Limit with respect to a measure on $\mathbb{N}$

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Outline

Aim of the talk

- The aim of this talk is a brief analysis and discussion of the concept of limit of sequences and its extension in fuzzy direction.
- This extension has a quantitative character. We define the degree of convergence of a given sequence to a given point as a number in the interval $[0, 1]$ where the value 1 means the most perfect convergence.
- It follows from the results that this degree of convergence can be interpreted as the logical value of the statement "The sequence (x_n) converges to the point x_0" in $[0, 1]$-valued fuzzy logic.

Standard convergence of sequences

- A sequence (x_n) of points in a metric space (X, d) converges to a point $x_0 \in X$ if and only if

$$\forall\, \varepsilon > 0\ \exists\, n_0 \in \mathbb{N} \text{ such that } \forall\, n > n_0 :\ d(x_n, x_0) < \varepsilon.$$

- Every metric space is topologically equivalent to a metric space in which all possible distances belong to the interval $[0, 1]$. This means that, without loss of generality, complete information and decision on convergence is in the sequence $(d(x_n, x_0))$ in $[0, 1]$.
- Decision must be: Either converges or does not.

Standard convergence of sequences

- Roughly spoken, a sequence (x_n) of points in a metric space (X, d) converges to a point $x_0 \in X$ if and only if all sets

$$L_\varepsilon = \{n \in \mathbb{N} \mid d(x_n, x_0) < \varepsilon\}, \ \varepsilon > 0$$

are **extremely big** (i.e. cofinite).

- Even less formally, a sequence (x_n) of points in a metric space (X, d) converges to a point $x_0 \in X$ if and only if **almost all** points of the sequence are undistinguishable from the limit point by **arbitrary precise** measurements.
- It suggests two kind of possible generalizations: **how many points** and by **how precise** measurements.

Convergence with respect to a filter

- The first kind of possible generalization: a sequence (x_n) converges to a point $x_0 \in X$ if and only if all sets

$$L_\varepsilon = \{n \in \mathbb{N} \mid d(x_n, x_0) < \varepsilon\}, \ \varepsilon > 0$$

are **big in some sense**.

- Standard way how to characterize big sets is that they belong to a suitable **filter**, i.e. class of subsets of $\mathbb{N}$ closed with respect to supersets and finite intersections.
- A sequence (x_n) converges to a point $x_0 \in X$ **with respect to a filter** $\mathcal{F}$ if and only if all sets

$$L_\varepsilon = \{n \in \mathbb{N} \mid d(x_n, x_0) < \varepsilon\}, \ \varepsilon > 0$$

belong to $\mathcal{F}$. It was introduced by H. Cartan (1937).

Convergence with respect to an ideal

- Dual formulation: a sequence (x_n) converges to a point $x_0 \in X$ if and only if all sets

$$G_\varepsilon = \{n \in \mathbb{N} \mid d(x_n, x_0) \geq \varepsilon\}, \ \varepsilon > 0$$

  are **small in some sense**.
- Standard way how to characterize small sets is that they belong to a suitable **ideal**, i.e. class of subsets of $\mathbb{N}$ closed with respect to subsets and finite unions.
- A sequence (x_n) converges to a point $x_0 \in X$ **with respect to an ideal** $\mathcal{I}$ if and only if all sets

$$G_\varepsilon = \{n \in \mathbb{N} \mid d(x_n, x_0) \geq \varepsilon\}, \ \varepsilon > 0$$

  belong to $\mathcal{I}$.

Statistical convergence

- The above concept (equivalent to the convergence with respect to a filter) was rediscovered in 1999 by Kostyrko, Šalát and Wilczyński. They were inspired by the statistical convergence.
- Convergence with respect to the ideal of sets of asymptotic density 0, i.e.

$$\mathcal{I}_d = \left\{ A \subset \mathbb{N} \mid \lim_{n \to \infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n} = 0 \right\}$$

  is called **statistical convergence.**
- It was introduced by Fast (1951) and has several applications in number theory.
- Convergence is still strict: either converges or does not.

Fuzzy convergence

- The second king of generalization of the standard convergence was introduced by M. Burgin in 2000.
- In his approach a sequence (x_n) is r-**convergent** to a point x_0 if for every $\varepsilon > 0$ the set of all indices n such that x_n belongs to the $r + \varepsilon$ ball around x_0 is cofinite, i.e.

$$L_{\varepsilon+r} = \{n \in \mathbb{N} \mid d(x_n, x_0) < r + \varepsilon\}$$

is cofinite.
- The fact that this kind of convergence depends on a positive real number r is probably the reason why he calls this kind of convergence a **fuzzy convergence**.

Motivating example

- Consider the sequence given by $x_n = \frac{1+(-1)^n}{2}$ and the following three statements.
- "The sequence does not converge".
- "The sequence has two accumulation points 0 and 1".
- "The sequence half converges to 0 and half converges to 1 with respect to asymptotic density".
- In our opinion, the last statement contains the most information.
- This is a motivation to introduce a partial convergence with respect to a measure on $\mathbb{N}$.

Outline

Measure on $\mathbb{N}$

- By a **measure** on $\mathbb{N}$ we will mean any monotone function $\mu\colon 2^{\mathbb{N}} \to [0,1]$ such that $\mu(F) = 0$ and $\mu(\mathbb{N} \setminus F) = 1$ for every finite $F \subset \mathbb{N}$.
- μ is said to be $\{0,1\}$-measure if $\mu(A) \in \{0,1\}$ for all $A \subset \mathbb{N}$.
- A measure μ is **subadditive** if

$$\mu(A \cup B) \leq \mu(A) + \mu(B)$$

for all $A, B \subset \mathbb{N}$, it is **superadditive** if

$$\mu(A \cup B) \geq \mu(A) + \mu(B)$$

for all **disjoint** $A, B \subset \mathbb{N}$.

- A measure is **additive** if it is both subadditive and superadditive.

Measure on $\mathbb{N}$

- We will need also some other properties of measures.
- A measure μ is **submodular (supermodular)** if

$$\mu(A \cup B) + \mu(A \cap B) \leq \ (\geq)\ \mu(A) + \mu(B)$$

for all $A, B \subset \mathbb{N}$,

- it is **intersective** if

$$\mu(A \cap B) \geq \mu(A) + \mu(B) - 1$$

holds for all $A, B \subset \mathbb{N}$.

- Super(submodularity) is stronger than super(subadditivity) Also, every supermodular measure is also intersective.

Degree of convergence

- Let (X, d) be a metric space, $\mathbf{x} = (x_n)$ a sequence in X and let $x_0 \in X$. Suppose that μ is a measure on $\mathbb{N}$. The number

$$\mathrm{L}_\mu(\mathbf{x}, x_0) = \inf\{\mu(\{n \in \mathbb{N} |\ d(x_n, x_0) < \varepsilon\}),\ \varepsilon > 0\}$$

  is called the **degree of convergence** of the sequence $\mathbf{x}$ to the point x_0 with respect to the measure μ.
- We call a point $x_0 \in X$ a $\mu-$**limit** of a sequence $\mathbf{x}$ if $\mathrm{L}_\mu(\mathbf{x}, x_0) = 1$ and we will denote this fact by $\mathbf{x} \xrightarrow{\mu} x_0$.
- It appears that statistical convergence and more general convergence with respect to a filter are very special cases of the above concept.
- Properties of such a convergence strongly depend on properties of the chosen measure.

Outline

$\{0,1\}$- measures

- Recall that a measure is $\{0,1\}$-measure if $\mu(A) \in \{0,1\}$ for all $A \subset \mathbb{N}$.
- In the class of $\{0,1\}$-measures some of the above defined properties coincide, especially
  - subadditive = submodular,
  - superadditive = supermodular = intersective.
- Let $\mathcal{I}$ be an ideal of subsets of $\mathbb{N}$. Then $\mu_{\mathcal{I}}$ defined by

$$\mu_{\mathcal{I}}(A) = \begin{cases} 0 & \text{if } A \in \mathcal{I} \\ 1 & \text{if } A \notin \mathcal{I} \end{cases}$$

is a submodular (hence also subadditive) measure. The smallest ideal on $\mathbb{N}$ is Fin, the ideal of all finite sets.

$\{0,1\}$- measures

- Let $\mathcal{F}$ be a filter of subsets of $\mathbb{N}$. Then $\mu_{\mathcal{F}}$ defined by

$$\mu_{\mathcal{F}}(A) = \begin{cases} 0 & \text{if } A \notin \mathcal{F} \\ 1 & \text{if } A \in \mathcal{F} \end{cases}$$

  is a supermodular (hence also superadditive and intersective) measure.
- The smallest filter is Cof, the filter of all cofinite sets.
- All subadditive $\{0,1\}$-measures are of the form $\mu_{\mathcal{I}}$ and all superadditive $\{0,1\}$- measures are off the form $\mu_{\mathcal{F}}$.
- $L_{\mu_{\mathcal{F}}}$ is in fact the convergence with respect to a filter $\mathcal{F}$ and $L_{\mu_{\mathrm{Cof}}}$ is exactly the standard convergence.
- μ_{Cof} is the smallest measure and μ_{Fin} is the largest measure. Consequently, the standard convergence is the strongest one.

$\{0,1\}$- measures

- If $\mathcal{F}$ is a filter then

$$\mathcal{I}_{\mathcal{F}} = \{\mathbb{N} \setminus F \mid F \in \mathcal{F}\}$$

  is the **dual ideal** to $\mathcal{F}$ and vice versa.
- Maximal filter is called *ultrafilter* and its dual ideal is maximal as well (and vice versa). $\mathcal{U}$ is an ultrafilter if and only if for every $A \subset \mathbb{N}$ either $A \in \mathcal{U}$ or $\mathbb{N} \setminus A \in \mathcal{U}$.
- Let $\mathcal{U}$ be an ultrafilter and $\mathcal{M}$ its dual ideal. Then for every $A \subset \mathbb{N}$ exactly one of the sets $A, \mathbb{N} \setminus A$ belongs to $\mathcal{U}$ and the other one belongs to $\mathcal{M}$. Consequently both measures $\mu_{\mathcal{U}}$ and $\mu_{\mathcal{M}}$ coincide, thus they are additive. Moreover, all additive $\{0,1\}$-measures are of such kind.

Asymptotic density measures

- For a set $A \subset \mathbb{N}$ the numbers

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap \{1,2,\dots,n\}|}{n}$$

and

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap \{1,2,\dots,n\}|}{n}$$

are called the **lower** and **upper asymptotic density** of the set A, respectively. If $\underline{d}(A) = \overline{d}(A)$ we call this common value $d(A)$ **asymptotic density.**

- $\overline{d}$ is a subadditive measure and $\underline{d}$ is a superadditive and intersective measure. We will see that convergence with respect to $\underline{d}$ coincides with the statistical convergence, hence it has "nice" properties.
- Convergence with respect to $\overline{d}$ is strange.

Convergence with respect to $\underline{d}$

- Recall that a sequence $\mathbf{x} = (x_n)$ in a metric space statistically converges to a point x_0 (i.e. $\mathbf{x} \xrightarrow{\text{stat}} x_0$) if and only if for every $\varepsilon > 0$, denoting

$$L_\varepsilon = \{n \in \mathbb{N} \mid d(x_n, x_0) < \varepsilon\}, \; G_\varepsilon = \{n \in \mathbb{N} \mid d(x_n, x_0) \geq \varepsilon\},$$

we have $\overline{d}(G_\varepsilon) = 0$, i.e.

$$\underline{d}(L_\varepsilon) = 1.$$

- This shows that that

$$\mathbf{x} \xrightarrow{\text{stat}} x_0 \qquad \Longleftrightarrow \qquad \mathbf{x} \xrightarrow{\mu_d} x_0,$$

i.e. convergence with respect to $\underline{d}$ and statistical convergence coincide.

Convergence with respect to $\overline{d}$

- There are sequences $\mathbf{x} = (x_n)$ in $[0, 1]$ (the so called maldistributed sequences) such that $x_n \stackrel{\overline{d}}{\to} a$ holds for every $a \in [0, 1]$. An example of such sequence is $(\{\log \log n\})$, the sequence of fractional parts of $\log \log n$.
- Although $[0, 1]$ is a compact space, there are sequences $\mathbf{x}$ in $[0, 1]$ such that

$$\mathrm{L}_{\overline{d}}(\mathbf{x}, a) = 0$$

for every $a \in [0, 1]$. For example every uniformly distributed sequence in $[0, 1]$ has this property. But here we have to note that in this example $\overline{d}$ can be replaced by $\underline{d}$.

Uniformly distributed sequences

- A sequence $\mathbf{x} = (x_n)$ in $[0, 1]$ is called **uniformly distributed** (u.d.) if

$$d(\{n \in \mathbb{N} \mid x_n \in [a, b)\}) = b - a$$

holds for every interval $[a, b) \subset [0, 1]$.

- Thus, if $\mathbf{x}$ is u.d., then for all $x_0 \in [0, 1]$ and $\varepsilon > 0$ we have

$$d(\{n \in \mathbb{N} \mid x_n \in (x_0 - \varepsilon, x_0 + \varepsilon)\}) = 2\varepsilon,$$

consequently

$$L_{\mu_{\underline{d}}}(\mathbf{x}, x_0) = L_{\mu_{\overline{d}}}(\mathbf{x}, x_0) = 0.$$

Pairs of dual measures

- Let μ be a measure on $\mathbb{N}$. The measure μ^* defined by

$$\mu^*(A) = 1 - \mu(\mathbb{N} \setminus A), \qquad A \subset \mathbb{N}$$

  is called the **dual measure** to μ. Note that $(\mu^*)^* = \mu$.

- If μ is submodular then μ^* is supermodular and $\mu \geq \mu^*$, hence also

$$L_\mu(\mathbf{x}, x_0) \geq L_{\mu^*}(\mathbf{x}, x_0)$$

  for all sequences **x** and points x_0 in a metric space.

- Let $\mathcal{I}$ be an ideal of subsets of $\mathbb{N}$ and let $\mathcal{F}$ be its dual filter. Then $\mu_\mathcal{I}$ and $\mu_\mathcal{F}$ form a pair of dual measures. Here $\mu_\mathcal{I}$ is submodular and $\mu_\mathcal{F}$ is supermodular, hence $\mu_\mathcal{I} \geq \mu_\mathcal{F}$.

Pairs of dual measures

- If μ is subadditive then μ^* is intersective (be careful, not necessary superadditive!) and $\mu \geq \mu^*$, hence also

$$L_\mu(\mathbf{x}, x_0) \geq L_{\mu^*}(\mathbf{x}, x_0)$$

for all sequences $\mathbf{x}$ and points x_0 in a metric space.

- Also $\overline{d}$ and $\underline{d}$ form a pair of dual measures where the first one is subadditive, the second one is intersective (also superadditive) and $\overline{d} \geq \underline{d}$.

Outline

Uniqueness of limits

- One of the most important property of limit is its uniqueness in Hausdorff, hence metric spaces. In our case we have the following more general theorem.

Theorem

Let μ be a superadditive measure on $\mathbb{N}$ and $\mathbf{x} = (x_n)$ a sequence in X. Then

$$\sum_{x_0 \in X} \mathrm{L}_\mu(\mathbf{x}, x_0) \leq 1.$$

Uniqueness of limits

Corollary

Let μ be a superadditive measure on $\mathbb{N}$. Then each sequence in X has at most one μ-limit.

- Note that additive measures are superadditive, thus both statements hold for all additive measures. Also note that both statements can be applied in the case of $\mu_{\mathcal{F}}$ where $\mathcal{F}$ is a filter. We have already seen that the standard convergence is the special case of convergence with respect to the filter Cof. The above Corollary shows that our results generalize the standard ones.

- If $\mathbf{x}$ is a sequence and $A \subset \mathbb{N}$ then by $\mathbf{x}_A$ we denote the subsequence of $\mathbf{x}$ with indices in A. By $\mathrm{T_{Luk}}$ we denote the Łukasiewicz t-norm, i.e. the binary operation on $[0,1]$ defined by

$$\mathrm{T_{Luk}}(a,b) = \max\{0, a+b-1\}$$

for all pairs $(a,b) \in [0,1]^2$.

Theorem

Let μ be an intersective measure on $\mathbb{N}$, let $\mathbf{x} = (x_n)$ be a sequence in X and $A \subset \mathbb{N}$. Then for every $x_0 \in X$ the inequalities

$$\mathrm{T_{Luk}}(\mathrm{L}_\mu(\mathbf{x}, x_0), \mu(A)) \leq \mathrm{L}_\mu(\mathbf{x}_A, x_0) \leq \min\{\mathrm{L}_\mu(\mathbf{x}, x_0), \mu(A)\}.$$

hold.

Convergence of subsequences

Corollary

Let μ be an intersective measure on $\mathbb{N}$, let $\mathbf{x} = (x_n)$ be a sequence in X and $A \subset \mathbb{N}$. If $\mathbf{x} \xrightarrow{\mu} x_0$ then

$$\mathrm{L}_\mu(\mathbf{x}_A, x_0) = \mu(A).$$

Corollary

Let μ be an intersective measure on $\mathbb{N}$, let $\mathbf{x} = (x_n)$ be a sequence in X such that $\mathbf{x} \xrightarrow{\mu} x_0$ and let $A \subset \mathbb{N}$ be such that $\mu(A) = 1$. Then

$$\mathbf{x}_A \xrightarrow{\mu} x_0.$$

Preservation of operations

Theorem

Let μ be an intersective measure on $\mathbb{N}$, let $\mathbf{x} = (x_n), \mathbf{y} = (y_n)$ be two real sequences and $A \subset \mathbb{N}$, Let $*$ be any of operations of addition, substraction, multiplication or division (except by 0). Then for every real x_0 and y_0 the inequality

$$\mathrm{L}_\mu(\mathbf{x} * \mathbf{y}, x_0 * y_0) \geq \mathrm{T}_{\mathrm{Luk}}(\mathrm{L}_\mu(\mathbf{x}, x_0), \mathrm{L}_\mu(\mathbf{y}, y_0)) \qquad (1)$$

holds.

- Note that each additive measure is also intersective, hence the Theorem is valid for all additive measures as well as for all supermodular measures.
- Consequently, in the special case of additive or supermodular $\{0, 1\}$- measures we obtain the standard result.

Convergence in compact spaces

Theorem

Let μ be a subadditive $\{0,1\}$-measure and let (X,d) be a compact metric space. Then each sequence has a limit.

- The condition of being $\{0,1\}$- measure is substantial.

Corollary

Let μ be an additive $\{0,1\}$-measure and let (X,d) be a compact metric space. Then each sequence has a unique limit.

- **Remark** The last corollary is well known. As the only additive $\{0,1\}$-measures are of the form $\mu_{\mathcal{U}}$ for an ultrafilter $\mathcal{U}$ (see the last slide $\{0,1\}$-measures), it says that every sequence in compact metric space has a unique limit along an ultrafilter.

Outline

Sequences with infinitely many positive degrees of convergence

- **Example** For every $n \in \mathbb{N}$ denote by $i(n)$ the largest power of 2 such that n is divisible by $2^{i(n)}$. Note that $i(n) = 0$ for all odd positive integers. Now define the sequence $\mathbf{x} = (x_n)$ by

$$x_n = 2^{-i(n)-1}.$$

- One can easily check that $L_{\overline{d}}(\mathbf{x}, 2^{-k}) = L_{\underline{d}}(\mathbf{x}, 2^{-k}) = 2^{-k} > 0$ for all positive integers k. This shows that the set of points at which the degree of convergence is positive can be infinite also for superadditive measures, e.g. the measure $\underline{d}$.

Sequences with arbitrary prescribed degree of convergence

- **Example** Let $\alpha \in (0,1)$, let (X,d) be a metric space, $x_0 \neq y_0$ be two different points in X and (p_n) be a sequence in X converging to x_0. Define $x_n = p_k$ if $n = \lfloor \frac{k}{\alpha} \rfloor$ for some $k \in \mathbb{N}$, otherwise put $x_n = y_0$.
- Each neighbourhood of x_0 contains terms $x_{\lfloor \frac{k_0+1}{\alpha} \rfloor}, x_{\lfloor \frac{k_0+2}{\alpha} \rfloor}, \dots$ for some $k_0 \in \mathbb{N}$. Thus

$$d(\{n \in \mathbb{N} |\ d(x_n, x_0) < \varepsilon\}) = \lim_{n\to\infty} \frac{n}{\lfloor \frac{k_0+n}{\alpha} \rfloor} = \alpha$$

for every $\varepsilon > 0$, proving

$$L_{\underline{d}}((x_n), x_0) = L_{\overline{d}}((x_n), x_0) = \alpha.$$