Plabic graphs in physics

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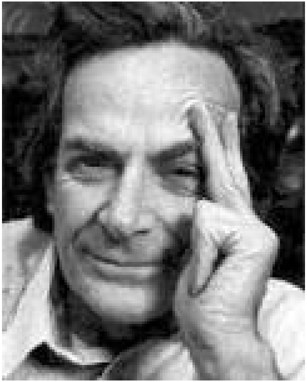
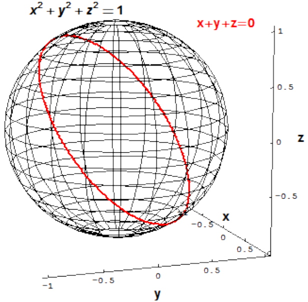
Richard P. Feynman (1918 - 1988)



"One cannot understand... the universality of laws of nature, the relationship of things, without an understanding of mathematics. There is no other way to do it."

Algebraic geometry



We could describe **algebraic geometry** as the study of **polynomial functions** and the spaces on which they are defined, which we call **algebraic varieties.** It found applications, for example, in number theory (solution of Fermat's last theorem) in mathematics and in quantum gravity and in quantum field theory in theoretical physics (twistor theory, amplituhedron).

Outline

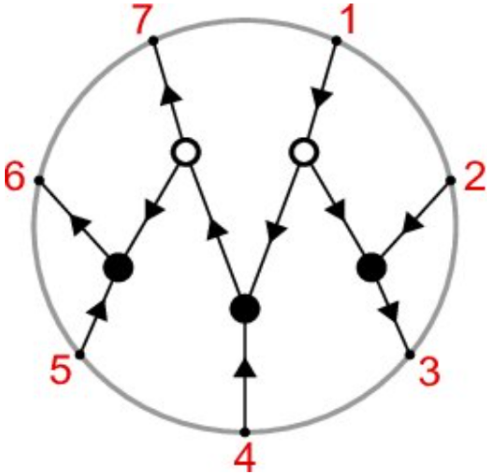
- plabic graphs
  1. Le-diagrams, decorated permutations
  2. formal boundary measurements
  3. winding index
- amplituhedron
- string theory
- ring paradigm

Planar directed network

Definition

A planar directed graph G is a directed graph drawn inside a disk. We will assume that G has finitely many vertices and edges. We allow G to have loops and multiple edges. We assume that G has n boundary vertices on the boundary of the disk labeled $b_1, ..., b_n$ clockwise. The remaining vertices, which we call the internal vertices, are located strictly inside the disk. We always assume that each boundary vertex b_i is either a source or a sink. Even if b_i is an isolated boundary vertex, we will assign b_i to be a source or a sink. A planar directed network $N = (G, x)$ is a planar directed graph G as above together with strictly positive real weights $x_e > 0$ assigned to all edges e of G.

Plabic graph



Boundary measurement

Definition

For such network N, the source set $I \subset [n]$ and the sink set $\bar{I} \equiv [n] \setminus I$ of N are the sets such that $b_i, i \in I$, are the sources of N and the $b_j, j \in \bar{I}$, are the boundary sinks.

Definition

If the network N is acyclic, then, for any $i \in I$ and $j \in \bar{I}$ we define the boundary measurement M_{ij} as the finite sum

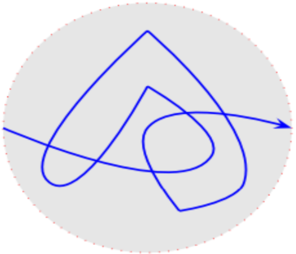
$$M_{ij} \equiv \sum_{P:b_i \to b_j} \prod_{e \in P} x_e, \tag{1}$$

where the sum is over all directed paths P in N from the boundary source b_i to the boundary sink b_j, and the product is over all edges e in P.

Winding index

We define the **winding index** for a path P from a boundary vertex b_i to a boundary vertex b_j: we assume that all edges of the network are given by smooth curves; thus the path P is given by a continuous piecewise-smooth curve. It is possible to slightly modify the path and smoothen it around each junction , so it is given by a smooth curve $f : [0, 1] \to \mathbb{R}^2$, and furthermore make the initial tangent vector $f'(0)$ to have the same direction as the final tangent vector $f'(1)$. We could now define the winding index $wind(P) \in \mathbb{Z}$ of the path P as the signed number of full 360° turns that the tangent vector $f'(t)$ makes as we go from b_i to b_j. Smiliarly, we define the winding index $wind(C)$ for a closed directed path C in the graph.

Winding index - picture



We will write now a recursive combinatorical procedure for calculation of the winding index for a path P with vertices $v_1, v_2, ..., v_l$. In the case that the path P has no self-intersections, then $wind(P) = 0$. For a counterclockwise closed path C without self-intersections, we have $wind(C) = 1$ (similarly for clockwise $wind(C) = -1$).

We will suppose now that P has at least one self-intersection, that is $v_i = v_j = v$ for $i < j$. Let C be the closed segment of P with the vertices $v_i, v_{i+1}, ..., v_j$, and let P' be the path P with erased segment C, P' has the vertices $v_1, ..., v_i, v_{j+1}, ..., v_l$. Consider the four edges $e_1 = (v_{i-1}, v_i)$, $e_2 = (v_i, v_{i+1})$, $e_3 = (v_{j-1}, v_j)$, $e_4 = (v_j, v_{j+1})$ in the path P, which are incident to the vertex v. We will define now the number $\epsilon = \epsilon(e_1, e_2, e_3, e_4) \in \{-1, 0, 1\}$, as follows. If the edges are arranged as e_1, e_2, e_3, e_4 clockwise, then set $\epsilon = -1$; otherwise set $\epsilon = 0$. In particular, if some of the edges e_1, e_2, e_3, e_4 are the same, then $\epsilon = 0$.

Lemma

We have $wind(P) = wind(P') + wind(C) + \epsilon$.

$$wind(P) = \begin{cases} wind(P') + 1, & \text{if } C \text{ is a counterclockwise cycle and } \epsilon = 0 \\ wind(P') - 1, & \text{if } C \text{ is a clockwise cycle and } \epsilon = 0 \\ wind(P'), & \end{cases}$$

Essential cycles

If $\epsilon = 0$, then we say that C is an **essential cycle.** Now it is possible to express the winding index in terms of the erased cycles as $\mathit{wind}(P) \equiv \#\{\mathit{counterclockwise\ essential\ cycles}\} - \#\{\mathit{clockwise\ essential\ cycles}\}$.

Let N be a planar directed network with graph G as above (it is allowed that it contains cycles). We will assume for a moment that the weights x_e of edges in N are formal variables. For a path P in G with the edges $e_1, ..., e_l$, we will write $x_P \equiv x_{e_1}...x_{e_l}$. For a source $b_i,\ i \in I$, and sink $b_j,\ j \in \overline{I}$, we define the formal boundary measurement M_{ij}^{form} as the formal power series in the x_e

$$M_{ij}^{form} \equiv \sum_{P:b_i \to b_j} (-1)^{wind(P)} x_P, \qquad (2)$$

where we take the sum over all directed paths P in N from b_i to b_j.

Subtraction-free rational expression

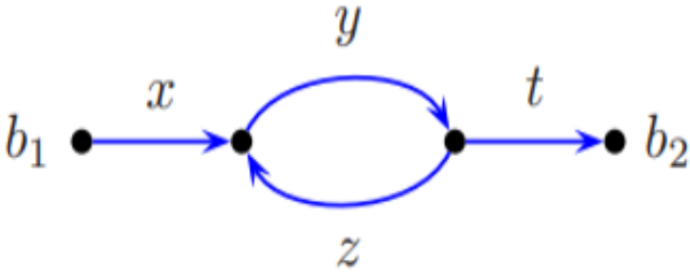
Recall that a **substraction-free rational expression** is an expression with positive integer coefficients that can be written with the operations of addition, multiplication, and division (but subtraction is strictly forbidden). We could equivalently write that it is an expression that can be written as a quotient of two polynomials with positive coefficients. For example,

$$\frac{x + \frac{y^2}{x}}{z^2 + 25y/(x+t)} = \frac{(x^2+y^2)(x+t)}{xz^2(x+t) + 25xy}$$

is subtraction-free.

Boundary measurement

It is possible to define the **boundary measurement** M_{ij} as the specializations of the formal boundary measurements M_{ij}^{form}, written as subtraction-free expressions, when we assign the x_e to be the positive real weights of edges e in the network N. Therefore the boundary measurements M_{ij} are well-defined non-negative real numbers for an arbitrary network.



y
x
t
b_1
b_2
z

Inverse boundary problem

- What information about planar directed network could be recovered from the collection of boundary measurements M_{ij}?
- How can be this information recovered?
- Describe all possible collections of boundary measurements.
- Describe transformation of networks that preserve the boundary measurement.

Gauge transformations

We describe the gauge transformations of the weights x_e. Let us pick a collection of positive real numbers $t_v > 0$, for each internal vertex in N; and also assume that $t_{b_i} = 1$ for each boundary vertex b_i. Let N' be the network with the same directed graph as the network N and with the weights

$$x'_e = x_e t_u t_v^{-1}, \quad (3)$$

for each directed edge $e = (u, v)$. Explained in other words, for each internal vertex v we multiply by t_v the weights of all edges outgoing from v, divide by t_v the weights of all edges incoming to v. Then the network N' has the same boundary measurements as the network N.

Boundary measurement map

We will now describe the set of all possible collections of boundary measurements. For a network N with k boundary sources $b_i, i \in I$, and $n-k$ boundary sinks $b_j, j \in \bar{I}$, it will be convenient to encode the $k(n-k)$ boundary measurements $M_{ij}, i \in I, j \in \bar{I}$, as a certain point in the Grassmannian Gr_{kn}. We recall that $\Delta_J(A)$ is the maximal minor of a matrix A in the column set J. The collection of all Δ_J, for k-subsets $J \subset [n]$, form projective Plücker coordinates on Gr_{kn}.

Definition

Let Net_{kn} be the set of planar directed networks with k boundary sources and $n-k$ boundary sinks. We define the boundary measurement map

$$Meas : Net_{kn} \to Gr_{kn},$$

as follows. For a network $N \in Net_{kn}$ with the source set I and with the boundary measurement M_{ij}, the point $Meas(N) \in Gr_{kn}$ is given in terms of its Plücker coordinates $\{\Delta_J\}$ by the conditions that $\Delta_I \neq 0$ and

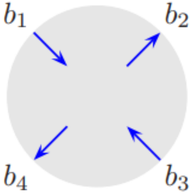
$$M_{ij} = \Delta_{(I\setminus\{i\})\cup\{j\}}/\Delta_I \text{ for any } i \in I \text{ and } j \in \bar{I}.$$

More explicitely, if $I = \{i_1 < ... < i_k\}$, then the point $Meas(N) \in Gr_{kn}$ is represented by the boundary measurement matrix $A(N) = (a_{ij}) \in Mat_{kn}$ such that

1. The submatrix $A(N)_I$ in the column set I is the identity matrix Id_k.
2. The remaining entries of $A(N)$ are $a_{rj} = (-1)^s M_{i_r,j}$ for $r \in [k]$ and $j \in \bar{I}$, where s is the number of elements of I strictly between i_r and j.

Network with four boundary vertices - example

$N =$



$$A(N) = \begin{pmatrix} 1 & M_{12} & 0 & -M_{14} \\ 0 & M_{32} & 1 & M_{34} \end{pmatrix}$$

Main theorem

Theorem

The image of the boundary measurement map Meas is exactly the totally nonnegative Grassmannian:

$$Meas(Net_{kn}) = Gr_{kn}^{tnn}, \tag{4}$$

where $Gr_{kn}^{tnn} = Mat_{kn}^{tnn}/GL_k^+$ *and* Mat_{kn}^{tnn} *is the set of real* $k \times n$*-matrices* A *of rank* k *with nonnegative maximal minors* $\Delta_I(A) \geq 0$ *and* GL_k^+ *is the group of* $k \times k$*-matrices with positive determinant.*

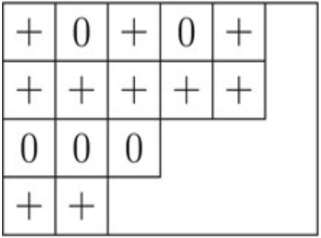
Decorated permutation

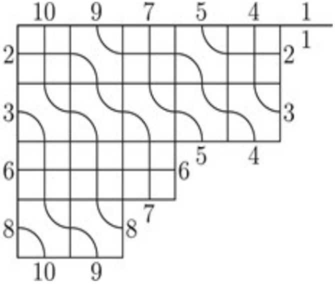
A decorated permutation of the set $[n]$ is a bijection $\pi : [n] \to [n]$ whose fixed points are colored either black or white. We denote a black fixed point by $\pi(i) = \underline{i}$ and white fixed point by $\pi(i) = \bar{i}$. An anti-excendance of the decorated permutation π is an element $i \in [n]$ such that either $\pi^{-1}(i) > i$ or $\pi(i) = \bar{i}$.

Le-diagram

Let us define a Le-diagram D of shape λ and type (k, n) as a filling of boxes of the Young diagram of shape λ contained in a $k \times (n-k)$ rectangle with 0's and $+$'s such that for any three boxes indexed (i', j), (i', j'), (i, j'), where $i < i'$ and $j < j'$, filled with a, b, c, correspondingly, if $a, c \neq 0$ then $b \neq 0$ (Le-property is satisfied).

Le-diagram





Bijection between Le-diagrams and decorated permutations

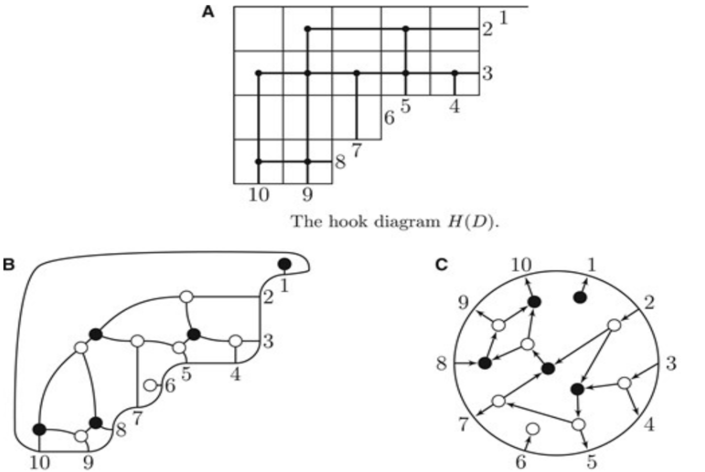
1. Replace each cross $+$ in the Le-diagram D with an elbow joint, and exchange each 0 with a cross.
2. Note that the southeast border of Y_λ gives rise to a length-n path from the northeast corner to the southwest corner of the $k \times (n-k)$ rectangle. Label the edges of this path with the numbers 1 through n.
3. Label the edges of the north and west border of Y_λ so that opposite horizontal edges and opposite vertical edges have the same label.
4. View the resulting 'pipe dream' as a permutation $\pi = \pi(D)$ on $[n]$, by following the 'pipes' from the southeaster border to the northwest border of the Young diagram. If the pipe originating at label i ends at the label j, we define $\pi(i) \equiv j$.
5. If $\pi(i) = i$ and i labels two horizontal (vertical) edges of Y_λ, then $\pi(i) \equiv \underline{i}$ ($\pi(i) \equiv \bar{i}$).

Perfect orientation of a plabic graph

A perfect orientation O of a plabic graph G is a choice of orientation of each of its edges such that each black internal vertex u is incident to exactly one edge directed away from u, and each white internal vertex v is incident to exactly one edge directed towards v. A plabic graph is called **perfectly orientable** if it admits a perfect orientation.

Let D be a Le-diagram and π its decorated permutation. Delete the 0's of D and replace each $+$ with a vertex. From each vertex we construct a hook, which goes east and south, to the border of the Young diagram. The resulting diagram is called the **hook diagram** $H(D)$. After replacing the edges along the southeast border of the Young diagram with boundary vertices labeled by $1, ..., n$, we obtain a planar graph in a disk, with n boundary vertices and one internal vertex for each $+$ of D. Then we replace the local region around each internal vertex as in Figure, and add black (respectively ,white) lollipop for each black (white) fixed point of π. This gives rise to a plabic graph which we call $G(D)$. By orienting the edges of $G(D)$ down and to the left, we obtain a perfect orientation.

Hook diagram

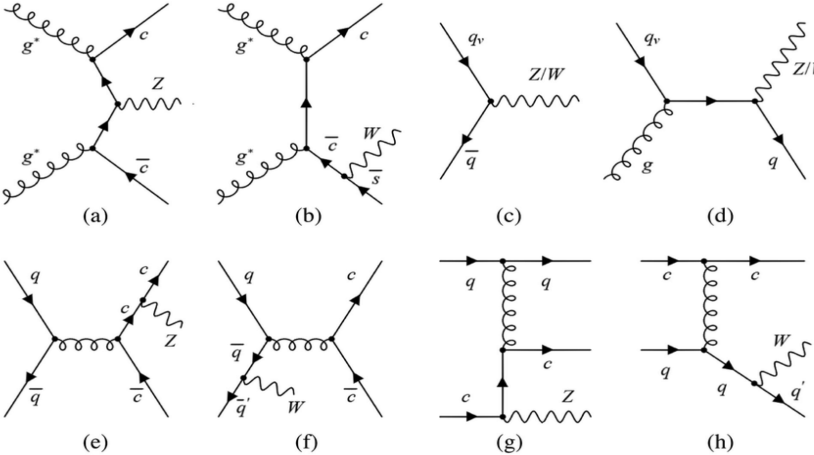


The hook diagram $H(D)$.

The plabic graph $G(D)$.

The plabic graph $G(D)$ redrawn and perfectly oriented.

Perturbative description of Nature in QFT -Feynman diagrams



Amplituhedron

- perturbation theory $\rightarrow$ locality and unitarity as manifest as possible
- toy model in supersymmetric QFT $\rightarrow$ locality and unitarity do not play a central role, but emerge as derived features from a different starting point

Amplituhedron - definition

Let Z be a $(k+m) \times n$ real matrix whose maximal minors are all positive, where $m \geq 0$ is fixed with $k+m \leq n$. Then it induces a map

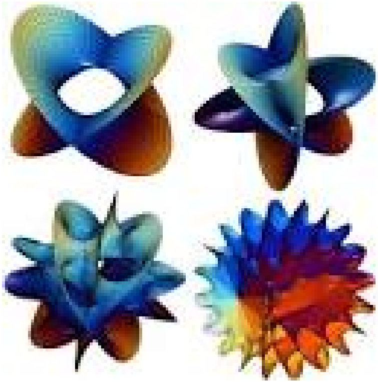
$$\tilde{Z} : Gr_{k,n}^{\geq 0} \to Gr_{k,k+m} \tag{5}$$

defined by

$$\tilde{Z}(\langle v_1, ..., v_k \rangle) \equiv \langle Z(v_1), Z(v_2), ..., Z(v_k) \rangle, \tag{6}$$

where $\langle v_1, ..., v_k \rangle$ is an element of $Gr_{k,n}^{\geq 0}$ written as the span of k basis vectors. The (tree) amplituhedron $A_{n,k,m}(Z)$ is defined to be the image $\tilde{Z}(Gr_{k,n}^{\geq 0})$ inside $Gr_{k,k+,m}$.

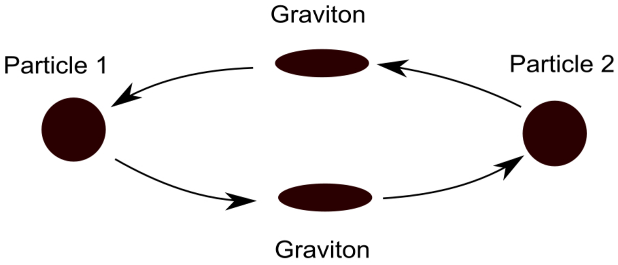
String theory and algebraic geometry



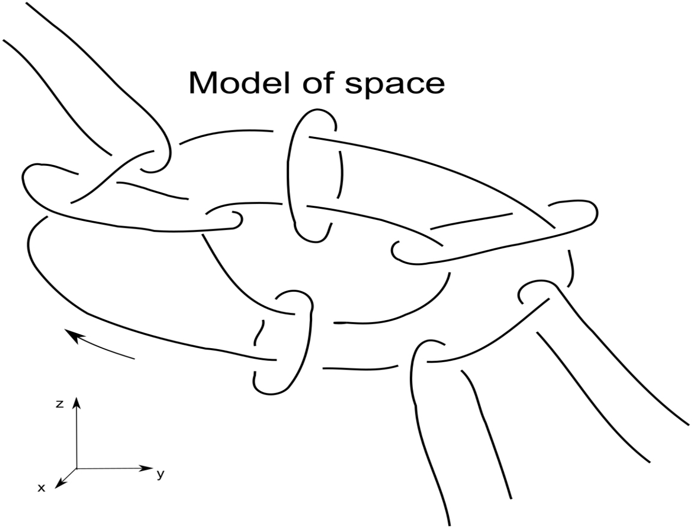
An N-dimensional **Calabi-Yau variety** is an N-dimensional Kähler manifold with (holomorphically, rather than just topologically) trivial canonical bundle. This is equivalent to saying that it is real Riemannian manifold of even dimension $2N$ which has special holonomy in the subgroup.

Ring Paradigm

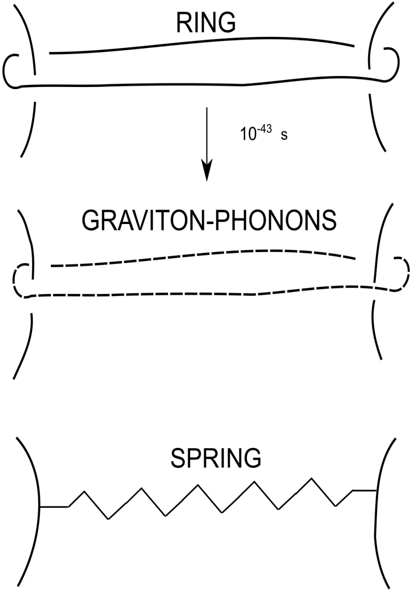
The usual picture about the gravitational interaction was that some quantum (graviton) is exchanging between every particle in the Universe. We suggest a different scheme in RP.



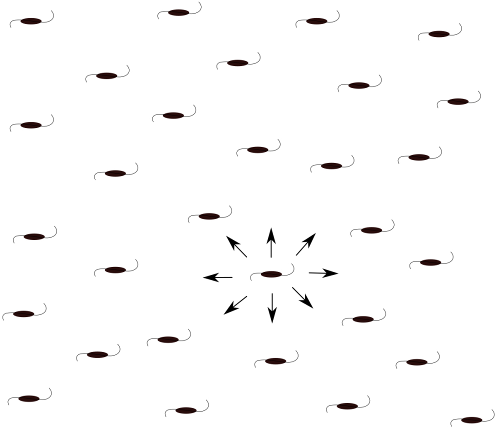
Model of space



We substitute the picture of the gravitons carrying the initial impulse by the creation of a gravitational ring, which tightens the objects in Planck time.

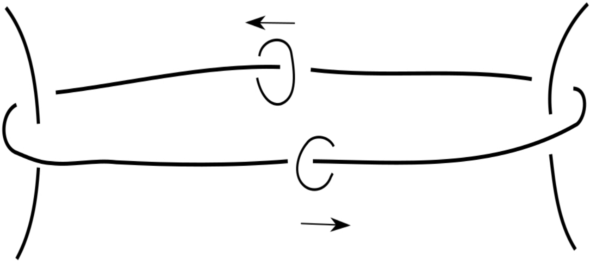

RING
10^{-43} s
GRAVITON-PHONONS
SPRING

Symmetries



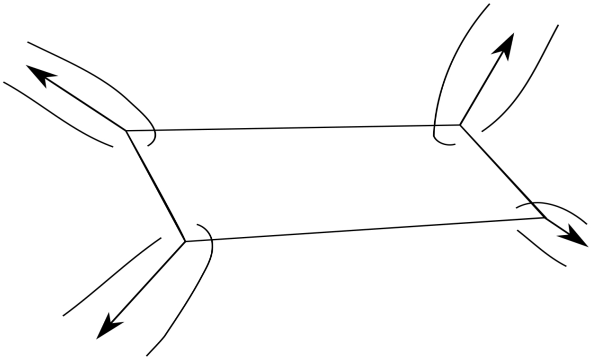
Unchanged particle sector

The elementary particles of the standard model could move only around gravitational rings.

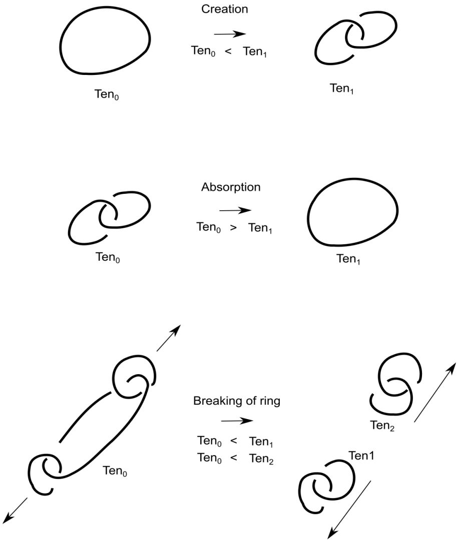


Variational principle

The ring has the shortest length from all possible configurations in space, which means a variational principle must be applied in the derivation of the field equations of RP.

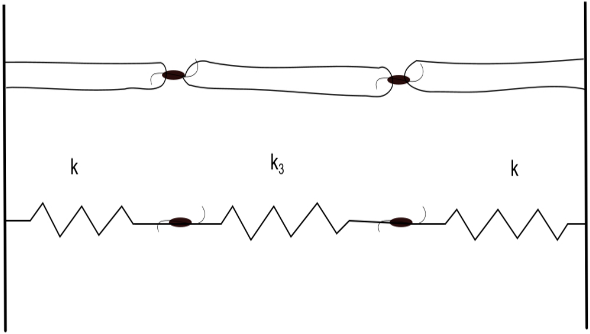


Processes with rings



Graviton as a phonon

The creation of rings in Planck time effectively gives rise to springs between the galaxies. We quantize their longitudinal vibrations and obtain the graviton-phonons, which mediate the Newtonian force.



$$H = \sum_{i=1}^{2} \frac{1}{2m} P_i^2 + \sum_{i,j=1}^{2} V_{ij} Q_i Q_j, \tag{7}$$

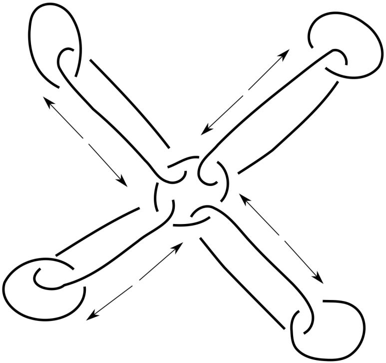
where

$$V = \begin{pmatrix} \frac{1}{2}k + \frac{1}{2}k_3 & -\frac{1}{2}k_3 \\ -\frac{1}{2}k_3 & \frac{1}{2}k + \frac{1}{2}k_3 \end{pmatrix},$$

$k, k_3 > 0$.

Accelerated expansion of the Universe

The classical description is that the gravitational rings are effectively made from some material, which has an inner dependence on the deformation due to the stress. The "gravitational"material breaks at *Mpc* distances, which causes accelerated expansion in the Universe.



Modification of gravity

$$\mathscr{R}_{\mu\nu} - \frac{1}{2}\mathscr{R}\mathscr{G}_{\mu\nu} + \Lambda_r \mathscr{G}_{\mu\nu} = \frac{8\pi G \mathscr{T}_{\mu\nu}}{c_g^4}, \tag{8}$$

where $\mathscr{G}_{\mu\nu}$ is the metric and also all the other quantities have an analogous meaning as in GR. The cosmological constant Λ_r could be computed from QFT. We neglect the RHS with respect to the LHS, so

$$\mathscr{R}_{\mu\nu} - \frac{1}{2}\mathscr{R}\mathscr{G}_{\mu\nu} + \Lambda_r \mathscr{G}_{\mu\nu} = 0. \tag{9}$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{10}$$

A new cosmological constant term Λ appeared approximately 8 billion years after Big Bang due to the QG phenomenon (actually $\Lambda = \Lambda_b$ in our previous notation):

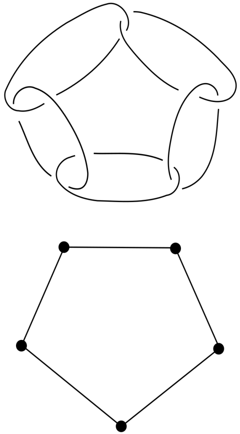
$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{11}$$

Mathematical problem

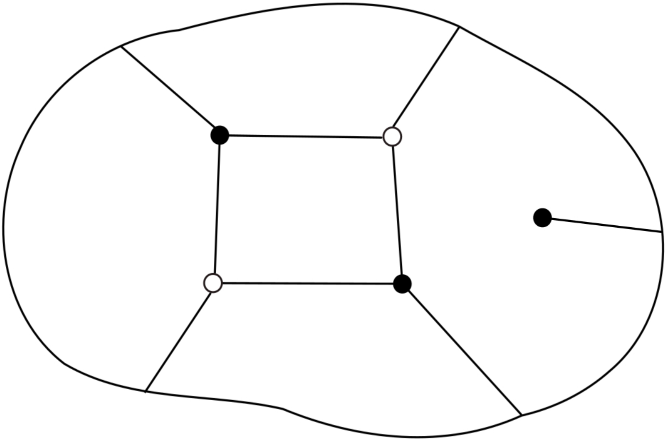
We take a finite collection of P rings (simple closed curves) S^1 in $\mathbb{R}^3$, which do not touch; Give a complete characterization of all non-homeomorphic structures, that can be constructed from this set of rings. Every two rings are linked maximally once, they could not be knotted or twisted (in the case, when we have a differentiable structure). We do not consider any Brunnian type of link (Whitehead link, Borromean rings, etc.) and we study only a connected component.

Identification with graphs

Every ring of the crystal could be identified with a vertex, and we put an edge on the graph if the corresponding rings would be Hopf-linked.

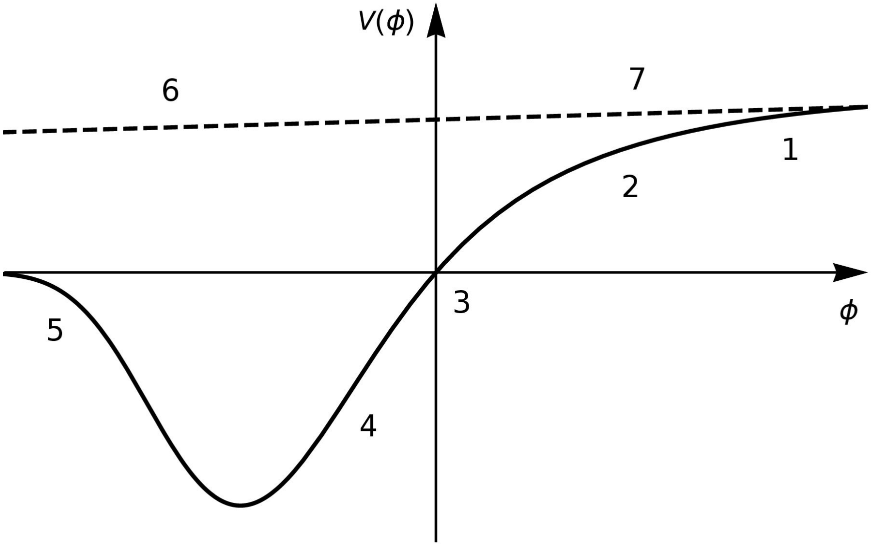


We defined RP on the crystal made of rings, which can be identified with the plabic graphs.



Application of the paradigm

1. singularity theorems
2. cyclic universes
3. black hole information paradox
4. dimensional reduction
5. curvature of the universe
6. EPR-paradox
7. determinism of physical theories



$V(\phi)$
6
7
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2
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ϕ
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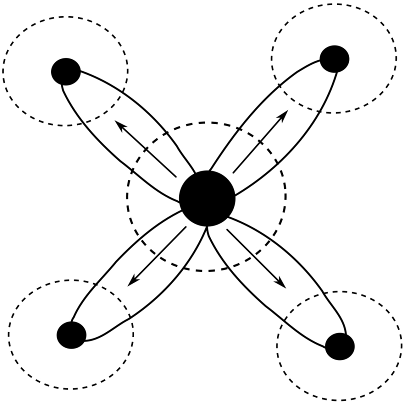
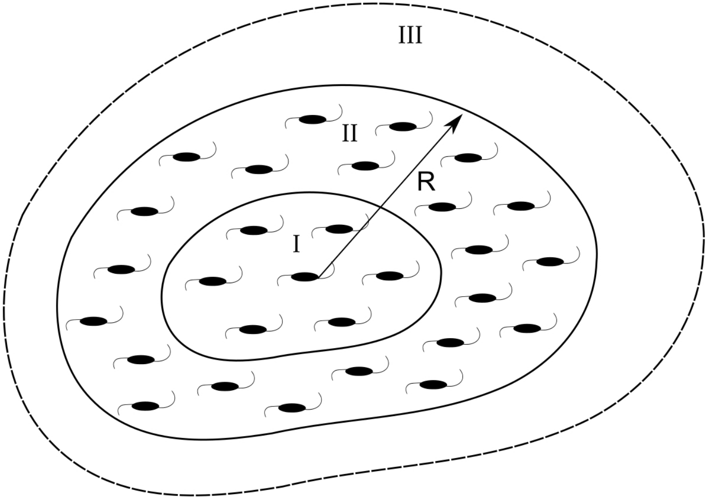
4

RP is a highly non-local theory, and the rings are sticking out of the horizon for any black hole. It means that the information could travel at superluminal speed from the interior of the black hole. This gives us a full solution of the information paradox on the non-perturbative QG level.




III
II
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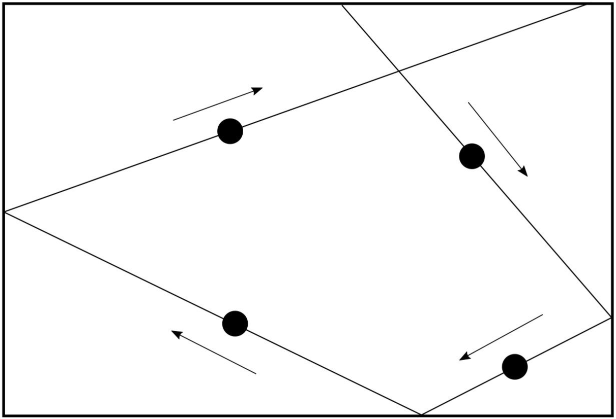
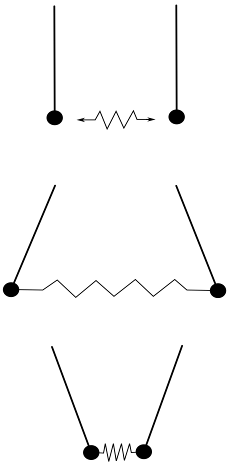
I

RP is built on the postulate that the elementary particles move only at the pre-prepared lanes. This could have serious consequences for the determinism of physical theories.



The gravitational rings are mediating gravity by a velocity $c_g > c$. It is equal to the maximal allowed velocity, how information can actually be transmitted according to QG.



Generalization of transformations

$$t' = \frac{t - \frac{x}{v}\frac{v^2}{c^2}\epsilon - \frac{x}{v}\frac{v^2}{c_g^2}}{\sqrt{1 - \frac{v^2}{c^2}\epsilon - \frac{v^2}{c_g^2}}},$$

$$x' = \frac{x - tv}{\sqrt{1 - \frac{v^2}{c^2}\epsilon - \frac{v^2}{c_g^2}}},$$

where $\epsilon = \epsilon(v)$ denotes some step function defined by the prescription

$$\epsilon(v) = \begin{cases} 1 & \text{for } v \leq c, \\ 0 & \text{for } v > c. \end{cases}$$

Work for future

- scalar field in classical cosmology
- Lorentz violating theories
- conformal field theory in 2 dimension

Literature

1. - Total positivity, Grassmanians and networks, Alexander Postnikov, math/0609764
   - The m = 1 amplituhedron and cyclic hyperplane arrangements, Steven N. Karp and Lauren K. Williams
2. The Amplituhedron, Nima Arkani-Hamed, Jaroslav Trnka, 1312.2007
3. Graviton as a phonon and dark energy problem, JN, sent to Classical and Quantum Gravity

Book Two faces of Johny Newman

- 3 popularization articles: Adventrure of modern cosmology, Adventure of quantum gravity, Are we alone?
- plugged to the story of american theoretical physicist Johny Newman and american pianist Kate Goldberg

Thank You for paying attention! (Some pictures were taken from the web and some were created by myself.)

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