &\leq \delta \|u\|_1 && (3.3)\\
\|Jf\|_0 \leq 2\|f\|_0, & \qquad \|J'u\|_0 \leq 2\|u\|_0, && (3.4)\\
\|(f - J'Jf)\|_0 &\leq \delta \|f\|_1, && (3.5)\\
\|J'u - J_1'u\|_0 &\leq \delta \|u\|_1, && (3.6)\\
|a(f, J_1'u) - a'(J_1 f, u)| &\leq \delta \|f\|_k \|u\|_1, && (3.7)
\end{aligned}
$$

We denote $d_{\text{Haussdorff}}(A; B)$ the Hausdorff distance for subsets $A; B \subset \mathbb{R}$

$$d_{\text{Haussdorff}}(A, B) := \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}, \qquad (3.8)$$

where $d(a, B) := \inf_{b \in B} |a - b|$. We set

$$\bar{d}(A, B) := d_{Hausdorff} \left( (A + 1)^{-1}, (B + 1)^{-1} \right) \qquad (3.9)$$

for closed subsets of $[0, \infty)$.
We denote

$$(A + 1)^{-1} = \left\{ (1 + x)^{-1} : x \in A \right\}$$

and

$$(B + 1)^{-1} = \left\{ (1 + y)^{-1} : y \in B \right\}$$

# Theorem

There exists $\eta(\delta) > 0$ with $\eta(\delta) \to 0$ as $\eta(\delta)$ such that

$$\bar{d}\left(\sigma_\bullet(A), \sigma_\bullet(A')\right) \leq \eta(\delta) \tag{3.10}$$

for all pairs of non-negative operators and Hilbert spaces $(H, A)$ and $(H', A')$ which are $\delta$-close. Here, $\sigma_\bullet(A)$ denotes either the entire spectrum, the essential or the discrete spectrum of $A$. Furthermore, the multiplicity of the discrete spectrum, $\sigma_{\text{disc}}$, is preserved, i.e. if $\lambda \in \sigma_{\text{disc}}$ has multiplicity $\mu > 0$, then there exist $\mu$ eigenvalues (not necessarily all distinct) of operator $A'$ belonging to interval $(\lambda - \eta(\delta), \lambda + \eta(\delta))$.

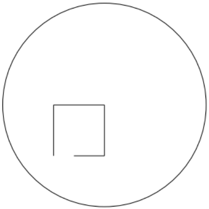
The starting point is to consider a bounded domain $\Omega \subset \mathbb{R}^2$ and a compact set $K \subset \Omega$ with zero Lebesgue measure (e.g. an interval or a piece of a curve). We denote $\Omega_K := \Omega \backslash K$. The Neumann Laplacian $-\Delta_N^{\Omega_K}$ $N$ is defined on the Sobolev space $\mathcal{H}^1(\Omega_K)$ via the quadratic form

$$\int_\Omega |\nabla u|^2 dxdy, \quad u \in \mathcal{H}^1(\Omega_K).$$

In case if $K$ is empty set then unperturbed Neumann Laplacian denoted by $-\Delta_N^{\Omega}$ which is defined via the same form
$\displaystyle\int_\Omega |\nabla u|^2 dxdy, \quad u \in \mathcal{H}^1(\Omega).$

During our paper we suppose the additional property *: for any $(x_0; y_0) \in \Omega_K$ at least one of the following conditions takes place

- The line $l(x_0) = \{x = x_0\}$. Then at least one of the half-lines $l(x_0) \cap \{y \geq y_0\}$ and $l(x_0) \cap \{y \leq y_0\}$ has no intersection with $K$.
- The line $h(y_0) = \{y = y_0\}$. Then at least one of the half-lines $h(y_0) \cap \{x \geq x_0\}$ and $h(y_0) \cap \{x \leq x_0\}$ has no intersection with $K$.



The main result of this section is the following theorem.

# Theorem 3.1

Let $\Omega$ be an open bounded domain in $\mathbb{R}^2$ and let $p \in \Omega$ be some fixed point. Suppose that $B_\varepsilon \subset \Omega$ is a ball with center at $p$ and radius $\varepsilon > 0$. Let $K = K(\varepsilon) \subset B_\varepsilon$ be a compact set with zero Lebesgue measure (e.g. an interval or a piece of a curve). Moreover, suppose that $\Omega_K$ satisfies property*. Let $-\Delta_N^{\Omega}$ and $-\Delta_N^{\Omega_{K_\varepsilon}}$ be the Neumann Laplacians defined on $\Omega$ and $\Omega_{K_\varepsilon}$, respectively. Then for small enough $\varepsilon$ the Neumann Laplacians $-\Delta_N^{\Omega}$ and $-\Delta_N^{\Omega_{K_\varepsilon}}$ are $\mathcal{O}(\varepsilon^{1/6})$ close of order 2.

# Theorem 3.2

Using the above assumptions, let $-\Delta_N^{\Omega}$ and $-\Delta_N^{\Omega_{K_\varepsilon}}$ be the Neumann Laplacians defined on $\Omega$ and $\Omega_{K_\varepsilon}$, respectively. Then there exists $\eta(\varepsilon) > 0$ with $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that the following spectral convergence takes place

$$\bar{d}\left(\sigma_\bullet(-\Delta_N^{\Omega_{K_\varepsilon}}), \sigma_\bullet(-\Delta_N^{\Omega})\right) \leq \eta(\varepsilon)$$

where $\bar{d}$ is defined in (9) and $\sigma_\bullet(.)$ denotes either the entire spectrum, the essential or the discrete spectrum. Moreover, the multiplicity of the discrete spectrum is preserved.

## Corollary 3.3

Suppose that $-\Delta_N^{\Omega}$ has purely discrete spectrum denoted by $\lambda_k(\Omega)$ (repeated according to multiplicity). Then the infimum of the essential spectrum of $-\Delta_N^{\Omega_{K_\varepsilon}}$ tends to infinity and there exists $\eta_k(\varepsilon) > 0$ with $\eta_k(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$|\lambda_k(\Omega) - \lambda_k(\Omega_{K_\varepsilon})| \leq \eta_k(\varepsilon)$$

for small enough $\varepsilon$. Here, $\lambda_k(\Omega_{K_\varepsilon})$ denotes the discrete spectrum of $-\Delta_N^{\Omega_{K_\varepsilon}}$ (below the essential spectrum) repeated according to multiplicity.

## Corollary 3.4

The Hausdorff distance between the spectra of $-\Delta_N^{\Omega_{K_\varepsilon}}$ and $-\Delta_N^{\Omega}$ converges to zero on any compact interval $[0; \Lambda]$

STEP 1: Construction of the mappings $J, J', J_1, J_1'$ . It is easy to notice that $H = H' = L^2(\Omega)$, $A = A' = -\Delta$; $H^1, H_1'$ correspond to Sobolev spaces $\mathcal{H}^1(\Omega)$ and $\mathcal{H}^1(\Omega_{K_\varepsilon})$ and $H_2 = \mathrm{Dom}(-\Delta_N^\Omega)$. The norm $\|.\|_0$ corresponds with the $L^2$ norm and

$$\|u\|_1 = (\|u\|_0^2 + \|\nabla u\|_0^2)^{1/2}, \quad \|f\|_2 = \| - \Delta f + f\|_0$$

Since $H = H'$ and $H_1 \subset H_1'$ we choose $J = J' = I$, where $I$ is the identity operator and $J_1$ is the restriction operator: $J_1 u = u|_{\Omega_{K_\varepsilon}}$ for $u \in H^1$. Let us now construct the mapping $J_1' : H_1' \to H^1$. Without loss of generality, assume that the ball $B_\varepsilon$ mentioned in Theorem 3.1 is centered at the origin. Let $\epsilon \in (\varepsilon, 2\varepsilon)$ be a number to be chosen later and let $B_\epsilon \supset B_\varepsilon$ be the ball with center again at the origin and radius $\epsilon$, $\Omega_\epsilon := \Omega \backslash B_\epsilon$. We are going to construct mapping $J_1'$ first for smooth functions. For any $v \in C^\infty(\Omega_{K_\varepsilon})$ we define

$$J_1' v := \begin{cases} v, & on \quad \Omega_\epsilon \\ \frac{r}{\epsilon} \tilde{v}(\epsilon, \varphi) & on \quad B_\epsilon, \end{cases}$$

where $\tilde{v}(r, \varphi) = v(r \cos \varphi, r \sin \varphi)$.

Now let us construct the mapping $J_1' u$ for any $u \in H_1'$ . Employing the approximation method described in (5,Thm.2, 5.3.2), for the fixed sequence $\{\eta_k\}_{k=1}^{\infty}$ converging to zero we construct the sequence $v_{\eta k} \in C^{\infty}(\Omega_{K_\varepsilon})$ which satisfies

$$\|u - v_{\eta_k}\|_1 = \|u - v_{\eta_k}\|_{\mathcal{H}^1(\Omega_{K_\varepsilon})} \leq \eta_k \|u\|_1 \tag{5.1}$$

Let us mention that in view of the inequalities (4.9) and (4.16) which will be proved later it follows for any smooth function $v$.

$$\|J_1' v\|_1^2 = \int_\Omega |\nabla J_1' v|^2 dxdy + \int_\Omega |J_1' v|^2 dxdy \leq \bar{C}(\varepsilon)\|v\|_1^2,$$

where $\bar{C}(\varepsilon)$ is some constant. Therefore using the completeness of space $H_1$ we are able to define

$$J_1' u = \lim_{k\to\infty} J_1' v_{\eta_k}. \tag{5.2}$$

STEP 2: The conditions (3.1)-(3.7) hold for the mappings $J$, $J'$, $J_1$, $J_1'$.

- we have that the estimates (3.1)-(3.5) are satisfied with $\delta = 0$.
- We prove (2.6), i.e. under the assumptions stated in Theorem 3.1 inequality (3.6) is satisfied with $\delta = \mathcal{O}(\sqrt{\varepsilon})$ for small enough $\varepsilon$.
- We give the proof of the estimate (3.7), i.e. under the assumptions stated in Theorem 3.1 inequality (3.7) takes place with $k = 2$ and $\delta = \mathcal{O}(\varepsilon^{1/6})$ for small enough $\varepsilon$.

Indeed, we have

$$|a(f, J_1'u) - a'(J_1 f, u)| \leq 2^{2/3}(\tilde{C}(4C'' + 34\varepsilon))^{1/2}\varepsilon^{1/6}\|f\|_2\|u\|_1. \quad (5.3)$$

It is easy to notice that the right-hand side of inequality (5.3) for small enough $\varepsilon$ satisfies

$$r.h.s.(5.3) = \mathcal{O}\left(\varepsilon^{1/6}\right)\|f\|_2\|u\|_1,$$

which ends the proof.

# Extended Results

## Estimation on the rate of convergence

### Theorem

Let $-\Delta_N^{\Omega}$ and $-\Delta_N^{\Omega_{K_\varepsilon}}$ be the Neumann Laplacians defined on $\Omega$ and $\Omega_{K_\varepsilon}$, respectively. Then

$$\bar{d}\left(\sigma_\bullet(-\Delta_N^{\Omega_{K_\varepsilon}}), \sigma_\bullet(-\Delta_N^{\Omega})\right) \leq 4\sqrt{2}\varepsilon^{1/6},$$

where $C > 0$ is a constant independent of $\varepsilon$.

## The dependence on the distance of the hole from the boundary to the speed of spectral convergence

### Theorem

Let $\Omega$ be an open bounded domain in $\mathbb{R}^2$ and let $p \in \Omega$. Suppose that $B_\varepsilon \in \Omega$ is a ball with center at $p$ and redius $\varepsilon$ which contains an interval or a piece of a curve. Suppose that $\Omega_K$ satisfies property *. Let $-\Delta_N^{\Omega}$ and $-\Delta_N^{\Omega_{K_\varepsilon}}$ be the Neumann Laplacians defined on $\Omega$ and $\Omega_{K_\varepsilon}$, respectively. Then the spectral convergence takes place faster when moving the ball $B_\varepsilon$ to the center of $\Omega$ or moving the ball to the position such that the center $p$ lies on the circle $B(w, \varepsilon)$ where $w$ satisfies

$$\text{dist}(w, \partial\Omega) = 1/\text{diam}(\Omega).$$

# An asymptotic formula for the eigenvalues of the Laplacian in a perturbed domain

Let $\Omega$ be an open bounded domain in $\mathbb{R}^2$ and let $w \in \Omega$ be some fixed point. Suppose that $\mathbb{B}_\varepsilon \in \Omega$ is a ball with center at $w(x_0, y_0)$ and radius $\varepsilon > 0$. Remove from $\Omega$ the horizontal slit,
$K_\varepsilon = \{(x,y)|y = y_0, |x - x_0| \leq \varepsilon\}$
We denote $\Omega_{K_\varepsilon} := \Omega \backslash K_\varepsilon$. We consider the following eigenvalue problems

$$\begin{aligned} -\Delta_x u(x) &= \lambda(\varepsilon)u(x), \quad x \in \Omega_{K_\varepsilon}, \\ u(x) &= 0, \quad x \in \partial\Omega, \\ \frac{\partial u}{\partial \nu}(x) &= 0, x \in K, \end{aligned} \tag{6.1}$$

where $\partial/\partial\nu$ denotes the derivative along the inner normal vector at $x$ with respect to the domain $\Omega_{K_\varepsilon}$.

$$-\Delta_x u(x) = \lambda u(x), \quad x \in \Omega \tag{6.2}$$

Let $0 < \mu_1(\varepsilon) \leq \mu_2(\varepsilon) \leq \ldots$ be the eigenvalues of (6.1). Let $0 < \mu_1 \leq \mu_2 \leq \ldots$ be the eigenvalues of (6.2).

**Theorem**

Assume that $\mu_i$ is a simple eigenvalue. Then,

$$\mu_i(\varepsilon) = \mu_i - 2\pi\varepsilon^2 |\mathrm{grad}\varphi_i(w)|^2 + \pi\mu_i\varphi_i(w)^2\varepsilon^2 + O(\varepsilon^{5/2-s})$$

for an arbitrary $s > 0$, holds as $\varepsilon$ tends to zero, where $\varphi_i(x)$ denotes the eigenfunction associated with $\mu_i$ satisfying

$$\int_\Omega \varphi_i(x)^2 = 1.$$

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Thank you very much for your attention!