# Ideal clock as the fundamental relativistic rotator

Tobiasz Pietrzak

Complex Systems Theory Department (NZ44)
Institute of Nuclear Physics, Polish Academy of Sciences

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# Agenda

# Introduction

- I will use natural units in which the speed of light is unity.
- Einstein notation - Any index in a single term that appears twice once as a subscript and once as a superscript, implies summation of that term over all the values of the index.

**Example**

If $\alpha$ can range over the set $\{0, 1, 2, 3\}$ then we have

$$x_\alpha x^\alpha = x_0 x^0 + x_1 x^1 + x_2 x^2 + x_3 x^3.$$

# Introduction

## Minkowski spacetime

We shall call Minkowski spacetime the 4-tuple $(\mathcal{E}, \mathbf{g}, \mathcal{I}^+, \epsilon)$ where:

- $\mathcal{E}$ is an affine space of dimension four over $\mathbb{R}$, the underlying vector space being denoted by $E$ and $\mathcal{E}$ is called spacetime and its elements are called events,
- $\mathbf{g}$ is a bilinear form on $E$ that is symmetric, nondegenerate and has the signature $(+, -, -, -)$ and $\mathbf{g}$ is called the pseudometric tensor,
- $\mathcal{I}^+$ is one of the two sheets of $\mathbf{g}$'s null cone, called the future null cone,
- $\epsilon$ is a four-linear form on $E$ that is antisymmetric and results in $\pm 1$ when applied to any basis that is orthonormal with respect to $\mathbf{g}$ and $\epsilon$ is called the Levi–Civita symbol associated with the pseudometric $\mathbf{g}$.

# Introduction

The signature $(+, -, -, -)$ prevents $\mathbf{g}$ to be positive definite. The scalar product of a vector $\mathbf{v}$ with itself can take any sign and be null without $\mathbf{v}$ being zero. Accordingly, the vectors are classified in three types (apart from the zero vector). A vector $\mathbf{v} \in E$ is said:

- timelike if and only if $\mathbf{g}(\mathbf{v}, \mathbf{v}) > 0$,
- spacelike if and only if $\mathbf{g}(\mathbf{v}, \mathbf{v}) < 0$,
- null or lightlike if and only if $\mathbf{v} \neq 0$ and $\mathbf{g}(\mathbf{v}, \mathbf{v}) = 0$.

**Massive particle**

Any massive particle is represented by a piecewise twice continuously differentiable curve $\mathcal{L}$ (worldline) of Minkowski spacetime such that any vector tangent to $\mathcal{L}$ is timelike.

# Intoduction

## Proper time

If $A$ and $B$ are two events of some worldline $\mathcal{L}$ of a given massive particle and if $\phi$ is a parametrization of $\mathcal{L}$ such that $A = \phi(\lambda_1)$ and $B = \phi(\lambda_2)$, we set

$$\tau(A, B) = \int_{\lambda_1}^{\lambda_2} \sqrt{\mathbf{g}(\mathbf{v}(\lambda), \mathbf{v}(\lambda))}d\lambda$$

where $\tau(A, B)$ is proper time between events $A$ and $B$ along a worldline $\mathcal{L}$ and $\mathbf{v}$ is the tangent vector field associated with the parametrization $\phi$. Proper time does not depend on the choice of the parametrization. On the other hand, it depends on the worldline connecting $A$ to $B$.

# Introduction

## Clock hypothesis

The clock hypothesis asserts that an ideal clock measures its proper time.This means that the number of consecutive cycles registered by the clock increases steadily with the affine parameter of the worldline of the clock's center of mass.

## Ideal clock

Purely mathematical construct, with its own intrinsic non-quantum clocking mechanism, experiencing no fatigue and friction.

# Definitions

## Relativistic rotator

A relativistic rotator is a dynamical system described by position $x$ and a single null direction $k$ and, additionally, by two parameters, $m$ (mass) and $l$ (length).

## Remark

The most general relativistically invariant action for a system described by position $x$ and null direction $k$ has the following form:

$$S = -\int m\sqrt{\dot{\mathbf{x}}\dot{\mathbf{x}}}\, f\left(-l^2 \frac{\dot{\mathbf{k}}\dot{\mathbf{k}}}{(\mathbf{k}\dot{\mathbf{x}})^2}\right) d\lambda, \tag{1}$$

where $\lambda$ is an arbitrary parameter, a dot denotes differentiation with respect to $\lambda$ and $f$ is an arbitrary, non-constant, positive function.

# Definitions

## Fundamental dynamical system

A dynamical system described by a relativistically invariant action, is said to be fundamental, if its both Casimir invariants of the Poincaré group are parameters with fixed numerical values rather than arbitrary constants of motion.

## Poincaré group

The Poincaré group is the group of Minkowski spacetime isometries. It is a ten-dimensional Lie group.

## Casimir operator

A Casimir invariant is a distinguished element of the center of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$.

# Fundamental relativistic rotators

For action (1) we calculate the Noether constants of motion:

$$P_\mu = p_\mu, \tag{2}$$

$$M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + k_\mu \pi_\nu - k_\nu \pi_\mu. \tag{3}$$

The momenta canonically conjugated with $x$ and $k$ are, respectively:

$$p_\mu = -\frac{\partial L}{\partial \dot{x}^\mu} = m\left[\frac{\dot{x}_\mu}{\sqrt{\dot{x}^\alpha \dot{x}_\alpha}} f(\xi) - 2\frac{\sqrt{\dot{x}^\alpha \dot{x}_\alpha}\, k_\mu}{k^\beta \dot{x}_\beta} \xi f'(\xi)\right], \tag{4}$$

$$\pi_\mu = -\frac{\partial L}{\partial \dot{k}^\mu} = 2m \frac{\sqrt{\dot{x}^\alpha \dot{x}_\alpha}}{\dot{k}^\alpha \dot{k}_\alpha} \xi f'(\xi) \dot{k}_\mu, \tag{5}$$

where

$$\xi = -l^2 \frac{\dot{k}_\alpha \dot{k}^\alpha}{(k_\beta \dot{x}^\beta)^2}.$$

# Fundamental relativistic rotators

Casimir invariants of the Poincaré group are:

$$C_1 = P_\mu P^\mu = m^2 \Big[ f^2(\xi) - 4\xi f(\xi) f'(\xi) \Big], \tag{6}$$

$$C_2 = W_\mu W^\mu = -4m^4 l^2 \xi f^2(\xi) [f'(\xi)]^2, \tag{7}$$

where $W^\mu = -\frac{1}{2}\epsilon^{\mu\alpha\beta\gamma} M_{\alpha\beta} P_\gamma$ is the Pauli-Lubański-Mathisson (space-like) vector.

# Fundamental relativistic rotators

By requiring that $C_1 = m^2$ and $C_2 = -\frac{1}{4}m^4 l^2$ we get two differential equations of the form:

$$f^2(\xi) - 4\xi f(\xi) f'(\xi) - 1 = 0, \tag{8}$$

$$16\xi f^2(\xi)[f'(\xi)]^2 - 1 = 0, \tag{9}$$

It is worth to notice that equations (8) and (9) have a common solution given by the formula

$$f(\xi) = \pm\sqrt{1 \pm \sqrt{\xi}}. \tag{10}$$

# Fundamental relativistic rotators

## Hamilton's action

There are only two relativistic rotators which are fundamental. Their Hamilton's action has the form

$$S = -\int m\sqrt{\dot{\mathbf{x}}\dot{\mathbf{x}}}\sqrt{1 \pm \sqrt{-l^2 \frac{\dot{\mathbf{k}}\dot{\mathbf{k}}}{(\mathbf{k}\dot{\mathbf{x}})^2}}}\,d\lambda. \tag{11}$$

For those rotators:

$$C_1 = m^2, \tag{12}$$

$$C_2 = -\frac{1}{4}m^4 l^2. \tag{13}$$

# Lagrangian singularity

The basic equation for the calculus of variations is the Euler-Lagrange equation. This equation in physics can be used to describe both particles and fields. For systems with a finite number of degrees of freedom, this equation has the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^{\mu}} - \frac{\partial L}{\partial q^{\mu}} = 0. \tag{14}$$

The Euler-Lagrange equations can be recast in the general form

$$A_{\mu\nu}\dot{q}^{\nu} + H_{\mu\nu}\ddot{q}^{\nu} + \frac{\partial^2 L}{\partial t \partial \dot{q}^{\mu}} - \frac{\partial L}{\partial q^{\mu}} = 0, \tag{15}$$

where $A_{\mu\nu} = \frac{\partial^2 L}{\partial \dot{q}^{\mu} \partial q^{\nu}}$ and $H_{\mu\nu} = \frac{\partial^2 L}{\partial \dot{q}^{\mu} \partial \dot{q}^{\nu}}$.

# Lagrangian singularity

**Definition**

We say that the Lagrangian $L = L(q, \dot{q})$ is singular if:

$$\det\left[\frac{\partial^2 L}{\partial \dot{q}^\mu \partial \dot{q}^\nu}\right] = 0. \tag{16}$$

# Lagrangian singularity

The singularity of the Lagrangian is geometric in nature and independent of the choice of the coordinate system.

$$
\begin{aligned}
\det H' &= \det\left[\frac{\partial^2 L}{\partial \dot{q}'^i \partial \dot{q}'^j}\right] = \det\left[\frac{\partial}{\partial \dot{q}'^i}\left(\frac{\partial \dot{q}^l}{\partial \dot{q}'^j}\frac{\partial L}{\partial \dot{q}^l}\right)\right] = \\
&= \det\left[\frac{\partial}{\partial \dot{q}'^i}\left(\frac{\partial \dot{q}^l}{\partial \dot{q}'^j}\right)\frac{\partial L}{\partial \dot{q}^l} + \frac{\partial \dot{q}^k}{\partial \dot{q}'^i}\frac{\partial \dot{q}^l}{\partial \dot{q}'^j}\frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^l}\right] = \\
&= \det\left[\frac{\partial^2 \dot{q}^l}{\partial \dot{q}'^i \partial \dot{q}'^j}\frac{\partial L}{\partial \dot{q}^l} + \frac{\partial q^k}{\partial q'^i}\frac{\partial q^l}{\partial q'^j}\frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^l}\right] = \\
&= \det\left[\frac{\partial q^k}{\partial q'^i}\frac{\partial q^l}{\partial q'^j}\frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^l}\right] = \\
&= \det\left[\frac{\partial q^k}{\partial q'^i}\right]\det\left[\frac{\partial q^l}{\partial q'^j}\right]\det\left[\frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^l}\right] = \\
&= J^2 \det H.
\end{aligned}
\tag{17}
$$

# Lagrangian singularity

Due to the geometric nature of the concept of regularity, further calculations will be made in the selected map without loss of generality. We use Cartesian map for the space-time position and spherical angels for the null direction, and the arbitrary parameter is chosen as $\lambda = t$ thus:

$$\mathsf{x}(t) \to [t, x, y, z],$$

$$\mathbf{k}(t) \to [1, \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta].$$

# Lagrangian singularity

After introducing the matrix:

$$\dot{X} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}, \qquad K = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}, \qquad \dot{K} = \begin{pmatrix} l\dot{\theta}^2 \\ l\dot{\varphi}^2\sin^2\theta \end{pmatrix}, \tag{18}$$

the Lagrangian of any relativistic rotator is expressed by the formula

$$L = -m\sqrt{1 - \dot{X}^T\dot{X}}f(\xi), \tag{19}$$

where:

$$\xi = \frac{\dot{K}^T\dot{K}}{(1 - K^T\dot{X})^2}. \tag{20}$$

# Lagrangian singularity

$$\frac{\partial^2 L}{\partial \dot{K}^T \partial \dot{K}} = -2m\xi \frac{\sqrt{1-\dot{X}^T\dot{X}}}{\dot{K}^T\dot{K}}\left[f'(\xi)\mathbb{I}_2 + 2\xi f''(\xi)\frac{\dot{K}\dot{K}^T}{\dot{K}^T\dot{K}}\right], \tag{21}$$

$$\frac{\partial^2 L}{\partial \dot{X}^T \partial \dot{X}} = \frac{mf(\xi)}{\sqrt{1-\dot{X}^T\dot{X}}}\left[\mathbb{I}_3 + \frac{\dot{X}\dot{X}^T}{1-\dot{X}^T\dot{X}} + 2\xi\frac{f'(\xi)}{f(\xi)}\left(\frac{K\dot{X}^T + \dot{X}K^T}{1-K^T\dot{X}} + \frac{-KK^T(1-\dot{X}^T\dot{X})}{(1-K^T\dot{X})}\left[2\xi\frac{f''(\xi)}{f'(\xi)}+3\right]\right)\right], \tag{22}$$

$$\frac{\partial^2 L}{\partial \dot{K}^T \partial \dot{X}} = -m\frac{2\xi f'(\xi)\sqrt{1-\dot{X}^T\dot{X}}}{\dot{K}^T\dot{K}}\left[\frac{-\dot{K}\dot{X}^T}{1-\dot{X}^T\dot{X}} + 2\left(\xi\frac{f''(\xi)}{f'(\xi)}+1\right)\frac{\dot{K}K^T}{1-K^T\dot{X}}\right]. \tag{23}$$

# Lagrangian singularity

If we denote:

$$L_1 = \frac{\partial^2 L}{\partial \dot{X}^T \dot{X}} \in M_{3\times 3}(\mathbb{R}), \tag{24}$$

$$L_2 = L_3^T \in M_{3\times 2}(\mathbb{R}), \tag{25}$$

$$L_3 = \frac{\partial^2 L}{\partial \dot{K}^T \dot{X}} \in M_{2\times 3}(\mathbb{R}), \tag{26}$$

$$L_4 = \frac{\partial^2 L}{\partial \dot{K}^T \dot{K}} \in M_{2\times 2}(\mathbb{R}), \tag{27}$$

then the Hesse matrix can be written in block form as follows

$$H = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} \in M_{5\times 5}(\mathbb{R}). \tag{28}$$

# Lagrangian singularity

For the matrix $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, such that the matrices $A, B, C, D$ are of $n \times n, n \times m, m \times n, m \times m$ dimensions, respectively and a matrix $D$ is non-singular, the equality is true

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbb{I}_n & \mathbb{O}_{n \times m} \\ -D^{-1}C & \mathbb{I}_m \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & B \\ \mathbb{O}_{m \times n} & D \end{pmatrix}, \tag{29}$$

where $\mathbb{O}_{p \times q}$ denotes a zero matrix of dimension $p \times q$. Calculating determinant from the left and right side of equation (29) we obtain

$$\det S = \det(A - BD^{-1}C) \det D \tag{30}$$

# Lagrangian singularity

$$\det L_4 = 4m^2\xi^2[f'(\xi)]^2\frac{1-\dot{X}^T\dot{X}}{(\dot{K}^T\dot{K})^2}\left[1+\frac{2\xi f''(\xi)}{f'(\xi)}\right], \tag{31}$$

$$\det(L_1 - L_2L_4^{-1}L_3) = \frac{m^3f^3(\xi)}{(1-\dot{X}^T\dot{X})^{\frac{5}{2}}}\left[1+\frac{2\xi[f'(\xi)]^2}{f(\xi)[f'(\xi)+2\xi f''(\xi)]}\right]. \tag{32}$$

In calculations, the Weinstein-Aronszajn identity $\det(\mathbb{I}_m + AB) = \det(\mathbb{I}_n + BA)$, which holds for matrices $A$ and $B$ of dimension $m\times n$ i $n \times m$ respectively is used. The $L_4^{-1}$ matrix was determined using the Cayley-Hamilton method

$$L_4^{-1} = \frac{1}{\det L_4}[\mathbb{I}_2\mathrm{tr}L_4 - L_4]. \tag{33}$$

# Lagrangian singularity

For rotators described by the Lagrangian associated with the action (1) the Hessian is expressed by the formula

$$\det H = \frac{4m^5 f^3(\xi)[f'(\xi)]^2}{(\sqrt{1-\dot{X}^T\dot{X}})^3(1-K^T\dot{X})^4}\left(1+2\xi\frac{f'(\xi)}{f(\xi)}+2\xi\frac{f''(\xi)}{f'(\xi)}\right), \tag{34}$$

where $\dot{X}$ and $K$ are the matrices of the spatial components of the velocity vector and the null vector in a given map.

# Lagrangian singularity

Zeroing out expression (34) leads to a differential equation of the form

$$1 + 2\xi\left(\frac{f'(\xi)}{f(\xi)} + \frac{f''(\xi)}{f'(\xi)}\right) = 0. \tag{35}$$

The solution to the above equation is the function:

$$f(\xi) = a\sqrt{1 \pm b\sqrt{\xi}}, \tag{36}$$

where $a$ and $b$ are integration constants.

# Lagrangian singularity

## Conclusion

When the Hessian matrix is singular, it is not invertible and therefore, the accelerations cannot be uniquely determined from the actual positions and velocities, at any instant of time. In other words, given positions and velocities, an infinite number of accelerations is available, from which the system can choose at each stage of its movement.

Thank you for your attention!