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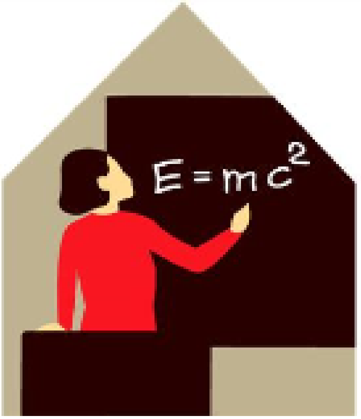

On the classical A. Buhl problem, its G. Pfeiffer solution and its connection with completely integrable heavenly type dynamical systems

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ABSTRACT

We will review a novel Lie-algebraic approach to studying integrable heavenly type multi-dimensional dynamical systems and its relationships to old and recent investigations of the classical M. A. Buhl problem of describing compatible linear vector field equations by a prominent ukrainian mathematician G. Pfeiffer and in modern times by members of the japanese Sato school. Eespecially we analyze the related Lie-algebraic structures and integrability properties of a very interesting class of nonlinear dynamical systems called the dispersionless heavenly type equations, which were initiated by Plebański and later analyzed in a series of articles. The AKS-algebraic and related R-structure schemes are used to study the orbits of the corresponding co-adjoint actions, which are intimately related to the classical Lie–Poisson structures on them.

It is demonstrated that their compatibility condition coincides with the corresponding heavenly type equations under consideration. It is shown that all these equations originate in this way and can be represented as a Lax compatibility condition for specially constructed loop vector fields on the torus. The infinite hierarchy of conservations laws related to the heavenly equations is described, and its analytical structure connected with the Casimir invariants, is mentioned. In addition, typical examples of such equations, demonstrating in detail their integrability via the scheme devised herein, are presented. The relationship of a very interesting Lagrange–d'Alembert type mechanical interpretation of the devised integrability scheme with the Lax–Sato equations is also discussed.

In 1922 the French mathematician M.A. Buhl posed [24] (published by E. Goursat in his "*Lecons sur le problème de Pfaff* (French Edition), 1922) and later analyzed in his works [11, 12, 13] the problem of classifying all infinitesimal symmetries of a given linear vector field equation

$$A\psi = 0, \tag{1.1}$$

where function $\psi \in C^2(\mathbb{R}^n; \mathbb{R})$, and

$$A := \sum_{j=\overline{1,n}} a_j(x) \frac{\partial}{\partial x_j} \tag{1.2}$$

is a vector field operator on $\mathbb{R}^n$ with coefficients $a_j \in C^1(\mathbb{R}^n;\mathbb{R}), j=\overline{1,n}$. It is easy to show that the problem under regard is reduced [47] to describing all possible vector fields

$$A^{(k)} := \sum_{j=\overline{1,n}} a_j^{(k)}(x)\frac{\partial}{\partial x_j} \tag{1.3}$$

with coefficients $a_j^{(k)} \in C^1(\mathbb{R}^n;\mathbb{R}), j,k=\overline{1,n}$, satisfying the Lax type commutator condition

$$[A, A^{(k)}] = 0 \tag{1.4}$$

for all $x \in \mathbb{R}^n$ and $k=\overline{1,n}$. This M.A. Buhl problem above for all $x \in \mathbb{R}^n$ and $k = \overline{1, n}$. This M.A. Buhl problem above was completely solved in 1931 by the Ukrainian mathematician G. Pfeiffer in the works [44, 45, 46, 47, 48, 49], where he has constructed explicitly the searched set of independent vector fields (1.3), having made use effectively of the full set of invariants for the vector field (1.2) and the related solution set structure of the Jacobi-Mayer system of equations, naturally following from (1.4). Some results, yet not complete, were also obtained by C. Popovici in [51].

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A prominent Ukrainian mathematician
Professor George V. Pfeiffer

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Director of the Institute of Mathematics at the Ukrainian Academy of Sciences in (1941-1944)

Ordinary Memeber of the Ukrainian Academy of Sciences

Some years ago the M.A. Buhl type equivalent problem was independently reanalyzed once more by Japanese mathematicians K. Takasaki and T. Takebe [63, 64] and later by L.V. Bogdanov, V.S. Dryuma and S.V. Manakov [8] for a very special case when the vector field operator (1.2) depends analytically on a "spectral" parameter $\lambda \in \mathbb{C}$:

$$\tilde{A} := \frac{\partial}{\partial t} + \sum_{j=\overline{1,n}} a_j(t,x;\lambda)\frac{\partial}{\partial x_j} + a_0(t,x;\lambda)\frac{\partial}{\partial \lambda}. \tag{1.5}$$

Based on the before developed Sato theory [55, 56], the authors mentioned above have shown for some special kinds of vector fields (1.5) that there exists an infinite hierarchy of the symmetry vector fields

$$\tilde{A}^{(k)} := \frac{\partial}{\partial \tau_k} + \sum_{j=\overline{1,n}} a_j^{(k)}(\tau, x; \lambda) \frac{\partial}{\partial x_j} + a_0^{(k)}(\tau, x; \lambda) \frac{\partial}{\partial \lambda}, \tag{1.6}$$

where $\tau = (t; \tau_1, \tau_2, \ldots) \in \mathbb{R}^{\mathbb{Z}_+}, k \in \mathbb{Z}_+$, satisfying the Lax-Sato type compatible commutator conditions

$$[\tilde{A}, \tilde{A}^{(k)}] = 0 = [\tilde{A}^{(j)}, \tilde{A}^{(k)}] \tag{1.7}$$

for all $k, j \in \mathbb{Z}_+$. Moreover, in the cases under regard, the compatibility conditions (1.7) proved to be equivalent to some very important for applications heavenly type dispersionless equations in partial derivatives.

We present interesting examples of the Lie-algebraic description of typical integrable heavenly equations amongst which the Mikhalev-Pavlov equation [34], the first and second reduced Shabat type [2] and Hirota [18] heavenly equations, the Liouville type [9] equations and some other.

We also generalized the Lie-algebraic scheme of [24, 52] subject to the loop Lie algebra $\widetilde{diff}(\mathbb{S}^{1|N})$ of superconformal vector fields on $\mathbb{S}^{1|N}$, being a Lie algebra of the Lie group of superconformal diffeomorphisms of the $1|N$-dimensional supertorus $\mathbb{S}^{1|N} \simeq \mathbb{S}^1 \times \Lambda_1^N$, where $\Lambda := \Lambda_0 \oplus \Lambda_1$ is an infinite-dimensional Grassmann algebra over $\mathbb{C}$, $\Lambda_0 \supset \mathbb{C}$. It is applied to constructing the Lax-Sato integrable superanalogs of the Mikhalev-Pavlov type heavenly super-equation for every $N \in \mathbb{N} \setminus \{4;5\}$. As a result of suitably chosen superconformal mappings in the space of variables $(z;\theta_1,\ldots,\theta_N) \in \mathbb{S}^{1|N}$ the superanalogs of Liouville type equations are obtained by means of using the loop Lie superalgebra $\widetilde{diff}(\mathbb{S}^{1|N})$. Some results are also presented for a special Lie-algebraic integrability scheme based on a metrized loop Lie algebra, generated by a semisimple sum of the holomorphic Lie algebra $\bar{\mathcal{G}} = diff_{hol}(\mathbb{C} \times \mathbb{T}^n)$ and its coadjoint space $\bar{\mathcal{G}}^*$.

2.1. A vector field on the torus and its invariants. Consider a simple vector field $X : \mathbb{R}\times\mathbb{T}^n \to T(\mathbb{R}\times\mathbb{T}^n)$ on the $(n+1)$-dimensional toroidal manifold $\mathbb{R}\times\mathbb{T}^n$ for arbitrary $n\in\mathbb{Z}_+$, which we will write in the slightly special form

$$A = \frac{\partial}{\partial t} + < a(t,x), \frac{\partial}{\partial x} >, \tag{2.1}$$

where $(t,x)\in\mathbb{R}\times\mathbb{T}^n, a(t,x)\in\mathbb{E}^n, \frac{\partial}{\partial x} := (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n})^\intercal$ and $<\cdot,\cdot>$ is the standard scalar product on the Euclidean space $\mathbb{E}^n$. With the vector field (2.1), one can associate the linear equation

$$A\psi = 0 \tag{2.2}$$

for some function $\psi\in C^2(\mathbb{R}\times\mathbb{T}^n;\mathbb{R})$, which we will call an *"invariant"* of the vector field.

Next, we study the existence and number of such functionally-independent invariants to the equation (2.2). For this let us pose the following Cauchy problem for equation (2.2): Find a function $\psi \in C^2(\mathbb{R} \times \mathbb{T}^n; \mathbb{R})$, which at point $t^{(0)} \in \mathbb{R}$ satisfies the condition $\psi(t,x)|_{t=t^{(0)}} = \bar{\psi}(x)$, $x \in \mathbb{R}^n$, for a given function $\bar{\psi} \in C^2(\mathbb{T}^n; \mathbb{R})$. For the equation (2.2) there is naturally related parametric vector field on the torus $\mathbb{T}^n$ in the form of the ordinary vector differential equation

$$dx/dt = a(t,x), \tag{2.3}$$

to which there corresponds the following Cauchy problem: find a function $x : \mathbb{R} \to \mathbb{T}^n$ satisfying

$$x(t)|_{t=t^{(0)}} = z \tag{2.4}$$

for an arbitrary constant vector $z \in \mathbb{T}^n$. Assuming that the vector-function $a \in C^1(\mathbb{R} \times \mathbb{T}^n; \mathbb{R}^n)$, it follows from the classical

Cauchy theorem [14] on the existence and unicity of the solution to (2.3) and (2.4), that we can obtain a unique solution to the vector equation (2.3) as some function $\Phi \in C^1(\mathbb{R}\times\mathbb{T}^n;\mathbb{T}^n)$, $x=\Phi(t,z)$, such that the matrix $\partial\Phi(t,z)/\partial z$ is nondegenerate for all $t\in\mathbb{R}$ sufficiently close to $t^{(0)}\in\mathbb{R}$. Hence, the Implicit Function Theorem [14, 15] implies that there exists a mapping $\Psi:\mathbb{R}\times\mathbb{T}^n\to\mathbb{T}^n$, such that

$$\Psi(t,x)=z \tag{2.5}$$

for every $z\in\mathbb{T}^n$ and all $t\in\mathbb{R}$ sufficiently enough to $t^{(0)}\in\mathbb{R}$. Supposing now that the functional vector $\Psi(t,x)=(\psi^{(1)}(t,x),\psi^{(2)}(t,x),\dots,\psi^{(n)}(t,x))^\intercal$, $(t,x)\in\mathbb{R}\times\mathbb{T}^n$, is constructed, from the arbitrariness of the parameter $z\in\mathbb{T}^n$ one can deduce that all functions $\psi^{(j)}:\mathbb{R}\times\mathbb{T}^n\to\mathbb{R}$, $j=\overline{1,n}$, are functionally independent invariants of the vector field equation (2.2), that is $A\psi^{(j)}=0, j=\overline{1,n}$. Thus, the vector field equation (2.2) that is $A\psi^{(j)} = 0, j = \overline{1,n}$. Thus, the vector field equation (2.2) has exactly $n \in \mathbb{Z}_+$ functionally independent invariants, which make it possible, in particular, to solve the Cauchy problem posed above. Namely, let a mapping $\alpha : \mathbb{R}^n \longrightarrow \mathbb{R}$ be chosen such that $\alpha(\Psi(t,x))|_{t=t^{(0)}} = \bar{\psi}(x)$ for all $x \in \mathbb{R}^n$ and a fixed $t^{(0)} \in \mathbb{R}$. Inasmuch as the superposition of functions $\alpha \circ \Psi : \mathbb{R} \times \mathbb{T}^n \longrightarrow \mathbb{R}$ is, evidently, also an invariant for the equation (2.2), it provides the solution to this Cauchy problem, which we can formulate as the following classical lemma.

Lemma 2.1. *The linear equation (2.2), generated by the vector field (2.3) on the toroidal manifold $\mathbb{R} \times \mathbb{T}^n$, has exactly $n \in \mathbb{Z}_+$ functionally independent invariants.*

Consider now a differential form $\chi^{(n)} \in \Lambda^n(\mathbb{T}^n)$, generated by the vector of independent invariants (2.5), additionally depending parametrically on the vector evolution parameter $t \in \mathbb{R}^n$:

$$\chi^{(n)} := d\psi^{(1)} \wedge d\psi^{(2)} \wedge \ldots \wedge d\psi^{(n)}, \tag{2.6}$$

where, by definition, for any $\psi \in C^2(\mathbb{R}^n \times \mathbb{T}^n; \mathbb{R})$ the differential

$$d\psi :=< \frac{\partial \psi}{\partial x}, dx > + \sum_{s=\overline{1,n}} \frac{\partial \psi}{\partial t_s} dt_s. \tag{2.7}$$

As follows from the Frobenius theorem [14, 23, 25], the Plucker type form (2.6) is for $t \in \mathbb{R}^n$ nonzero on the torus $\mathbb{T}^n$ owing to the functional independence of the invariants. It is easy to see that the following [48] Jacobi-Mayer type relationship

(2.8)

$$\left|\frac{\partial \Psi}{\partial x}\right|^{-1} d\psi^{(1)} \wedge d\psi^{(2)} \wedge \ldots \wedge d\psi^{(n)} = (dx_1 - \sum_{j=\overline{1,n}} a_j^{(1)}(t,x)dt_j) \wedge$$

$$\wedge (dx_2 - \sum_{j=\overline{1,n}} a_j^{(2)}(t,x)dt_j) \wedge \ldots \wedge (dx_n - \sum_{j=\overline{1,n}} a_j^{(m)}(t,x)dt_j),$$

holds on the manifold $\mathbb{T}^n$, where on the right-hand side one has the volume measure on the torus $\mathbb{T}^n$, which is naturally dependent on $t \in \mathbb{R}^n$ owing to the vector field relationships (2.3). Taking into

$t \in \mathbb{R}^n$ owing to the vector field relationships (2.3). Taking into account that for all invariants $\psi^{(j)} \in C^2(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{R}), j = \overline{1,n}$ there hold the differential expressions

$$(2.9) \qquad d\psi^{(j)} = < \frac{\partial \psi^{(j)}}{\partial x}, dx > + \sum_{k=0}^{\infty} \frac{\partial \psi^{(j)}}{\partial t_k} dt_k,$$

their substitution into (2.8) gives rise, owing to the independence of the differentials $dt_s, s = \overline{1,n}$, to the following set of the compatible vector field relationships

$$(2.10) \qquad \frac{\partial \Psi}{\partial t_s} + \sum_{j,k=\overline{1,n}} \left[\left(\frac{\partial \Psi}{\partial x} \right)^{-1}_{jk} \frac{\partial \psi^{(k)}}{\partial t_s} \right] \frac{\partial \Psi}{\partial x_j} = 0$$

for any $s = \overline{1,n}$. The latter property, as it was demonstrated by M.G. Pfeiffer in [48], makes it possible to solve effectively the M.A. Buhl problem and has interesting applications [8, 25] in the theory of completely integrable dynamical systems of heavenly type, which are considered in the next section.

2.2. Vector field hierarchies on the torus with "*spectral*" parameter and the Lax–Sato integrable heavenly dynamical systems. Consider some naturally ordered infinite set of parametric vector fields (2.1) on the infinite dimensional toroidal manifold $\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n$ in the form

(2.11)

$$A^{(k)} = \frac{\partial}{\partial t_k} + < a^{(k)}(t,x;\lambda), \frac{\partial}{\partial x} > + a_0^{(k)}(t,x;\lambda)\frac{\partial}{\partial \lambda} := \frac{\partial}{\partial t_k} + \mathrm{A}^{(k)},$$

where $t_k \in \mathbb{R}, k \in \mathbb{Z}_+, (t,x;\lambda) \in (\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n) \times \mathbb{C}$ are the evolution parameters, and the dependence of smooth vectors $(a_0^{(k)}, a^{(k)})^\intercal \in \mathbb{E} \times \mathbb{E}^n$, $k \in \mathbb{Z}_+$, on the "*spectral*" parameter $\lambda \in \mathbb{C}$ is assumed to be holomorphic. Suppose now that the infinite hierarchy of linear equations

$$A^{(k)}\psi = 0 \qquad (2.12)$$

for $k \in \mathbb{Z}_+$ has exactly $n+1 \in \mathbb{Z}_+$ common functionally independent invariants $\psi^{(j)}(\lambda) \in C^2(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{C}), j = \overline{0,n}$ on the torus $\mathbb{T}^n$, suitably depending on the parameter $\lambda \in \mathbb{C}$. Then, owing

$\mathbb{T}^n$, suitably depending on the parameter $\lambda \in \mathbb{C}$. Then, owing to the existence theory [14, 15] for ordinary differential equations depending on the *"spectral"* parameter $\lambda \in \mathbb{C}$, these invariants may be assumed to be such that allow analytical continuation in the parameter $\lambda \in \mathbb{C}$ both inside $\mathbb{D}^1_+ \subset \mathbb{C}$ of some disc $\mathbb{D}^1 \subset \mathbb{C}$ and subject to the parameter $\lambda^{-1} \in \mathbb{C}, |\lambda| \to \infty$, outside $\mathbb{D}^1_- \subset \mathbb{C}$ of this circle $\mathbb{D}^1 \subset \mathbb{C}$. This means that as $|\lambda| \to \infty$ we have the following expansions:

$$\psi^{(0)}(\lambda) \sim \lambda + \sum_{k=0}^{\infty} \psi_k^{(0)}(t,x)\lambda^{-k},$$

$$\psi^{(1)}(\lambda) \sim \sum_{k=0}^{\infty} \tau_k^{(1)}(t,x)\psi_0(\lambda)^k + \sum_{k=1}^{\infty} \psi_k^{(1)}(t,x)\psi_0(\lambda)^{-k}, \tag{2.13}$$

$$\psi^{(2)}(\lambda) \sim \sum_{k=0}^{\infty} \tau_k^{(2)}(t,x)\psi_0(\lambda)^k + \sum_{k=1}^{\infty} \psi_k^{(2)}(t,x)\psi_0(\lambda)^{-k},$$

$$\dots$$

$$\psi^{(n)}(\lambda) \sim \sum_{k=0}^{\infty} \tau_k^{(n)}(t,x)\psi_0(\lambda)^k + \sum_{k=1}^{\infty} \psi_k^{(n)}(t,x)\psi_0(\lambda)^{-k},$$

where we took into account that $\psi^{(0)}(\lambda) \in C^2(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{C})$, $\lambda \in \mathbb{C}$, is the basic invariant solution to the equations (2.12), the functions $\tau_l^{(s)} \in C^2(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{R})$ for all $s = \overline{1,n}, l \in \mathbb{Z}_+$, are assumed to be independent and $\psi_k^{(j)} \in C^2(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{R})$ for all

$k \in \mathbb{N}$, $j = \overline{0, n}$, are arbitrary. Write down now the condition (2.8) on the manifold $\mathbb{C} \times \mathbb{T}^n$ in the form

(2.14)

$$|\frac{\partial \Psi}{\partial \mathrm{x}}|^{-1} d\psi^{(0)} \wedge d\psi^{(1)} \wedge d\psi^{(2)} \wedge ... \wedge d\psi^{(n)} =$$

$$= (d\lambda - \sum_{j=\overline{1,n}} a_j^{(0)}(t, \mathrm{x}) dt_j) \wedge (dx_1 - \sum_{j=\overline{1,n}} a_j^{(1)}(t, \mathrm{x}) dt_j) \wedge \quad ,$$

$$\wedge (dx_2 - \sum_{j=\overline{1,n}} a_j^{(2)}(t, \mathrm{x}) dt_j) \wedge ... \wedge (dx_n - \sum_{j=\overline{1,n}} a_j^{(m)}(t, \mathrm{x}) dt_j)$$

where $\mathrm{x} := (\lambda, x) \in \mathbb{C} \times \mathbb{T}^n$, $|\frac{\partial \Psi}{\partial \mathrm{x}}|$ is the Jacobi determinant of the mapping $\Psi := (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, ..., \psi^{(n)})^\intercal \in C^2(\mathbb{C} \times (\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n); \mathbb{C}^{n+1})$ on the manifold $\mathbb{C} \times \mathbb{T}^n$. Inasmuch this mapping sub-

$\mathbb{T}^n); \mathbb{C}^{n+1})$ on the manifold $\mathbb{C} \times \mathbb{T}^n$. Inasmuch this mapping subject to the parameter $\lambda \in \mathbb{C}$ has analytical continuation [15] inside $\mathbb{D}^1_+ \subset \mathbb{C}$ of the disc $\mathbb{D}^1 \subset \mathbb{C}$ and subject to the parameter $\lambda^{-1} \in \mathbb{C}$ as $|\lambda| \to \infty$ outside $\mathbb{D}^1_- \subset \mathbb{C}$ of this disc $\mathbb{D}^1 \subset \mathbb{C}$, one can easily obtain from the vanishing differential expressions

$$d\psi^{(j)} =< \frac{\partial \psi^{(j)}}{\partial \mathrm{x}}, d\mathrm{x} > + \sum_{k=0}^{\infty} \frac{\partial \psi^{(j)}}{\partial \tau_k^{(j)}} d\tau_k^{(j)} = 0 \tag{2.15}$$

for all $j = \overline{0,n}$ and the relationship (2.14) on the extended manifold $\mathbb{C} \times \mathbb{T}^n \times \mathbb{T}^\infty$ of the independent variables $\mathrm{x} \in \mathbb{C} \times \mathbb{T}^n$, evolving analytically with respect to the parameters $\tau_k^{(j)} \in \mathbb{R}$, $j = \overline{1,n}, k \in \mathbb{Z}_+$, the following Lax-Sato criterion:

$$\left(|\frac{\partial \Psi}{\partial \mathrm{x}}|^{-1} d\psi^{(0)} \wedge d\psi^{(1)} \wedge d\psi^{(2)} \wedge ... \wedge d\psi^{(n)} \right)_- = 0, \tag{2.16}$$

where $(...)_-$ means the asymptotic part of an expression in the bracket, depending on the parameter $\lambda^{-1} \in \mathbb{C}$ as $|\lambda| \to \infty$. The bracket, depending on the parameter $\lambda^{-1} \in \mathbb{C}$ as $|\lambda| \to \infty$. The substitution of expressions (2.15) into (2.16) easily yields

(2.17)

$$-\frac{\partial \Psi}{\partial \tau_k^{(j)}} = \left[\left(\frac{\partial \Psi}{\partial \mathrm{x}} \right)^{-1}_{0j} \psi^{(0)}(\lambda)^k \right]_+ \frac{\partial \Psi}{\partial \lambda} + \sum_{s=1}^{n} \left[\left(\frac{\partial \Psi}{\partial \mathrm{x}} \right)^{-1}_{sj} \psi^{(0)}(\lambda)^k \right]_+ \frac{\partial \Psi}{\partial x_s}$$

for all $k \in \mathbb{Z}_+, j = \overline{1, n}$. These relationships (2.17) comprise an infinite hierarchy of Lax–Sato compatible [63, 64] linear equations, where $(...)_+$ denotes the asymptotic part of an expression in the bracket, depending on nonnegative powers of the complex parameter $\lambda \in \mathbb{C}$. As for the independent functional parameters $\tau_k^{(j)} \in C^2(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{R})$ for all $k \in \mathbb{Z}_+, j = \overline{1, n}$, one can state their functional independence by taking into account their *a priori* linear dependence on the independent evolution parameters $t_k \in \mathbb{R}$, $k \in \mathbb{Z}_+$. On the other hand, taking into account the explicit form

$k \in \mathbb{Z}_+$. On the other hand, taking into account the explicit form of the hierarchy of equations (2.17), following [8], it is not hard to show that the corresponding vector fields

(2.18)

$$A_k^{(j)} := \left[\left(\frac{\partial \Psi}{\partial x}\right)^{-1}_{0j} \psi^{(0)}(\lambda)^k\right]_+ \frac{\partial}{\partial \lambda} + \sum_{s=1}^{n} \left[\left(\frac{\partial \Psi}{\partial x}\right)^{-1}_{sj} \psi^{(0)}(\lambda)^k\right]_+ \frac{\partial}{\partial x_s}$$

on the manifold $\mathbb{C} \times \mathbb{T}^n$ satisfy for all $k, m \in \mathbb{Z}_+, j, l = \overline{1, n}$, the Lax-Sato compatibility conditions

$$\frac{\partial A_m^{(l)}}{\partial \tau_k^{(j)}} - \frac{\partial A_k^{(j)}}{\partial \tau_m^{(l)}} = [A_k^{(j)}, A_m^{(l)}], \tag{2.19}$$

which are equivalent to the independence of the all functional parameters $\tau_k^{(j)} \in C^1(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{R})$, $k \in \mathbb{Z}_+, j = \overline{1, n}$. As a corollary of the analysis above, one can show that the infinite parameters $\tau_k^{(j)} \in C^1(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n;\mathbb{R}),\ k \in \mathbb{Z}_+, j = \overline{1,n}$. As a corollary of the analysis above, one can show that the infinite hierarchy of vector fields (2.11) is a linear combination of the basic vector fields (2.18) and also satisfies the Lax type compatibility condition (2.19). Inasmuch the coefficients of vector fields (2.18) are suitably smooth functions on the manifold $\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n$, the compatibility conditions (2.19) yield the corresponding sets of differential-algebraic relationships on their coefficients, which have the common infinite set of invariants, thereby comprising an infinite hierarchy of completely integrable so called *heavenly* nonlinear dynamical systems on the corresponding multidimensional functional manifolds.

2.3. Example: the vector field representation for the Mikhalev–Pavlov heavenly type equation. The Mikhalev–Pavlov equation was first constructed in [34] and has the form

(2.22)
$$u_{xt} + u_{yy} = u_y u_{xx} - u_x u_{xy},$$

where $u \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^1; \mathbb{R})$ and $(t, y; x) \in \mathbb{R}^2 \times \mathbb{T}^1$. Assume now [8] that the following two functions

(2.23)
$$\psi^{(0)} = \lambda, \quad \psi^{(1)} \sim \sum_{k=3}^{\infty} \lambda^k \tau_k - \lambda^2 t + \lambda y + x + \sum_{j=1}^{\infty} \psi_j^{(1)}(t, y, \tau; x)\, \lambda^{-j},$$

where $\psi_1^{(1)}(t, y, \tau; x) = u$, $(t, y, \tau; x) \in \mathbb{R}^2 \times \mathbb{R}^\infty \times \mathbb{T}^1$, are invariants of the set of vector fields (2.12) for an infinite set of constant parameters $\tau_k \in \mathbb{R}$, $k = \overline{3, \infty}$, as the complex parameter $\lambda \to \infty$.

By applying to the invariants (2.23) the criterion (2.16) in the form

$$((\partial\psi^{(1)}/\partial x)^{-1}d\psi^{(1)})_{-} = 0, \tag{2.24}$$

one can easily obtain the following compatible linear vector field equations

$$\frac{\partial\psi}{\partial t} + (\lambda^2 + \lambda u_x - u_y)\frac{\partial\psi}{\partial x} = 0 \tag{2.25}$$

$$\frac{\partial\psi}{\partial y} + (\lambda + u_x)\frac{\partial\psi}{\partial x} = 0,$$

$$\dots$$

$$\frac{\partial\psi}{\partial\tau_k} + P_k(u;\lambda)\frac{\partial\psi}{\partial x} = 0,$$

where $P_k(u;\lambda), k = \overline{3,\infty}$, are independent differential-algebraic polynomials in the variable $u \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^\infty \times \mathbb{T}^1)$ and algebraic polynomials in the spectral parameter $\lambda \in \mathbb{C}$, calculated from the expressions (2.17). Moreover, as one can check, the compatibility expressions (2.17). Moreover, as one can check, the compatibility condition (2.19) for the first two vector field equations of (2.25) yields exactly the Mikhalev–Pavlov equation (2.22).

2.4. Example: The Dunajski metric nonlinear equation. The equations for the Dunajski metric [17] are

$$u_{x_1 t} + u_{y x_2} + u_{x_1 x_1} u_{x_2 x_2} - u_{x_1 x_2} - v = 0, \tag{2.26}$$

$$v_{x_1 t} + v_{x_2 y} + u_{x_1 x_1} v_{x_2 x_2} - 2 u_{x_1 x_2} v_{x_1 x_2} = 0,$$

where $(u, v) \in C^\infty(R^2 \times T^2; R^2)$, $(y, t; x_1, x_2) \in R^2 \times T^2$. One can construct now, by definition, the following asymptotic expansions

where $(u,v) \in C^\infty(R^2 \times T^2; R^2)$, $(y,t;x_1,x_2) \in R^2 \times T^2$. One ca
construct now, by definition, the following asymptotic expansions

(2.27)

$$\psi^{(0)} \sim \lambda + \sum_{j=1}^{\infty} \psi_j^{(0)}(t,y;x)\lambda^{-j},$$

$$\psi^{(1)} \sim \sum_{k=2}^{\infty} (\psi^{(0)})^k \tau_k^{(1)} - \psi^{(0)} y + x_1 + \sum_{j=1}^{\infty} \psi_j^{(1)}(t,y;x)\ (\psi^{(0)})^{-j},$$

$$\psi^{(2)} \sim \sum_{k=2}^{\infty} (\psi^{(0)})^k \tau_k^{(2)} + \psi^{(0)} t + x_2 + \sum_{j=1}^{\infty} \psi_j^{(1)}(t,y;x)\ (\psi^{(0)})^{-j},$$

where $\partial u/\partial x_1 := \psi_1^{(2)}, \partial u/\partial x_2 := \psi_1^{(1)}, v := \psi_1^{(0)}$ and $\tau_k^{(s)} \in R$, $s = \overline{1,2}$, $k = \overline{2,\infty}$, are constant parameters. Then the condition (2.16)

$$\left(\left| \frac{\partial(\psi^{(0)}, \psi^{(1)}, \psi^{(2)})}{\partial(\lambda, x_1, x_2)} \right|^{-1} d\psi^{(0)} \wedge d\psi^{(1)} \wedge d\psi^{(2)} \right)_- = 0 \qquad (2.28)$$

yield a compatible hierarchy of the following Lax-Sato type linear vector field equations:

yield a compatible hierarchy of the following Lax-Sato type linear vector field equations:

(2.29)

$$A^{(t_0)}\psi := \frac{\partial\psi}{\partial t} + \mathrm{A}^{(t_0)}\psi = 0, \quad \mathrm{A}^{(t_0)} := u_{x_2x_2}\frac{\partial}{\partial x_1} - (\lambda + u_{x_1x_2})\frac{\partial}{\partial x_2} + v_{x_2}\frac{\partial}{\partial\lambda} = 0,$$

$$A^{(t_1)}\psi := \frac{\partial\psi}{\partial y} + \mathrm{A}^{(t_1)}\psi = 0, \quad \mathrm{A}^{(t_1)} := (\lambda - u_{x_1x_2})\frac{\partial}{\partial x_1} + u_{x_1x_1}\frac{\partial}{\partial x_2} - v_{x_1}\frac{\partial}{\partial\lambda} = 0,$$

$$A^{(t_k^{(s)})}\psi := \frac{\partial\psi}{\partial\tau_k^s} + < P_k^s(u;\lambda), \frac{\partial\psi}{\partial \mathrm{x}} >= 0,$$

where $P_k^s(u,v;\lambda) \in \mathbb{E}^3, s = \overline{1,2}, k \in \mathbb{N}\backslash\{1\}$, are some independent vector-valued differential-algebraic polynomials [10] in the variables $(u,v) \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^2;\mathbb{R}^2)$ and algebraic polynomials in the spectral parameter $\lambda \in \mathbb{C}$, calculated from the expressions (2.17). In particular, the compatibility condition (2.19) for the first two equations of (2.28) is equivalent to the Dunajski metric nonlinear equations (2.26).

The description of the Lax–Sato equations presented above, especially their alternative differential-geometric interpretation (2.20) and (2.21), makes it possible to realize that the structure group $Diff_{hol}(\mathbb{C}\times\mathbb{T}^n)$ should play an important role in unveiling the hidden Lie-algebraic nature of the integrable heavenly dynamical systems. This is actually the case, and a detailed analysis is presented in the sequel.

3. Heavenly equations: the Lie-algebraic integrability scheme

Let $\tilde{G}_{\pm} := \widetilde{Diff}_{\pm}(\mathbb{T}^n)$, $n \in \mathbb{Z}_+$, be subgroups of the loop diffeomorphisms group $\widetilde{Diff}(\mathbb{T}^n) := \{\mathbb{C} \supset \mathbb{S}^1 \longrightarrow Diff(\mathbb{T}^n)\}$, holomorphically extended in the interior $\mathbb{D}^1_+ \subset \mathbb{C}$ and in the exterior $\mathbb{D}^1_- \subset \mathbb{C}$ regions of the unit disc $\mathbb{D}^1 \subset \mathbb{C}^1$, such that for any $g(\lambda) \in \tilde{G}_{\pm}$, $\lambda \in \mathbb{D}^1_-$, $g(\infty) = 1 \in Diff(\mathbb{T}^n)$. The corresponding Lie subalgebras $\tilde{\mathcal{G}}_{\pm} := \widetilde{diff}_{\pm}(\mathbb{T}^n)$ of the loop subgroups $\tilde{G}_{\pm}$ are vector fields on $\mathbb{T}^n$ holomorphic, respectively, on $\mathbb{D}^1_{\pm} \subset \mathbb{C}^1$, where for any $\tilde{a}(\lambda) \in \tilde{\mathcal{G}}_-$ the value $\tilde{a}(\infty) = 0$. The split loop Lie algebra $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ + \tilde{\mathcal{G}}_-$ can be naturally identified with a dense subspace of the dual space $\tilde{\mathcal{G}}^*$ through the pairing

$$(\tilde{l}, \tilde{a}) := \frac{1}{2\pi i} \oint_{\mathbb{S}^1} (l(x,\lambda), a(x,\lambda))_{H^q} \frac{d\lambda}{\lambda^p}, \tag{3.1}$$

for some fixed $p, q \in \mathbb{Z}_+$. We took above, by definition [14, 52], for some fixed $p, q \in \mathbb{Z}_+$. We took above, by definition [14, 52], a loop vector field $\tilde{a} \in \Gamma(\tilde{T}(\mathbb{T}^n))$ and a loop differential 1-form $\tilde{l} \in \tilde{\Lambda}^1(\mathbb{T}^n)$ given as

$$
(3.2) \qquad \begin{aligned} \tilde{a} &= \sum_{j=1}^{n} a^{(j)}(x,\lambda)\frac{\partial}{\partial x_j} := \left\langle a(x;\lambda), \frac{\partial}{\partial x}\right\rangle, \\ \tilde{l} &= \sum_{j=1}^{n} l_j(x,\lambda)dx_j := \langle l(x;\lambda), dx\rangle, \end{aligned}
$$

introduced for brevity the gradient operator $\frac{\partial}{\partial x} := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)^{\intercal}$ in the Euclidean space $\mathbb{E}^n$ and chose the Sobolev type metric $(\cdot,\cdot)_{H^q}$ on the space $C^\infty(\mathbb{T}^n;\mathbb{R}^n) \subset H^q(\mathbb{T}^n;\mathbb{R}^n)$ for some $q \in \mathbb{Z}_+$ as

(3.3)

$$
(l(x;\lambda), a(x;\lambda))_{H^q} := \sum_{j=1}^{n}\sum_{|\alpha|=0}^{q}\int_{\mathbb{T}^n} dx \left(\frac{\partial^{|\alpha|} l_j(x;\lambda)}{\partial x^\alpha}\frac{\partial^{|\alpha|} a^{(j)}(x;\lambda)}{\partial x^\alpha}\right),
$$

where $\partial x^\alpha := \partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_2^{\alpha_n}, |\alpha| = \sum_{j=1}^{n}\alpha_j$ for $\alpha \in \mathbb{Z}_+^n$, generalizing the metric used before in [35]. The Lie commutator of eralizing the metric used before in [35]. The Lie commutator of vector fields $\tilde{a}, \tilde{b} \in \tilde{\mathcal{G}}$ is calculated the standard way and equals

$$(3.4) \qquad [\tilde{a}, \tilde{b}] = \tilde{a}\tilde{b} - \tilde{b}\tilde{a} = \left\langle \left\langle a(x;\lambda), \frac{\partial}{\partial x} \right\rangle b(x;\lambda), \frac{\partial}{\partial x} \right\rangle - \\ - \left\langle \left\langle b(x;\lambda), \frac{\partial}{\partial x} \right\rangle a(x;\lambda), \frac{\partial}{\partial x} \right\rangle .$$

The Lie algebra $\tilde{\mathcal{G}}$ naturally splits into the direct sum of two Lie subalgebras

$$(3.5) \qquad \tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-,$$

for which one can identify the dual spaces

$$\tilde{\mathcal{G}}_+^* \simeq \lambda^{p-1}\tilde{\mathcal{G}}_-, \qquad \tilde{\mathcal{G}}_-^* \simeq \lambda^{p-1}\tilde{\mathcal{G}}_+,$$

where for any $l(\lambda) \in \tilde{\mathcal{G}}_-^*$ one has the constraint $\tilde{l}(0) = 0$. Having where for any $l(\lambda) \in \tilde{\mathcal{G}}_{-}^{*}$ one has the constraint $\tilde{l}(0) = 0$. Having defined now the projections

$$P_{\pm}\tilde{\mathcal{G}} := \tilde{\mathcal{G}}_{\pm} \subset \tilde{\mathcal{G}}, \tag{3.6}$$

one can construct a classical $\mathcal{R}$-structure [65, 54, 61] on the Lie algebra $\tilde{\mathcal{G}}$ as the endomorphism $\mathcal{R} : \tilde{\mathcal{G}} \to \tilde{\mathcal{G}}$, where

$$\mathcal{R} := (P_{+} - P_{-})/2, \tag{3.7}$$

which allows to determine on the vector space $\tilde{\mathcal{G}}$ the new Lie algebra structure

$$[\tilde{a}, \tilde{b}]_{\mathcal{R}} := [\mathcal{R}\tilde{a}, \tilde{b}] + [\tilde{a}, \mathcal{R}\tilde{b}] \tag{3.8}$$

for any $\tilde{a}, \tilde{b} \in \tilde{\mathcal{G}}$, satisfying the standard Jacobi identity.

Let $D(\tilde{\mathcal{G}}^*)$ denote the space of smooth functions on $\tilde{\mathcal{G}}^*$. Then for any $f, g \in \mathcal{D}(\tilde{\mathcal{G}}^*)$ one can write the canonical [65, 54, 51, 4] Lie–Poisson bracket

$$\{f, g\} := (\tilde{l}, [\nabla f(\tilde{l}), \nabla g(\tilde{l})]), \tag{3.9}$$

where $\tilde{l} \in \tilde{\mathcal{G}}^*$ is a seed element and $\nabla f, \nabla g \in \tilde{\mathcal{G}}$ are the standard functional gradients at $\tilde{l} \in \tilde{\mathcal{G}}^*$ with respect to the metric (3.1). The related to (3.9) space $I(\tilde{\mathcal{G}}^*)$ of Casimir invariants is defined as the set $I(\tilde{\mathcal{G}}^*) \subset D(\tilde{\mathcal{G}}^*)$ of smooth independent functions $h_j \in D(\tilde{\mathcal{G}}^*), j = \overline{1, n}$, for which

$$ad^*_{\nabla h_j(\tilde{l})} \tilde{l} = 0, \tag{3.10}$$

where for any seed element

$$\tilde{l} = < l, dx > \tag{3.11}$$

the gradients

$$\nabla h_j(\tilde{l}) := \left\langle \nabla h_j(l), \frac{\partial}{\partial x} \right\rangle \tag{3.12}$$

and the coadjoint action (3.10) can be equivalently rewritten, for instance, in the case $q = 0$, as

$$\left\langle \frac{\partial}{\partial x}, \nabla h_j(l) \right\rangle l + \left\langle l, (\frac{\partial}{\partial x} \nabla h_j(l)) \right\rangle = 0 \tag{3.13}$$

for any $j = \overline{1, n}$. If one takes two smooth functions $h^{(y)}, h^{(t)} \in I(\tilde{\mathcal{G}}^*) \subset \mathrm{D}(\tilde{\mathcal{G}}^*)$, their second Poisson bracket

$$\{h^{(y)}, h^{(t)}\}_{\mathcal{R}} := (\tilde{l}, [\nabla h^{(y)}, \nabla h^{(t)}]_{\mathcal{R}}) \tag{3.14}$$

on the space $\tilde{\mathcal{G}}^*$ vanishes, that is on the space $\tilde{\mathcal{G}}^*$ vanishes, that is

$$\{h^{(y)}, h^{(t)}\}_{\mathcal{R}} = 0 \tag{3.15}$$

at any seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$. Since the functions $h^{(y)}, h^{(t)} \in I(\tilde{\mathcal{G}}^*)$, the following coadjoint action relationships hold:

$$ad^*_{\nabla h^{(y)}(\tilde{l})}\tilde{l} = 0, \qquad ad^*_{\nabla h^{(t)}(\tilde{l})}\tilde{l} = 0, \tag{3.16}$$

which can be equivalently rewritten (as above in the case $q = 0$) as

(3.17)

$$\langle \tfrac{\partial}{\partial x}, \nabla h^{(y)}(l)\rangle l + \ \langle l, (\tfrac{\partial}{\partial x}\nabla h^{(y)}(l))\rangle \ =$$

$$= \langle \nabla h^{(y)}(l), \tfrac{\partial}{\partial x}\rangle l + \langle (\tfrac{\partial}{\partial x}, \nabla h^{(y)}(l))\rangle l + \langle l, (\tfrac{\partial}{\partial x}\nabla h^{(y)}(l))\rangle :=$$

$$= (\nabla h^{(y)}(\tilde{l}) + B_{\nabla h^{(y)}})l$$

and similarly

and similarly

(3.18)

$$\langle \tfrac{\partial}{\partial x}, \nabla h^{(t)}(l) \rangle l + \langle l, (\tfrac{\partial}{\partial x}, \nabla h^{(t)}(l)) \rangle := (\nabla h^{(t)}(\tilde{l}) + B_{\nabla h^{(t)}}) l,$$

where the expressions

(3.19)

$$\nabla h^{(y)}(\tilde{l}) := \left\langle \nabla h^{(y)}(l), \frac{d}{dx} \right\rangle, \quad \nabla h^{(t)}(\tilde{l}) := \left\langle \nabla h^{(t)}(l), \frac{\partial}{\partial x} \right\rangle$$

are true vector fields on $\mathbb{T}^n$, yet the expressions

(3.20)

$$B_{\nabla h^{(y)}} := \left\langle (\frac{\partial}{\partial x}, \nabla h^{(y)}(l)) \right\rangle + \left(\frac{\partial}{\partial x} \nabla h^{(y)}(l) \right),$$

$$B_{\nabla h^{(t)}} := \left\langle (\frac{\partial}{\partial x}, \nabla h^{(t)}(l)) \right\rangle + \left(\frac{\partial}{\partial x} \nabla h^{(t)}(l) \right),$$

are the usual matrix homomorphisms of the Euclidean space $\mathbb{E}^n$.

Consider now the following Hamiltonian flows on the space $\mathcal{G}^*$:

(3.21) $$\partial\tilde{l}/\partial y := \{h^{(y)}, \tilde{l}\}_{\mathcal{R}} = -ad^*_{\nabla h^{(y)}(\tilde{l})_+}\tilde{l},$$

$$\partial\tilde{l}/\partial t := \{h^{(t)}, \tilde{l}\}_{\mathcal{R}} = -ad^*_{\nabla h^{(t)}(\tilde{l})_+}\tilde{l},$$

where $h^{(y)}, h^{(t)} \in I(\tilde{\mathcal{G}}^*)$ and $y, t \in \mathbb{R}$ are the corresponding evolution parameters. Since $h^{(y)}, h^{(t)} \in I(\tilde{\mathcal{G}}^*)$ are Casimirs, the flows (3.21) commute. Thus, taking into account the representations (3.17), one can recast the flows (3.21) as

(3.22)

$$\partial l/\partial t = -(\nabla h^{(t)}(\tilde{l})_+ + B_{\nabla h_+^{(t)}})l, \quad \partial l/\partial y = -(\nabla h^{(y)}(\tilde{l}) + + B_{\nabla h_+^{(y)}})l,$$

where

(3.23)

$$\nabla h^{(t)}(\tilde{l})_+ := \left\langle \nabla h^{(t)}(l)_+, \frac{\partial}{\partial x} \right\rangle, \quad \nabla h^{(y)}(\tilde{l})_+ := \left\langle \nabla h^{(y)}(l)_+, \frac{\partial}{\partial x} \right\rangle.$$

Lemma 3.1. *The compatibility of commuting flows (3.22) is equivalent to the Lax type vector fields relationship*

$$\frac{\partial}{\partial y}\nabla h^{(t)}(\tilde{l})_+ - \frac{\partial}{\partial t}\nabla h^{(y)}(\tilde{l})_+ + [\nabla h^{(y)}(\tilde{l})_+, \nabla h^{(t)}(\tilde{l})_+] = 0, \tag{3.24}$$

which holds for all $y, t \in \mathbb{R}$ *and arbitrary* $\lambda \in \mathbb{C}$.

Proof. The compatibility of commuting flows (3.22) implies that $\partial^2 l/\partial t\partial y - \partial^2 l/\partial y\partial t = 0$ for all $y, t \in \mathbb{R}$ and arbitrary $\lambda \in \mathbb{C}$.

For the exact representatives of the functions $h^{(y)}, h^{(t)} \in I(\tilde{\mathcal{G}}^*)$, it is necessary to solve the determining equation (3.13), taking into account that if the chosen element $\tilde{l} \in \tilde{\mathcal{G}}^*$ is singular as $|\lambda| \to \infty$, the related expansion

$$\nabla h^{(p)}(l) \simeq \lambda^p \sum_{j \in \mathbb{Z}_+} \nabla h_j(l) \lambda^{-j}, \tag{3.27}$$

where the degree $p \in \mathbb{Z}_+$ can be taken as arbitrary. Upon substituting (3.27) into (3.13) one can find recurrently all the coefficients $\nabla h(l)_j$, $j \in \mathbb{Z}_+$, and then construct functional gradients of the Casimir functions $h^{(y)}, h^{(t)} \in I(\tilde{\mathcal{G}}^*)$ projected on $\tilde{\mathcal{G}}_+$ as

$$\nabla h^{(t)}(l)_+ = (\lambda^{p_t} \nabla h(l))_+, \qquad \nabla h^{(y)}(l)_+ = (\lambda^{p_y} \nabla h(l))_+ \tag{3.28}$$

for some positive integers $p_y, p_t \in \mathbb{Z}_+$.

Remark 3.2. *As mentioned above, the expansion (3.27) is effective if a chosen seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$ is singular as $|\lambda| \to \infty$. In the case when it is singular as $|\lambda| \to 0$, the expression (3.27) should be replaced by the expansion*

$$\nabla h^{(p)}(l) \sim \lambda^{-p} \sum_{j \in \mathbb{Z}_+} \nabla h_j^{(p)}(l) \lambda^j \tag{3.29}$$

for an arbitrary $p \in \mathbb{Z}_+$, and the projected Casimir function gradients then are given by the expressions

$$\nabla h^{(y)}(l)_- = \lambda(\lambda^{-p_y-1} \nabla h(l))_-, \qquad \nabla h^{(t)}(l)_- = \lambda(\lambda^{-p_t-1} \nabla h(l))_- \tag{3.30}$$

for some positive integers $p_y, p_t \in \mathbb{Z}_+$. Then the corresponding flows are, respectively, written as

$$\partial \tilde{l} / \partial t = ad^*_{\nabla h^{(t)}(\tilde{l})_-} \tilde{l}, \quad \partial \tilde{l} / \partial y = ad^*_{\nabla h^{(y)}(\tilde{l})_-} \tilde{l}. \tag{3.31}$$

The above results, owing to 3.1, can be formulated as the following main proposition.

Proposition 3.3. *Let a seed vector field* $\tilde{l} \in \tilde{\mathcal{G}}^*$ *and* $h^{(y)}, h^{(t)} \in I(\tilde{\mathcal{G}}^*)$ *be Casimir functions subject to the metric* $(\cdot,\cdot)$ *on the loop Lie algebra* $\tilde{\mathcal{G}}$ *and the natural coadjoint action on the loop co-algebra* $\tilde{\mathcal{G}}^*$*. Then the following dynamical systems*

$$\partial\tilde{l}/\partial y = -ad^*_{\nabla h^{(y)}(\tilde{l})_+}\tilde{l}, \quad \partial\tilde{l}/\partial t = -ad^*_{\nabla h^{(t)}(\tilde{l})_+}\tilde{l} \tag{3.32}$$

are commuting Hamiltonian flows for all $y, t \in \mathbb{R}$*. Moreover, the compatibility condition of these flows is equivalent to the so called vector fields representation*

$$(\partial/\partial t + \nabla h^{(t)}(\tilde{l})_+)\psi = 0, \qquad (\partial/\partial y + \nabla h^{(y)}(\tilde{l})_+)\psi = 0, \tag{3.33}$$

where $\psi \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^n; \mathbb{C})$ *and the vector fields* $\nabla h^{(y)}(\tilde{l})_+, \nabla h^{(t)}(\tilde{l})_+ \in \tilde{\mathcal{G}}_+$*, given by the expressions (3.23) and (3.28), satisfy the Lax relationship (3.24).*

4. Integrable heavenly type equations: Examples

4.1. The Mikhalev–Pavlov heavenly type equation. This equation [34, 40] is

$$u_{xt} + u_{yy} = u_y u_{xx} - u_x u_{xy}, \tag{4.1}$$

where $u \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^1)$ and $(t, y; x) \in \mathbb{R}^2 \times \mathbb{T}^1$. Set $\tilde{\mathcal{G}}^* := \widetilde{diff}^*(\mathbb{T}^1)$ and take the corresponding seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$ as

$$\tilde{l} = (\lambda - u_x/2)dx \ . \tag{4.2}$$

It generates a Casimir invariant $h \in I(\tilde{\mathcal{G}}^*)$ for which the expansion (3.27) as $|\lambda| \to \infty$ is given by the asymptotic series

$$\nabla h(l) \sim 1 + u_x/\lambda - u_y/\lambda^2 + O(1/\lambda^3) \tag{4.3}$$

and so on. If further one defines

and so on. If further one defines

$$\nabla h^{(t)}(l)_+ := (\lambda^2 \nabla h)_+ = \lambda^2 + \lambda u_x - u_y, \tag{4.4}$$

$$\nabla h^{(y)}(l)_+ := (\lambda^1 \nabla h)_+ = \lambda + u_x,$$

it is easy to verify that

$$\nabla h^{(t)}(\tilde{l})_+ :=< \nabla h^{(t)}(l)_+, \frac{\partial}{\partial x} >= (\lambda^2 + \lambda u_x - u_y)\frac{\partial}{\partial x},$$

(4.5)

$$\nabla h^{(y)}(\tilde{l})_+ :=< \nabla h^{(y)}(l)_+, \frac{\partial}{\partial x} >= (\lambda + u_x)\frac{\partial}{\partial x}.$$

As a result of (4.5) and the commuting flows (3.32) on $\tilde{\mathcal{G}}^*$ we retrieve (the equivalent to the Mikhalev–Pavlov [40]) equation (4.1) vector field compatibility relationships

$$\frac{\partial \psi}{\partial t} + (\lambda^2 + \lambda u_x - u_y)\frac{\partial \psi}{\partial x} = 0 = \frac{\partial \psi}{\partial y} + (\lambda + u_x)\frac{\partial \psi}{\partial x}, \tag{4.6}$$

satisfied for $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^1; \mathbb{C})$, any $(y, t; x) \in \mathbb{R}^2 \times \mathbb{T}^1$ and all $\lambda \in \mathbb{C}$.

4.2. Example: The Witham heavenly type equation. Consider the following [20, 42, 32, 26] heavenly type equation:

$$u_{ty} = u_x u_{xy} - u_y u_{xx}, \tag{4.9}$$

where $u \in C^2(\mathbb{R}^2 \times \mathbb{T}^1; \mathbb{R})$ and $(t, y; x) \in \mathbb{R}^2 \times \mathbb{T}^1$. To prove the Lax-Sato type integrability of (4.9), let us consider a seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$, defined as

$$\tilde{l} = (u_y^{-2}\lambda^{-1} + 2u_x + \lambda)dx, \tag{4.10}$$

where $\lambda \in \mathbb{C}$ is a complex parameter. The following asymptotic expressions are gradients of the Casimir functionals $h^{(t)}, h^{(y)} \in I(\tilde{\mathcal{G}}^*)$, related with the holomorphic loop Lie algebra $\tilde{\mathcal{G}} = \widetilde{diff}(\mathbb{T}^1)$:

$$\nabla h^{(t)}(l) \sim \lambda[(u_x\lambda^{-1} - 1) + O(1/\lambda), \tag{4.11}$$

as $\lambda \longrightarrow \infty$, and

$$\nabla h^{(y)}(l) \sim u_y\lambda^{-1} + O(\lambda^2), \tag{4.12}$$

as $\lambda \longrightarrow 0$. Based on the expressions (4.11) and (4.12), one can as $\lambda \to 0$. Based on the expressions (4.11) and (4.12), one can construct [52] the following commuting to each other Hamiltonian flows

(4.13)

$$\frac{\partial}{\partial y}\tilde{l} = -ad^*_{\nabla h^{(y)}(\tilde{l})_-}\tilde{l}, \quad \frac{\partial}{\partial t}\tilde{l} = -ad^*_{\nabla h^{(t)}(\tilde{l})_+}\tilde{l}$$

with respect to the evolution parameters $y, t \in \mathbb{R}$, which give rise, in part, to the following equations:

(4.14)
$$u_{yt} = u_x u_{xy} - u_y u_{xx},$$

$$u_t = -u_y^{-2}/2 + 3u_x^2/2,$$

$$u_{yy} = -u_y^3[(u_x u_y)_x + u_x u_{xy}],$$

where the projected gradients $\nabla h_{-}^{(y)}, \nabla h_{+}^{(t)} \in \tilde{\mathcal{G}}$ are equal to the loop vector fields

$$\nabla h^{(t)}(\tilde{l})_{+} = (u_{x} - \lambda)\frac{\partial}{\partial x}, \quad \nabla h^{(y)}(\tilde{l})_{-} = \frac{u_{y}}{\lambda}\frac{\partial}{\partial x}, \tag{4.15}$$

satisfying for evolution parameters $y, t \in \mathbb{R}^{2}$ the Lax-Sato vector field compatibility condition:

$$\frac{\partial}{\partial y}\nabla h^{(y)}(\tilde{l})_{+} - \frac{\partial}{\partial t}\nabla h^{(y)}(\tilde{l})_{-} + [\nabla h^{(t)}(\tilde{l})_{+}, \nabla h^{(y)}(\tilde{l})_{-}] = 0. \tag{4.16}$$

As a simple consequence of the condition one finds exactly the first equation of the (4.14), coinciding with the heavenly type equation (4.9). Thereby, we have stated that this equation is a completely integrable heavenly type dynamical system with respect to both evolution parameters.

4.3. The Hirota heavenly equation. The Hirota equation describes [18, 33] three-dimensional Veronese webs and reads as

$$\alpha u_x u_{yt} + \beta u_y u_{xt} + \gamma u_t u_{xy} = 0 \tag{4.22}$$

for any evolution parameters $t, y \in \mathbb{R}$ and the spatial variable $x \in \mathbb{T}^1$, where α, β and $\gamma \in \mathbb{R}$ are arbitrary constants, satisfying the numerical constraint

$$\alpha + \beta + \gamma = 0. \tag{4.23}$$

To demonstrate the Lax-type integrability of the Hirota equation (4.22) we choose a seed vector field $\tilde{l} \in \tilde{\mathcal{G}}^* := \widetilde{diff}^*(\mathbb{R}^1)$ in the following rational form

$$\tilde{l} = \left(\frac{u_x^2}{u_t^2(\lambda + \alpha)} - \frac{u_x^2(u_y^2 + u_t^2)}{2\alpha u_t^2 u_y^2} + \frac{u_x^2}{u_y^2(\lambda - \alpha)} \right) dx. \tag{4.24}$$

$$\tilde{l} = \left(\frac{u_x^2}{u_t^2(\lambda + \alpha)} - \frac{u_x^2(u_y^2 + u_t^2)}{2\alpha u_t^2 u_y^2} + \frac{u_x^2}{u_y^2(\lambda - \alpha)} \right) dx. \tag{4.24}$$

The corresponding gradients for the Casimir invariants $\gamma^{(j)} \in I(\tilde{\mathcal{G}}^*)$, $j = \overline{1,2}$, are given by the following asymptotic expansions:

$$\nabla\gamma^{(1)}(l) \sim \sum_{j \in \mathbb{Z}_+} \nabla\gamma_j^{(1)}(l)\mu^j, \tag{4.25}$$

as $\lambda + \alpha := \mu \to 0$, and

$$\nabla\gamma^{(2)}(l) \sim \sum_{j \in \mathbb{Z}_+} \nabla\gamma_j^{(2)}(l)\mu^j, \tag{4.26}$$

as $\lambda - \alpha = \mu \to 0$. For the first case (4.25) one easily obtains that as $\lambda - \alpha = \mu \to 0$. For the first case (4.25) one easily obtains that

$$\nabla\gamma^{(1)}(l) \sim -2\gamma\frac{u_t}{u_x} + O(\mu^2), \tag{4.27}$$

and for the second one (4.26) one obtains

$$\nabla\gamma^{(2)}(l) \sim 2\beta\frac{u_y}{u_x} + O(\mu^2), \tag{4.28}$$

where we took into account that the following two Hamiltonian flows on $\tilde{\mathcal{G}}^*$

$$d\tilde{l}/dy = ad^*_{\nabla h^{(t)}(\tilde{l})_-}\tilde{l}, \qquad d\tilde{l}/dt = ad^*_{\nabla h^{(t)}(\tilde{l})_-}\tilde{l} \tag{4.29}$$

with respect to the evolution parameters $y, t \in \mathbb{R}$ hold for the following conservation laws gradients:

$$\nabla h^{(t)}(l)_- := \mu(\mu^{-2}\nabla\gamma^{(1)}(l))_-|_{\mu=\lambda+\alpha} = \frac{-2\gamma}{(\lambda+\alpha)}\frac{u_t}{u_x}, \tag{4.30}$$

$$\nabla h^{(y)}(l)_- := \mu(\mu^{-2}\nabla\gamma^{(2)}(l))_-|_{\mu=\lambda-\alpha} = \frac{2\beta}{(\lambda-\alpha)}\frac{u_y}{u_x}.$$

It is easy now to check that the compatibility condition (3.24) for a set of the vector fields (4.30) gives rise to the Hirota heavenly equation (4.22), whose equivalent Lax-Sato vector field representation reads as a system of the linear vector field equations

$$\frac{\partial\psi}{\partial t} - \frac{2\gamma u_t}{u_x(\lambda+\alpha)}\frac{\partial\psi}{\partial x} = 0, \quad \frac{\partial\psi}{\partial y} + \frac{2\beta u_y}{u_x(\lambda-\alpha)}\frac{\partial\psi}{\partial x} = 0, \tag{4.31}$$

satisfied for $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^1;\mathbb{C})$ for all $(y,t;x) \in \mathbb{R}^2 \times \mathbb{T}^1$ and $\lambda \in \mathbb{C}\backslash\{\pm\alpha\}$.

$$\alpha u_x u_{yt} + \beta u_y u_{xt} + \gamma u_t u_{xy} = 0 \tag{4.22}$$

4.8. **The Alonso–Shabat heavenly type equation.** This equation [2] has the form

$$u_{yx_2} - u_t u_{yx_1} + u_y u_{tx_1} = 0, \tag{4.62}$$

where $u \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{R})$, $(y,t) \in \mathbb{R}^2$ and $(x_1, x_2) \in \mathbb{T}^2$. To prove its Lax integrability, we define a seed element $\tilde{l} \in \tilde{\mathcal{G}}^* := \widetilde{diff}^*(\mathbb{T}^2)$ of the form

$$\tilde{l} = \quad z_{x_1}^2(\lambda+1)dx_1 + z_{x_1} z_{x_2}(\lambda+1)dx_2, \tag{4.63}$$

for a fixed function $z \in C^\infty(\mathbb{T}^2;\mathbb{R})$. Then one easily obtains asymptotic expansionsas $|\lambda| \to \infty$ for coefficients of the two independent Casimir functionals $h^{(1)}, h^{(2)} \in I(\tilde{\mathcal{G}}^*)$ gradients:

$$\nabla h^{(1)}(l) \sim (1/z_{x_1} + k z_{x_2}/z_{x_1}, -k)^\intercal + O(1/\lambda^2), \tag{4.64}$$

$$\nabla h^{(2)}(l) \sim (z_{x_2}/z_{x_1}, -1)^\intercal + O(1/\lambda^2),$$

4.6. **Plebański heavenly equation.** This equation [49] is

$$u_{tx_1} - u_{yx_2} + u_{x_1x_1}u_{x_2x_2} - u^2_{x_1x_2} = 0 \tag{4.45}$$

for a function $u \in C^\infty(\mathbb{R}^2; \mathbb{T}^2)$, where $(y, t; x_1, x_2) \in \mathbb{R}^2 \times \mathbb{T}^2$. We set $\tilde{\mathcal{G}}^* := \widetilde{diff}^*(\mathbb{T}^2)$ and take the corresponding seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$ as

$$\tilde{l} = (\lambda - u_{x_1x_2} + u_{x_1x_1})dx_1 + (\lambda - u_{x_2x_2} + u_{x_1x_2})dx_2. \tag{4.46}$$

This generates two independent Casimir functionals $h^{(1)}, h^{(2)} \in I(\tilde{\mathcal{G}}^*)$, whose gradient expansions (3.27) as $|\lambda| \to \infty$ are given by the expressions

$$\nabla h^{(1)}(l) \sim (0,1)^\intercal + (u_{x_2x_2}, -u_{x_1x_2})^\intercal \lambda^{-1} + O(\lambda^{-2}), \tag{4.47}$$

$$\nabla h^{(2)}(l) \sim (1,0)^\intercal + (u_{x_1x_2}, -u_{x_1x_2})^\intercal \lambda^{-1} + O(\lambda^{-2}),$$

and so on. Now, by defining

and so on. Now, by defining

$$\nabla h^{(y)}(l)_+ := (\lambda \nabla h^{(1)}(l))_+ = (u_{x_2x_2}, \lambda - u_{x_1x_2})^\intercal, \tag{4.48}$$

$$\nabla h^{(t)}(l)_+ := (\lambda \nabla h^{(2)}(l))_+ = (\lambda + u_{x_1x_2}, -u_{x_1x_1})^\intercal,$$

one obtains for (4.45) the following [49] vector field representation

$$\frac{\partial \psi}{\partial t} + u_{x_1x_1}\frac{\partial \psi}{\partial x_1} + (\lambda - u_{x_1x_2})\frac{\partial \psi}{\partial x_2} = 0, \tag{4.49}$$

$$\frac{\partial \psi}{\partial z} + (\lambda + u_{x_1x_2})\frac{\partial \psi}{\partial x_1} - u_{x_1x_1}\frac{\partial \psi}{\partial x_2} = 0,$$

satisfied for $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{C})$, any $(t, y; x_1, x_2) \in \mathbb{R}^2 \times \mathbb{T}^2$ and all $\lambda \in \mathbb{C}$.

4.7. General heavenly equation. This equation was first suggested and analyzed by Schief in [57, 58], where it was shown to be equivalent to the first Plebański heavenly equation, and later studied by Doubrov and Ferapontov [16]; it has the form

$$\alpha u_{yt}u_{x_1x_2} + \beta u_{tx_2}u_{yx_1} + \gamma u_{tx_1}u_{yx_2} = 0, \tag{4.50}$$

where $t, y \in \mathbb{R}, (x_1, x_2) \in \mathbb{T}^2, u \in C^\infty(\mathbb{T}^2; \mathbb{R})$ is a smooth functions and α, β and $\gamma \in \mathbb{R}$ are arbitrary constants, satisfying the constraint

$$\alpha + \beta + \gamma = 0. \tag{4.51}$$

To demonstrate the Lax integrability of the equation (4.50) we choose now a seed vector field $\tilde{l} \in \tilde{\mathcal{G}}^* := \widetilde{diff}^*(\mathbb{T}^2)$ in the following rational form

$$\tilde{l} = \left(\frac{\mu u_{x_1 x_2}^2}{\gamma(\mu+\beta)} + \frac{u_{x_1 x_2}^2}{\alpha} - \frac{\mu u_{x_1 x_2}^2}{\beta(\mu-\gamma)} \right) dx_1 + \tag{4.52}$$

$$+ \left(\frac{\mu u_{x_1 x_2} u_{x_2 x_2}}{\gamma(\mu+\beta)} + \frac{u_{x_1 x_2} u_{x_2 x_2}}{\alpha} - \frac{\mu u_{x_1 x_2} u_{x_2 x_2}}{\beta(\mu-\gamma)} \right) dx_2,$$

where is a smooth functions and $\mu \in \mathbb{C} \setminus \{\gamma, -\alpha, -\beta\}$ is a complex parameter. The corresponding equations for independent

The corresponding equations for independent

Casimir invariants $h^{(j)} \in I(\tilde{\mathcal{G}}^*), j = \overline{1,2}$, are given with respect to the standard metric $(\cdot,\cdot)$ by the following asymptotic expansions:

$$\nabla h^{(1)}(l) \sim \sum_{j \in \mathbb{Z}_+} \nabla h_j^{(1)}(l) \lambda^j, \tag{4.53}$$

as $\mu + \beta = \lambda \longrightarrow 0$ and

$$\nabla h^{(2)}(l) \sim \sum_{j \in \mathbb{Z}_+} \nabla h_j^{(2)}(l) \lambda^j, \tag{4.54}$$

as $\mu - \gamma = \lambda \longrightarrow 0$. For the first case (4.53) one obtains that

$$\nabla h^{(1)}(l) \sim \left(-\frac{\beta u_{tx_2}}{u_{x_1x_2}} + \frac{u_{tx_2}}{u_{x_1x_2}} \lambda, \frac{\beta u_{x_1x_1}}{u_{x_1x_2}} \right)^{\intercal} + O(\lambda^2) \tag{4.55}$$

and for the second one (4.54) one finds that

$$\nabla h^{(2)}(l) \sim \left(\frac{\gamma u_{yx_2}}{u_{x_1x_2}} + \frac{u_{yx_2}}{u_{x_1x_2}} \lambda, -\frac{\gamma u_{x_1x_1}}{u_{x_1x_2}} \right)^{\intercal} + O(\lambda^2). \tag{4.56}$$

Here we took into account that the following two Hamiltonian flows on $\tilde{\mathcal{G}}^*$

$$\partial\tilde{l}/\partial y = ad^*_{\nabla h^{(y)}(\tilde{l})_-}\tilde{l}, \qquad \partial\tilde{l}/\partial t = ad^*_{\nabla h^{(t)}(\tilde{l})_-}\tilde{l} \tag{4.57}$$

with respect to the evolution parameters $y, t \in \mathbb{R}$ hold for the following conservation laws gradients:

(4.58)

$$\nabla h^{(t)}(l)_- := \lambda(\lambda^{-2}\nabla h^{(1)}(l)_-\Big|_{\lambda=\mu+\beta} = \left(\frac{\mu u_{tx_2}}{u_{x_1x_2}(\mu+\beta)}, \frac{\beta u_{tx_1}}{u_{x_1x_2}(\mu+\beta)}\right)^{\intercal},$$

$$\nabla h^{(y)}(l)_- := \lambda(\lambda^{-2}\nabla h^{(2)}(l)_-\Big|_{\lambda=\mu-\gamma} = \left(\frac{\mu u_{yx_2}}{u_{x_1x_2}(\mu-\gamma)}, -\frac{\gamma u_{yx_1}}{u_{x_1x_2}(\mu-\gamma)}\right)^{\intercal}.$$

Owing to the compatibility condition of two commuting flows (4.58), one can easily rewrite it as the Lax relationship

(4.59) $\partial \nabla h^{(y)}(\tilde{l})_-/\partial t - \partial \nabla h^{(t)}(\tilde{l})_-/\partial y = [\nabla h^{(y)}(\tilde{l})_-, \nabla h^{(t)}(\tilde{l})_-],$

where

(4.60)

$$\nabla h^{(t)}(\tilde{l})_- := \left\langle \nabla h^{(t)}(l)_-, \frac{\partial}{\partial x} \right\rangle = \frac{\mu u_{tx_2}}{u_{x_1x_2}(\mu+\beta)} \frac{\partial}{\partial x_1} + \frac{\beta u_{tx_1}}{u_{x_1x_2}(\mu+\beta)} \frac{\partial}{\partial x_2},$$

$$\nabla h^{(y)}(\tilde{l})_- := \left\langle \nabla h^{(y)}(l)_-, \frac{\partial}{\partial x} \right\rangle = \frac{\mu u_{yx_2}}{u_{x_1x_2}(\mu-\gamma)} \frac{\partial}{\partial x_1} - \frac{\gamma u_{yx_1}}{u_{x_1x_2}(\mu-\gamma)} \frac{\partial}{\partial x_2}.$$

An easy calculation shows that the general heavenly equation (4.50) follows from the compatibility condition (4.59), whose equivalent vector field representation is given as

$$\text{(4.61)} \qquad \frac{\mu u_{tx_2}}{u_{x_1x_2}(\mu+\beta)}\frac{\partial\psi}{\partial x_1}+\frac{\beta u_{tx_1}}{u_{x_1x_2}(\mu+\beta)}\frac{\partial\psi}{\partial x_2}+\frac{\partial\psi}{\partial t}=0,$$

$$\frac{\mu u_{yx_2}}{u_{x_1x_2}(\mu-\gamma)}\frac{\partial\psi}{\partial x_1}-\frac{\gamma u_{yx_1}}{u_{x_1x_2}(\mu-\gamma)}\frac{\partial\psi}{\partial x_2}+\frac{\partial\psi}{\partial y}=0$$

for a function $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{C})$ for all $(y, t; x_1, x_2) \in \mathbb{R}^2 \times \mathbb{T}^2$.

4.8. **The Alonso–Shabat heavenly type equation.** This equation [2] has the form

$$u_{yx_2} - u_t u_{yx_1} + u_y u_{tx_1} = 0, \tag{4.62}$$

where $u \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{R}), (y,t) \in \mathbb{R}^2$ and $(x_1, x_2) \in \mathbb{T}^2$. To prove its Lax integrability, we define a seed element $\tilde{l} \in \tilde{\mathcal{G}}^* := \widetilde{diff}^*(\mathbb{T}^2)$ of the form

$$\tilde{l} = \quad z_{x_1}^2(\lambda+1)dx_1 + z_{x_1} z_{x_2}(\lambda+1)dx_2, \tag{4.63}$$

which can be rewritten as the compatibility condition for the following vector field equations:

$$\frac{\partial\psi}{\partial t}+\lambda u_t\frac{\partial\psi}{\partial x_1}-\lambda\frac{\partial\psi}{\partial x_2}=0,\qquad \frac{\partial\psi}{\partial y}+\lambda u_y\frac{\partial\psi}{\partial x_1}-k\lambda\frac{\partial\psi}{\partial x_2}=0, \tag{4.68}$$

satisfied for $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{C})$, any $(t, y; x_1, x_2) \in \mathbb{R}^2 \times \mathbb{T}^2$ and all $\lambda \in \mathbb{C}$. The resulting equation is then

$$u_{yx_2} - u_t u_{yx_1} + u_y u_{tx_1} + k u_{tx_2} = 0, \tag{4.69}$$

which reduces at $k = 0$ to the Alonso–Shabat heavenly equation (4.62).

Remark 4.2. It is interesting to observe that the seed elements $\tilde{l} \in \tilde{\mathcal{G}}^*$ of the examples presented above have the differential geometric structure:

$$\tilde{l} = \eta \, d\rho, \tag{4.70}$$

where η and $\rho \in C^\infty(\mathbb{R}^2 \times (\mathbb{C} \times \mathbb{T}^2); \mathbb{C})$ are some smooth functions. For instance,

$\tilde{l} = d(\lambda x - 2u)$ - Mikhalev–Pavlov type equation,

$\tilde{l} = d(\lambda x_1 + \lambda x_2 - u_{x_2} + u_{x_1})$ - Plebański heavenly equation,

$\tilde{l} = u_{x_1 x_2} \xi du_{x_2}, \xi := \left(\mu \, [\gamma(\mu+\beta)]^{-1} + \alpha^{-1} - \mu[\beta(\mu-\gamma)]^{-1}\right)$ - general heavenly equation, As an arbitrary heavenly equation is a Hamiltonian system with respect to both evolution parameters $t, y \in \mathbb{R}^2$ and $\lambda \in \mathbb{C}$, one can construct [3, 4, 5, 37, 51] its suitable Lagrangian (or quasi-Lagrangian) representation under some natural constraints. Thus,

it is possible to retrieve the corresponding Poisson structures related to both these evolution parameters $t, y \in \mathbb{R}^2$ and $\lambda \in \mathbb{C}$, which, as follows from the Lie-algebraic analysis in Section 3, are compatible to each other. In this way, one can show that any heavenly type equation is a bi-Hamiltonian integrable system on the corresponding functional manifold. It should be mentioned here that this property was introduced by Sergyeyev in (*arXiv:1501.01955*), published in [33], and rediscovered and applied in detail in [62] for investigating the integrability properties of the general heavenly equation (4.50), first suggested by Schief in [57] and later studied by Doubrov and Ferapontov in [16].

In his book "*Mecanique analytique*", v.1-2, published in 1788 in Paris, J.L. Lagrange formulated *one of the basic, most general, differential variational principles of classical mechanics, expressing necessary and sufficient conditions for the correspondence of the real motion of a system of material points, subjected by ideal constraints, to the applied active forces.* Within the d'Alembert–Lagrange principle the positions of the system in its real motion are compared with infinitely close positions permitted by the constraints at the given moment of time.

According to the d'Alembert–Lagrange principle, during a real motion of a system of $N \in \mathbb{Z}_+$ particles with massess $m_j \in \mathbb{R}_+, j = \overline{1,N}$, the sum of the elementary works performed by the given active forces $F^{(j)}, j = \overline{1,N}$, and by the forces of inertia for all *the possible particle displacements* $\delta x^{(j)} \in \mathbb{E}^3, j = \overline{1,N}$, **is equal to or less than zero:**

$$\sum_{j=\overline{1,N}} < F^{(j)} - m_j \frac{d^2 x^{(j)}}{dt^2}, \delta x^{(j)} > \leq 0 \tag{6.1}$$

at any moment of time $t \in \mathbb{R}$, where $< \cdot, \cdot >$ denotes the standard scalar product in the three-dimensional Euclidean space $\mathbb{E}^3$. *The equality in (6.1) is valid for the possible reversible displacements*, the symbol "$\leq$" is valid for *the possible irreversible displacements* $\delta x^{(j)} \in \mathbb{E}^3, j = \overline{1,N}$. Equation (6.1) is *the general equation of*

$\delta x^{(j)} \in \mathbb{E}^3, j = \overline{1, N}$. Equation (6.1) is *the general equation of the dynamics of systems with ideal constraints*; it comprises all the equations and laws of motion, so that one can say that all dynamics is reduced to this single general formula.

This principle, established by J.L. Lagrange by generalization of the principle of virtual displacements with the aid of the classical d'Alembert principle. For systems subject to bilateral constraints J.L. Lagrange based himself on formula (6.1) to deduce the general properties and laws of motion of bodies, as well as the equations of motion, which he applied to solve a number of problems in dynamics *including the problems of motions of non-compressible, compressible and elastic liquids, thus combining "dynamics and hydrodynamics as branches of the same principle and as conclusions drawn from a single general formula"*.

As it was first demonstrated in the work [24], in the last case of *generalized reversible motions* of a compressible elastic liquid, located in a one-connected open domain $\Omega_t \subset \mathbb{R}^n$ with the smooth boundary $\partial\Omega_t$, $t \in \mathbb{R}$, in space $\mathbb{R}^n, n \in \mathbb{Z}_+$, the expression (6.1) can be rewritten as

$$\delta W(t) := \int_{\Omega_t} < l(x(t);\lambda), \delta x(t) > d^n x(t) = 0 \tag{6.2}$$

for all $t \in \mathbb{R}$, where $l(x(t);\lambda) \in T^*(\mathbb{R}^n)$ is the corresponding virtual vector "*reaction force*", exerted by the ambient medium on the liquid and called a *seed element*, which is here assumed to depend meromorphically on a constant complex parameter $\lambda \in \mathbb{C}$.

If now to suppose that the evolution of liquid points $x(t) \in \Omega_t$ is determined for any parameters $\lambda \neq \mu \in \mathbb{C}$ by the generating *gradient type vector field*

$$\frac{dx(t)}{dt} = \frac{\mu}{\mu - \lambda} \nabla h(\, l(\mu))(t; x(t)) \tag{6.3}$$

and the Cauchy data

$$x(t)|_{t=0} = x^{(0)} \in \Omega_0$$

for an arbitrarily chosen open one-connected domain $\Omega_0 \subset \mathbb{T}^n$ with the smooth boundary $\partial\Omega_0 \subset \mathbb{R}^n$ and a smooth functional $h : \tilde{T}^*(\mathbb{R}^n) \to \mathbb{R}$, the Lagrange–d'Alembert principle says: the infinitesimal virtual work (6.2) equals zero for all moments of time, that is $\delta W(t) = 0 = \delta W(0)$ for all $t \in \mathbb{R}$. To check that time, that is $\delta W(t) = 0 = \delta W(0)$ for all $t \in \mathbb{R}$. To check that it is really zero, let us calculate the temporal derivative of the expression (6.2):

(6.4)

$$\frac{d}{dt}\delta W(t) = \frac{d}{dt}\int_{\Omega_t} < l(x(t);\lambda), \delta x(t) > d^n x(t) =$$

$$= \frac{d}{dt}\int_{\Omega_0} < l(x(t);\lambda), \delta x(t) > |\frac{\partial(x(t)}{\partial x_0}| d^n x^{(0)} = \int_{\Omega_0} \frac{d}{dt}(< l(x(t);\lambda), \delta x(t) > |\frac{\partial(x(t)}{\partial x_0}|) d^n x^{(0)} =$$

$$= \int_{\Omega_0} [\frac{d}{dt} < l(x(t);\lambda), \delta x(t) > + < l(x(t);\lambda), \delta x(t) > \operatorname{div} \tilde{K}(\mu)] |\frac{\partial(x(t)}{\partial x_0}| d^n x^{(0)} =$$

$$= \int_{\Omega_t} [\frac{d}{dt} < l(x(t);\lambda), \delta x(t) > + < l(x(t);\lambda), \delta x(t) > \operatorname{div} \tilde{K}(\mu)] d^n x(t) = 0,$$

if the condition

(6.5)

$$\frac{d}{dt} < l(x(t);\lambda), \delta x(t) > + < l(x(t);\lambda), \delta x(t) > \operatorname{div} \tilde{K}(\mu;\lambda) = 0$$

holds for all $t \in \mathbb{R}$, where

$$\tilde{K}(\mu;\lambda) := \frac{\mu}{\mu-\lambda} \nabla h(\ \tilde{l}(\mu)) = \frac{\mu}{\mu-\lambda} < \nabla h(\ l(\mu)), \frac{d}{dx} > \tag{6.6}$$

is a vector field on $\mathbb{R}^n$, corresponding to the evolution equations (6.3). Taking into account that the full temporal derivative tions (6.3). Taking into account that the full temporal derivative $d/dt := \partial/\partial t + L_{\tilde{K}(\mu;\lambda)}$, where $L_{\tilde{K}(\mu;\lambda)} = i_{\tilde{K}(\mu;\lambda)}d + di_{\tilde{K}(\mu;\lambda)}$ denotes the well known [1, 4, 23] Cartan expression for the Lie derivation along the vector field (6.6), can be represented as $\mu, \lambda \to \infty, |\lambda/\mu| < 1$ in the asymptotic form

$$\frac{d}{dt} \sim \sum_{j\in\mathbb{Z}_+} \mu^{-j} \frac{\partial}{\partial t_j} + \sum_{j\in\mathbb{Z}_+} \mu^{-j} L_{\tilde{K}_j(\lambda}, \tag{6.7}$$

the equality (6.5) can be *equivalently rewritten* as an infinite hierarchy of the following evolution equations

$$\partial \tilde{l}(\lambda)/\partial t_j := -ad^*_{\tilde{K}_j(\lambda)_+} \tilde{l}(\lambda) \tag{6.8}$$

for every $j \in \mathbb{Z}_+$ on the space of differential 1-forms $\tilde{\Lambda}^1(\mathbb{R}^n) \simeq \tilde{\mathcal{G}}^*$, where $\tilde{l}(\lambda) :=< l(x;\lambda), dx >\in \tilde{\Lambda}^1(\mathbb{R}^n) \simeq \tilde{\mathcal{G}}^*, \tilde{\mathcal{G}} := \widetilde{diff}(\mathbb{R}^n)$ is the Lie algebra of the corresponding loop diffeomorphism group $\widetilde{Diff}(\mathbb{R}^n)$. As from (6.6) one easily finds that

$$\tilde{K}_j(\lambda) = \nabla h^{(j)}(\ \tilde{l}) \tag{6.9}$$

for $\lambda \in \mathbb{C}$ and any $j \in \mathbb{Z}_+$, the evolution equations (6.8) transform equivalently into

$$\partial \tilde{l}(\lambda)/\partial t_j := -ad^*_{\nabla h^{(j)}(\tilde{l})_+} \tilde{l}(\lambda), \tag{6.10}$$

allowing to formulate the following important Adler-Kostant-Symes type [5, 4, 6, 65, 54, 53] proposition.

Proposition 6.1. *The evolution equations (6.10) are completely integrable commuting to each other Hamiltonian flows on the adjoint loop space $\tilde{\mathcal{G}}^*$ for a seed element $\tilde{l}(\lambda) \in \tilde{\mathcal{G}}^*$, generated by Casimir functionals $h^{(j)} \in I(\tilde{\mathcal{G}}^*)$, naturally determined by conditions $ad^*_{\nabla h^{(j)}(\tilde{l})} \tilde{l}(\lambda) = 0$, $j \in \mathbb{Z}_+$, with respect to the modified Lie-Poisson bracket on the adjoint space $\tilde{\mathcal{G}}^*$*

$$\{(\tilde{l}, \tilde{X}), (\tilde{l}, \tilde{Y})\} := (\tilde{l}, [\tilde{X}, \tilde{Y}]_{\mathcal{R}}),$$

Lie-Poisson bracket on the adjoint space $\tilde{\mathcal{G}}^*$

$$\{(\tilde{l}, \tilde{X}), (\tilde{l}, \tilde{Y})\} := (\tilde{l}, [\tilde{X}, \tilde{Y}]_{\mathcal{R}}),$$

defined for any $\tilde{X}, \tilde{Y} \in \tilde{\mathcal{G}}$ *by means of the canonical* $\mathcal{R}$*-structure on the loop Lie algebra* $\tilde{\mathcal{G}}$:

(6.11) $$[\tilde{X}, \tilde{Y}]_{\mathcal{R}} := [\tilde{X}_+, \tilde{Y}_+] - [\tilde{X}_-, \tilde{Y}_-],$$

where "$\tilde{Z}_\pm$*" means the positive (+)/(-)-negative part of a loop Lie algebra element* $\tilde{Z} \in \tilde{\mathcal{G}}$ *subject to the loop parameter* $\lambda \in \mathbb{C}$.

If, for instance, to consider the first two flows from (6.10) in the form

$$\partial\, \tilde{l}(\lambda)/\partial t_1 := \partial\, \tilde{l}(\lambda)/\partial y = -ad^*_{\nabla h^{(y)}(\ \tilde{l})_+}\ \tilde{l}(\lambda), \tag{6.12}$$

$$\partial\, \tilde{l}(\lambda)/\partial t_2 := \partial\, \tilde{l}(\lambda)/\partial t = -ad^*_{\nabla h^{(t)}(\ \tilde{l})_+}\ \tilde{l}(\lambda),$$

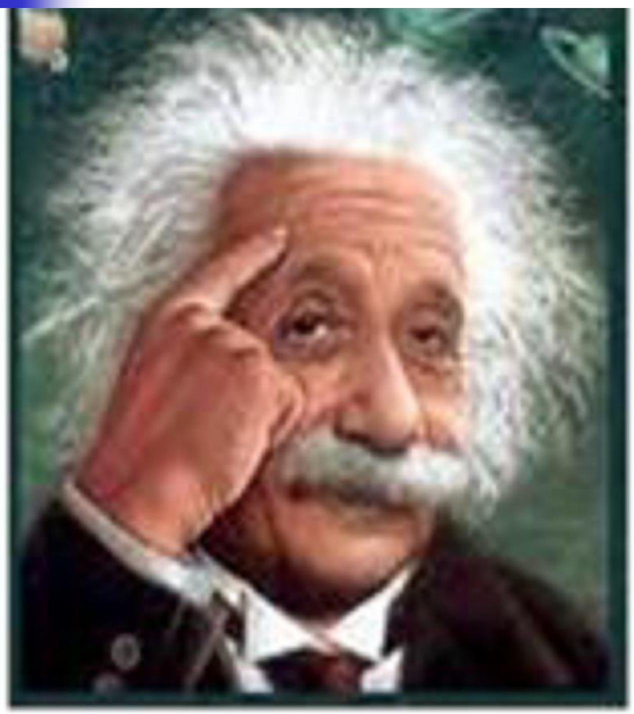
which are, by construction, commuting to each other, from their compatibility condition ensues some system of nonlinear equations in partial derivatives on the coefficients of the seed element $\tilde{l}(\lambda) \in \tilde{\mathcal{G}}^*$. As the latter is, evidently, equivalent to the Lax-Sato compat-

$\tilde{\mathcal{G}}^*$. As the latter is, evidently, equivalent to the Lax-Sato compatibility condition of the corresponding vector fields $\nabla h^{(y)}(\tilde{l})_+ := < \nabla h^{(1)}(l)_+, \partial/\partial x >, \nabla h^{(t)}(\tilde{l})_+ := < \nabla h^{(2)}(l)_+, \partial/\partial x > \in \tilde{\mathcal{G}}$:

$$[\partial/\partial y + \nabla h^{(y)}(\tilde{l})_+,\ \partial/\partial t + \nabla h^{(t)}(\tilde{l})_+] = 0, \tag{6.13}$$

a resulting from (6.13) system of nonlinear equations in partial derivatives is often called of heavenly type and was before actively analyzed in a series of articles [29, 8, 38, 39, 40, 57, 58, 33, 63, 64] and recently in [10, 8, 9, 25]. These works are closely related to the problem of constructing a hierarchy of commuting to each other vector fields, analytically depending on a complex parameter $\lambda \in \mathbb{C}$, which was in general form studied and completely solved by Pfeiffer in his classical work [44, 45, 48].

Velmi děkuji za Vaši pozornost!



Дякую сердечно за увагу!

Thanks so much for your attention!