Fractional characteristic functions and fractional moments of random variables

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Overview

1. Fractional CHF
2. Fractional moments
3. Applications

Main objectives

- Introduce a fractional variant of the characteristic function (CHF) of a random variable;
- To express fractional moments in terms of Riemann-Liouville integrals and derivatives of the fractional CHF.
- To give some illustrative examples for fractional pdf whose associated integer moments do not exist.

Fractional CHF

For a real random variable X with probability density (pdf) $p(x)$, the (classical) characteristic function of X is defined by

$$\varphi(t) = \mathbb{E}(e^{itx}) = \int_{-\infty}^{\infty} e^{itx} p(x) dx$$

$$\varphi(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \int_{-\infty}^{\infty} x^k p(x) dx = \sum_{k=0}^{\infty} (\mathbb{E}X^k) \frac{(it)^k}{k!},$$

where $\mathbb{E}X^k = \varphi^{(k)}(0)$ are the integer moments.

Definition

The fractional CHF $\varphi_\alpha(t)$, $0 < \alpha < 1$, $t \in \mathbb{R}$, of the random variable X is defined via the Mittag-Leffler transform of $p(x)$, i.e.

$$\varphi_\alpha(t) = \mathbb{E}(E_\alpha(i(tX)^\alpha)) = \int_{-\infty}^{\infty} E_\alpha(i(tx)^\alpha)p(x)dx.$$

Here $E_\alpha(z) = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$ denotes the one-parameter Mittag-Leffler function. Since $E_\alpha(0) = 1$, it is clear that $\varphi_\alpha(0) = 1$. For $\alpha = 1$, it is obvious that $\varphi_1(t) = \varphi(t)$ and $\overline{\varphi_1(t)} = \varphi(-t)$.

Definition

The fractional moment generating function (MGF) $M_\alpha(t)$, $0 < \alpha < 1$, $t \in \mathbb{R}$, of the random variable X is defined by

$$M_\alpha(t) = \mathbb{E}(E_\alpha(tX)^\alpha) = \int_{-\infty}^{\infty} E_\alpha((tx)^\alpha)p(x)dx.$$

In general, the fractional MGF has complex values. For non-negative random variables, it is real-valued on $\mathbb{R}^+$. If $\frac{1}{2} < \alpha \leq 1$, then the fractional CHF and the fractional MGF are related by

$$M_\alpha(t) = \varphi_\alpha(e^{-i\pi/(2\alpha)}t), \quad t \geq 0.$$

This follows from the equality $(te^{-i\pi/(2\alpha)})^\alpha = t^\alpha e^{-i\pi/2} = -it^\alpha$, which requires $\arg(t) > \pi(\frac{1}{2\alpha} - 1)$.

Proposition

The fractional CHF $\varphi_\alpha(t)$ exists for all $t \in \mathbb{R}$.

Proposition

The fractional MGF exists exactly for those $t \in \mathbb{R}$ for which the classical MGF exists.

Proposition

The fractional CHF of any random variable is uniformly continuous on $\mathbb{R}$.

We now comment on the related paper ([1] Jumarie G. 2006) , which defines the fractional Laplace transform of a function f by

$$\alpha \lim_{b\to\infty} \int_0^b (b-x)^{\alpha-1} E_\alpha(-s^\alpha x^\alpha) f(x) dx.$$

For $0 < \alpha < 1$, any function with bounded support, such as the density of a bounded random variable, has a fractional Laplace transform that is identically zero under this definition, which raises doubts about its usefulness. Moreover, the inversion formula presented in [1] rests on the identity $E_\alpha((x+y)^\alpha) = E_\alpha(x^\alpha) E_\alpha(y^\alpha)$, which is wrong for $\alpha \neq 1$; see ([2] Peng J. Li K.) for a discussion.

Examples

Let X have the half-Cauchy distribution with parameter $\beta > 0$, i.e. the pdf $p(x) = \frac{2}{\pi}\frac{\beta}{x^2+\beta^2}$, $x > 0$. $g(x) = E_\alpha(ix^\alpha)$ respectively. Since

$$E_\alpha(i(tx)^\alpha) \overset{\mathcal{M}}{\longleftrightarrow} \frac{1}{\alpha}\frac{\Gamma(1-\frac{s}{\alpha})\Gamma(\frac{s}{\alpha})}{\Gamma(1-s)}(e^{-i\frac{\pi}{2}}t^\alpha)^{-\frac{s}{\alpha}}$$

and (see [3], I. S. Gradshteyn and I. M. Ryzhik, p. 322, integral 3.241 (4))

$$\frac{1}{x^2+\beta^2} \overset{\mathcal{M}}{\longleftrightarrow} \frac{\beta^{s-2}}{2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right), \quad 0 < s < 2,\ \beta \neq 0,$$

by application of Parseval's convolution equality (see for example, [4] Luchko Y., Kiryakova V.S, FCAA 2013)

$$\int_0^{+\infty} p(x)g(xt)dx = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} p^*(1-s)g^*(s)t^{-s}ds,$$

we obtain

Example

$$\varphi_\alpha(t) = \frac{2\beta}{\pi} \int_0^\infty \frac{E_\alpha(i(tx)^\alpha)}{x^2+\beta^2} dx$$

$$= \frac{1}{\alpha\beta\pi} \frac{1}{2\pi i} \int_{\rho-i\infty}^{\rho+i\infty} \frac{\Gamma(1-\frac{s}{\alpha})\Gamma(\frac{s}{\alpha})\Gamma(1-\frac{s}{2})\Gamma(\frac{s}{2})}{\Gamma(1-s)} (\beta e^{i\frac{\pi}{2\alpha}} t^{-2})^s ds$$

$$= \frac{1}{\alpha\beta\pi} H^{2,2}_{3,3}\left[\frac{\beta e^{i\frac{\pi}{2\alpha}}}{t^2}\middle| \begin{matrix} (1,\frac{1}{\alpha}) & (1,\frac{1}{2}) & (1,0) \\ (1,\frac{1}{\alpha}) & (1,\frac{1}{2}) & (0,-1) \end{matrix}\right],$$

where $H^{2,2}_{3,3}$ is the Fox H-function.

Example

On the other hand,

$$M_\alpha(t) = \frac{2\beta}{\pi} \int_0^\infty \frac{E_\alpha((tx)^\alpha)}{x^2+\beta^2} dx = \infty, \quad t > 0,$$

by second Proposition, becouse $x \mapsto e^{tx}/x^2$ is not integrable for $t > 0$.

Example

The fundamental solution of space-time fractional diffusion model with space Riesz-Feller fractional derivative of order α is given in [([5] Mainardi F., Luchko Yu , Pagnini G. 2001); ([6] Mainardi F., Pagnini G. 2008); Tomovski, Ž., Sandev T., Metzler, R., Dubbeldam, J., Generalized space–time fractional diffusion equation with composite fractional time derivative, Physica A: Statistical Mechanics and its Applications, Vol. 391 (8) (2012) 2527-2542.]
by

$$L_\alpha^\theta(x) = \frac{1}{\alpha(\eta t)^{\frac{1}{\alpha}}} H_{2,2}^{1,1}\left[\frac{(\eta t)^{\frac{1}{\alpha}}}{|x|}\middle| \begin{array}{cc} (1,1) & (c,c) \\ (\frac{1}{\alpha},\frac{1}{\alpha}) & (c,c) \end{array}\right] du, \quad 0 < \alpha < 1, |\theta| \leq \alpha.$$

Example

Another form of this density is given by

$$L_{\alpha}^{\theta}(x) = \frac{1}{\alpha(\eta t)^{\frac{1}{\alpha}}} H_{2,2}^{1,1}\left[\frac{|x|}{(\eta t)^{\frac{1}{\alpha}}}\middle| \begin{matrix} (1-\frac{1}{\alpha},\frac{1}{\alpha}) & (1-c,c) \\ (0,1) & (1-c,c) \end{matrix}\right] du, \quad 1<\alpha<2, |\theta| \leq 2-\alpha.$$

In both cases $c = \frac{\alpha-\theta}{2\alpha} \neq 0$ and η is a diffusion constant.

Example

The Mellin transform of $L_\alpha^\theta(x)$ can be found in ([6] Mainardi F., Pagnini G. 2008):

$$L_\alpha^\theta(x) \overset{\mathcal{M}}{\longleftrightarrow} \frac{1}{\alpha} \frac{\Gamma(s)\Gamma(\frac{1-s}{\alpha})}{\Gamma(c - cs)\Gamma(1 - c + cs)}.$$

From this we can compute the "upper" fractional CHF φ_α^+, where we only integrate over $\mathbb{R}^+$. Assuming $0 < \rho < 1$ (see [6], Mainardi F., Pagnini G. 2008 for details), we have

Example

$$\varphi_\alpha^+(t) = \int_0^\infty E_\alpha(i(tx)^\alpha) L_\alpha^\theta(x)dx$$

$$= \frac{1}{2\pi i\alpha^2}\int_{\rho-i\infty}^{\rho+i\infty} \frac{\Gamma(1-\frac{s}{\alpha})\Gamma(\frac{s}{\alpha})\Gamma(\frac{s}{\alpha})}{\Gamma(cs)\Gamma(1-cs)} (e^{i\frac{\pi}{2\alpha}} t^{-2})^s ds$$

$$= \frac{1}{\alpha^2} H_{3,2}^{1,2}\left[\frac{e^{i\frac{\pi}{2\alpha}}}{t^2}\middle| \begin{matrix} (1,\frac{1}{\alpha}) & (1,\frac{1}{\alpha}) & (0,c) \\ (1,\frac{1}{\alpha}) & (0,-c) & \end{matrix}\right].$$

Fractional moments

If the fractional moments $\mathbb{E}((iX^{\alpha})^k)$ do not grow too fast, namely

$$|\mathbb{E}((iX^{\alpha})^k)| \leq c^k \Gamma(\alpha k + 1)$$

for some $c > 0$, then they appear in the fractional power series expansion of the fractional CHF:

$$\varphi_{\alpha}(t) = \sum_{k=0}^{\infty} \frac{(it^{\alpha})^k}{\Gamma(\alpha k + 1)} \int_{-\infty}^{\infty} x^{\alpha k} p(x) dx = \sum_{k=0}^{\infty} \mathbb{E}((iX^{\alpha})^k) \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)}. \quad (1)$$

This is correct for small $|t|$, under the assumption that $t \geq 0$ or that $X \geq 0$. The fractional RL integral $(I_\pm^\gamma f)(x)$ resp. derivative $(D_\pm^\gamma f)(x)$ are defined by

$$(I_\pm^\gamma f)(x) = \frac{1}{\Gamma(\gamma)} \int_0^\infty u^{\gamma-1} f(x \mp u)\, du, \quad \gamma > 0,$$

$$(D_\pm^\gamma f)(x) = \frac{d}{dx}(I_\pm^{1-\gamma} f)(x)$$

$$= \frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \int_0^\infty u^{-\gamma} f(x \mp u)\, du, \quad 0 < \gamma < 1.$$

Hilfer 1998 [7] gave the following representation of the RL fractional derivative:

$$(D_{\pm}^{\gamma} f)(x) = \frac{\gamma}{\Gamma(1-\gamma)} \int_0^{\infty} \left[f'(x \mp u) \int_u^{\infty} \frac{d\nu}{\nu^{1+\gamma}} \right] du.$$

Interchanging the order of integration, we get

$$(D_{\pm}^{\gamma} f)(x) = \frac{1}{\Gamma(-\gamma)} \int_0^{\infty} \frac{f(x \mp u) - f(x)}{u^{1+\gamma}} du. \tag{2}$$

proposition

Proposition

If $0 < \gamma < \alpha$ *and the double integral is absolutely convergent, then*

$$(I_{\pm}^{\gamma}\varphi_{\alpha})(0) = \frac{\sin(\pi\gamma)e^{i\frac{\pi\gamma}{2\alpha}}}{\alpha\sin(\frac{\pi}{\alpha}\gamma)}\mathbb{E}((\mp X)^{-\gamma})$$

and

$$(D_{\pm}^{\gamma}\varphi_{\alpha})(0) = \frac{\sin(\pi\gamma)e^{i\frac{\pi(-\gamma)}{2\alpha}}}{\alpha\sin(\frac{\pi}{\alpha}\gamma)}\mathbb{E}((\mp X)^{\gamma}).$$

For $\alpha = 1$ we get the result given by Cottone G., Di Paola M., Metzler R 2010 [8]:

$$(I_{\pm}^{\gamma}\varphi_1)(0) = (I_{\pm}^{\gamma}\varphi)(0) = e^{i\frac{\pi}{2}\gamma}\mathbb{E}((\mp X)^{-\gamma}),$$
$$(D_{\pm}^{\gamma}\varphi_1)(0) = (D_{\pm}^{\gamma}\varphi)(0) = e^{-i\frac{\pi}{2}\gamma}\mathbb{E}((\mp X)^{\gamma}).$$

$$\Gamma(\gamma)(I_{\pm}^{\gamma}\varphi_\alpha)(0) = \mathcal{M}\{\varphi_\alpha(\mp u); \gamma\}$$
$$\Gamma(-\gamma)(D_{\pm}^{\gamma}\varphi_\alpha)(0) = \mathcal{M}\{(\varphi_\alpha(\mp u) - \varphi_\alpha(0)); -\gamma\}$$

Applying the inverse Mellin transform and recalling the condition $\varphi_\alpha(0) = 1$, we obtain two representations of the fractional CHF:

$$\begin{aligned}\varphi_\alpha(-u) &= \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} u^{-\gamma}\Gamma(\gamma)(I_+^\gamma\varphi_\alpha)(0)d\gamma \\ &= \frac{1}{2\pi i\alpha}\int_{\rho-i\infty}^{\rho+i\infty} u^{-\gamma}\frac{\Gamma(\gamma)}{\sin(\pi\gamma)}\sin(\frac{\pi}{\alpha}\gamma)e^{i\frac{\pi}{2\alpha}\gamma}\mathbb{E}((-X)^{-\gamma})d\gamma\end{aligned}$$

$$\varphi_\alpha(-u) = 1 + \frac{1}{2\pi i\alpha}\int_{\rho-i\infty}^{\rho+i\infty} u^\gamma \frac{\Gamma(-\gamma)}{\sin(\pi\gamma)}\sin(\frac{\pi}{\alpha}\gamma)e^{-i\frac{\pi}{2\alpha}\gamma}\mathbb{E}((-X)^\gamma)d\gamma,$$

where $u > 0$ and the integrals are performed vertically with fixed real part ρ, belonging to the so-called fundamental strip of the Mellin transform of the function φ_α. The latter equations are integral extensions of (1).

The density function is restored in [8], with $\alpha = 1$, by using inverse Fourier transform:

$$p(x) = \frac{1}{(2\pi)^2} \int_{\rho - i\infty}^{\rho + i\infty} \Gamma(\gamma)\Gamma(1-\gamma)\{\mathbb{E}((-iX)^{-\gamma})(ix)^{\gamma-1} + \mathbb{E}((iX)^{-\gamma})(-ix)^{\gamma-1}\}d\gamma$$

Example

We will calculate the fractional CHF and MGF for a random variable with pdf $p(x) = E_\alpha(-x^\alpha)$, $0 \le \alpha \le 1$, defined by Pollard 1948 [9] and Mainardi 2014 [10]. Let $p^*(s)$ and $g^*(s)$ be the Mellin transforms of the functions $p(x) = E_\alpha(-x^\alpha)$, $x > 0$ and $g(x) = E_\alpha(ix^\alpha)$.

By application of Parseval's convolution equality ,

$$\int_0^{+\infty} p(x)g(xt)dx = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} p^*(1-s)g^*(s)t^{-s}ds,$$

we obtain

Example

$$\varphi_\alpha(t) = \int_0^\infty E_\alpha(i(tx)^\alpha)E_\alpha(-x^\alpha)dx$$

$$= \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} \frac{1}{\alpha}\frac{\Gamma(\frac{1-s}{\alpha})\Gamma(1-\frac{1-s}{\alpha})}{\Gamma(s)}\frac{1}{\alpha}\frac{\Gamma(\frac{s}{\alpha})\Gamma(1-\frac{s}{\alpha})}{\Gamma(1-s)}(-it^\alpha)^{\frac{-s}{\alpha}}t^{-s}ds$$

$$= \frac{1}{2\pi i\alpha^2}\int_{\rho-i\infty}^{\rho+i\infty} \frac{\Gamma(\frac{1-s}{\alpha})\Gamma(1-\frac{1-s}{\alpha})\Gamma(\frac{s}{\alpha})\Gamma(1-\frac{s}{\alpha})}{\Gamma(s)\Gamma(1-s)}(e^{i\frac{\pi}{2\alpha}}t^{-2})^s ds$$

$$= \frac{1}{\alpha^2}H_{3,3}^{2,2}\left[\frac{e^{i\frac{\pi}{2\alpha}}}{t^2}\middle|\begin{matrix}(\frac{1}{\alpha},-\frac{1}{\alpha}) & (0,-\frac{1}{\alpha}) & (0,1)\\ (\frac{1}{\alpha},\frac{1}{\alpha}) & (1,\frac{1}{\alpha}) & (0,-1)\end{matrix}\right],$$

Example

where $H^{2,2}_{3,3}$ is the Fox H-function. To investigate the fractional moments, we first find the fractional integral of CHF, and use Proposition 3.4 and the Mellin transform formula for the H-function: Mathai A.M., Saxena R. K., Haubold H. J [11] 2010:

$$(I^{\gamma}_{-}\varphi_{\alpha})(0) = \frac{1}{\alpha^2\Gamma(\gamma)}\int_0^{\infty} u^{\gamma-1} H^{2,2}_{3,3}\left[\frac{e^{i\frac{\pi}{2\alpha}}}{u^2}\middle| \begin{matrix} (\frac{1}{\alpha}, -\frac{1}{\alpha}) & (0,-\frac{1}{\alpha}) & (0,1) \\ (\frac{1}{\alpha}, \frac{1}{\alpha}) & (1,\frac{1}{\alpha}) & (0,-1) \end{matrix}\right] du$$

$$= \frac{e^{-i\frac{\pi\gamma}{4\alpha}}}{2\alpha^2}\,\frac{\Gamma(\frac{1}{\alpha}-\frac{\gamma}{2\alpha})\Gamma(1-\frac{1}{\alpha}-\frac{\gamma}{2\alpha})\Gamma^2(1-\frac{\gamma}{2\alpha})}{\Gamma(\gamma)\Gamma(-\frac{\gamma}{2})\Gamma(1-\frac{\gamma}{2})}.$$

Example

Hence, by last Proposition, we get

$$\mathbb{E}X^{-\gamma} = \frac{e^{-i\frac{3\pi\gamma}{4\alpha}}\sin(\frac{\pi\gamma}{\alpha})}{\alpha\pi}\frac{\Gamma(-\gamma)\Gamma(\frac{1}{\alpha}-\frac{\gamma}{2\alpha})\Gamma(1-\frac{1}{\alpha}-\frac{\gamma}{2\alpha})\Gamma^2(1-\frac{\gamma}{2\alpha})}{\Gamma^2(-\frac{\gamma}{2})}.$$

Example

We consider the Mittag-Leffler waiting time density $p(x) = x^{\alpha-1}E_{\alpha,\alpha}(-x^{\alpha})$, $x > 0$ defined in Physical Review E 51, R848 (1995), by R. Hilfer and L. Anton. Since

$$x^{\alpha-1}E_{\alpha,\alpha}(-x^{\alpha}) \stackrel{\mathcal{M}}{\longleftrightarrow} \frac{1}{\alpha}\frac{\Gamma(\frac{s+\alpha-1}{\alpha})\Gamma(1-\frac{s+\alpha-1}{\alpha})}{\Gamma(1-s)},$$

we obtain

Example

$$\varphi_\alpha(t) = \int_0^\infty E_\alpha(i(tx)^\alpha) x^{\alpha-1} E_{\alpha,\alpha}(-x^\alpha) dx$$

$$= \frac{1}{2\pi i \alpha^2} \int_{\rho-i\infty}^{\rho+i\infty} \frac{\Gamma(1-\frac{s}{\alpha})\Gamma(\frac{s}{\alpha})}{\Gamma(1-s)} \frac{\Gamma(1-\frac{s}{\alpha})\Gamma(\frac{s}{\alpha})}{\Gamma(s)} (e^{i\frac{\pi}{2\alpha}} t^{-2})^s ds$$

$$= \frac{1}{\alpha^2} H_{3,3}^{2,2}\left[\frac{e^{i\frac{\pi}{2\alpha}}}{t^2} \middle| \begin{matrix} (1,\frac{1}{\alpha}) & (1,\frac{1}{\alpha}) & (0,1) \\ (1,\frac{1}{\alpha}) & (1,\frac{1}{\alpha}) & (0,-1) \end{matrix} \right].$$

Example

Then,

$$(I_-^{\gamma}\varphi_\alpha)(0) = \frac{1}{\alpha^2\Gamma(\gamma)}\int_0^\infty u^{\gamma-1} H_{3,3}^{2,2}\left[\frac{e^{i\frac{\pi}{2\alpha}}}{u^2}\middle| \begin{matrix} (1,\frac{1}{\alpha}) & (1,\frac{1}{\alpha}) & (0,1) \\ (1,\frac{1}{\alpha}) & (1,\frac{1}{\alpha}) & (0,-1)\end{matrix}\right] du$$

$$= -e^{\frac{-i\pi\gamma}{4\alpha}} \frac{\pi^2}{\gamma\alpha^2\Gamma^2(-\frac{\gamma}{2})\sin^2\frac{\pi\gamma}{2\alpha}},$$

and hence

$$\mathbb{E}X^{-\gamma} = -e^{-i\frac{3\pi\gamma}{4\alpha}} \frac{\pi^2\cot(\frac{\pi\gamma}{2\alpha})}{\gamma\alpha\Gamma^2(-\frac{\gamma}{2})\sin(\pi\gamma)}.$$

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