Invariants of curves in conformal manifolds

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Introduction - curves in Euclidean space

$(\gamma_1(t), \gamma_2(t), \gamma_3(t))$ ... curve in $\mathbb{R}^3$

$U_1 := (\gamma_1'(t), \gamma_2'(t), \gamma_3'(t))$ ... $\sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2 + \gamma_3'(t)^2}$ velocity

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$U_i := (\gamma_1^{(i)}(t), \gamma_2^{(i)}(t), \gamma_3^{(i)}(t))$ ... $\alpha_i := \gamma_1^{(i)}(t)^2 + \gamma_2^{(i)}(t)^2 + \gamma_3^{(i)}(t)^2$
Invariants of a parametrized curve ... depend on parametrization

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Gram matrix

$$Gram(U_1, U_2, U_3) = \begin{pmatrix} \alpha_1 & \frac{1}{2}\alpha_1' & \frac{1}{2}\alpha_1'' - \alpha_2 \\ \frac{1}{2}\alpha_1' & \alpha_2 & \frac{1}{2}\alpha_2' \\ \frac{1}{2}\alpha_1'' - \alpha_2 & \frac{1}{2}\alpha_2' & \alpha_3 \end{pmatrix}$$

Linear dependence of U_1, U_2, U_3 ... relative invariants
$\Delta_i = det(Gram(U_1, \dots, U_i))$
Change of parametrization $\tilde{U}_1 = f'(t)^{-1} U_1 => \tilde{\Delta}_i = f'(t)^{-i(i+1)} \Delta_i$

Introduction - Frenet frame

If $\Delta_2 = 0$, then the curve is a line up to parametrization.
If $\Delta_3 = 0$, then the curve is a contained in a plane.
If $\Delta_3 \neq 0$, Gram–Schmidt orthogonalization
$U_1, U_2, U_3 \mapsto V_1, V_2, V_3$ ... Frenet frame

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$$\begin{pmatrix} V_1' \\ V_2' \\ V_3' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

$\kappa = \frac{\sqrt{\Delta_2}}{\sqrt{\Delta_1^3}}, \tau = \frac{\sqrt{\Delta_3}}{\Delta_2}$

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$\kappa = \frac{\sqrt{\Delta_2}}{\sqrt{\Delta_1^3}}, \tau = \frac{\sqrt{\Delta_3}}{\Delta_2}$

$\kappa = 0$ ... line
$\kappa = const, \tau = 0$ ... circle
$\kappa = const, \tau = const$ ... helix

Introduction - conformal invariants?

Metrics conformally related to Euclidean metric...
$g(U,V)(x) = \exp(2\sigma(x))U \cdot V$ for function σ
Velocity ... $\sqrt{g(U,U)} = \exp(\sigma(x))||U||$
$=>$ conformal densities of weight k ... functions depending on a choice of metric in conformal class by multiple $\exp(k\sigma(x))$
Angle of tangent vectors is independent on choice of metric in conformal class

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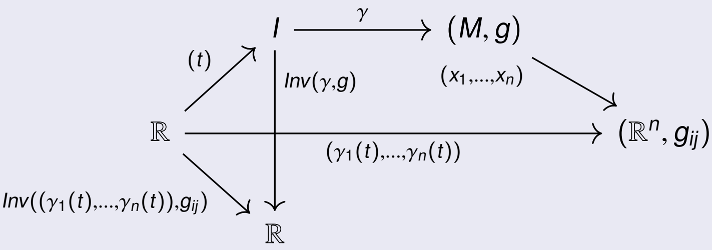
Conformal isometries ... map lines on circles and vice versa $=>$ invariants are different than in Euclidean case ... need to mix lower and higher derivatives $=>$ systematic approach?

General picture - Riemannian manifolds



Invariant = function depending on the curve γ and Riemannian manifold (M, g) that is independent on the choice of coordinates and parametrization of the curve, i.e.,
$Inv(\gamma, g)(t) = Inv((\gamma_1(t), \ldots, \gamma_n(t)), g_{ij})$

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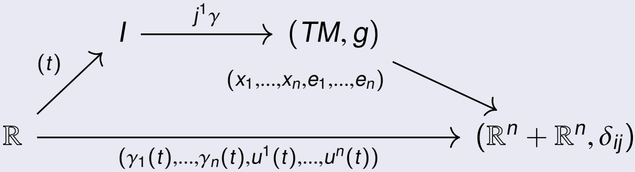

Coordinates on TM = choice of orthonormal frame $e_1, \dots, e_n$ ...
coordinates $U := (u^1(t), \dots, u^n(t))$ in the orthonormal frame $=>$
$\sqrt{u^1(t)^2 + \cdots + u^n(t)^2}$ velocity

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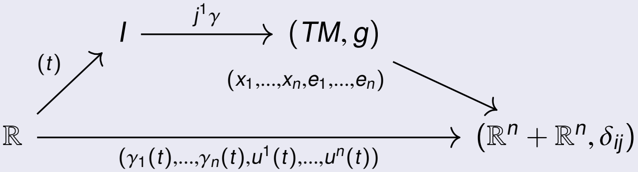

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In general, orthonormal frame is nonholonomic, i.e.,
$[e_i, e_j] = c_{ij}{}^k(x)e_k$ $=>$ Levi Civita connection
$\nabla_i(y^1, \dots, y^n) = (e_i.y^1 + \Gamma^1_{ij}y^j \dots, e_i.y^n + \Gamma^n_{ij}y^j)$ ...Christoffel symbols $\Gamma^k_{ij}e_k = \nabla_{e_i}e_j = \frac{1}{2}(c_{ij}{}^k(x) - c_j{}^k{}_i(x) + c^k{}_{ij})e_k$
$\nabla g = 0$

General picture - Frenet frame

$U' := \nabla_U U, U^{(i)} := \nabla_U U^{(i-1)}$
$U'(t) = (u^1(t)' + \Gamma^1_{ij} u^i(t) u^j(t), \dots, u^n(t)' + \Gamma^n_{ij} u^i(t) u^j(t))$
Again $\Delta_i = det(Gram(U, U', \dots, U^{(i)}))$ are relative invariants related to linear independence $U, U', \dots, U^{(i)}$.
$\Delta_i \neq 0 =>$ Gram–Schmidt orthogonalization
$U_1, \dots, U_i \mapsto V_1, \dots, V_i$ ... Frenet frame

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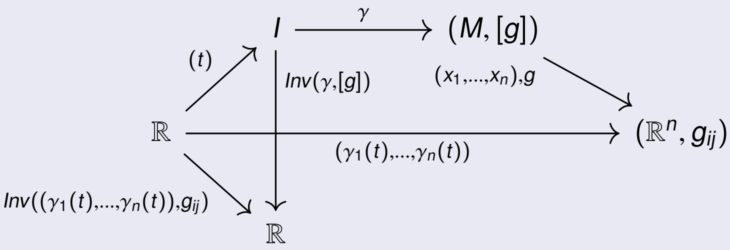
$$\begin{pmatrix} V_1' \\ \vdots \\ V_i' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_1 & 0 & \ddots & 0 \\ 0 & \ddots & 0 & \kappa_{i-1} \\ 0 & 0 & -\kappa_{i-1} & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ \vdots \\ V_i \end{pmatrix}$$

$$\kappa_1 = \frac{\sqrt{\Delta_2}}{\sqrt{\Delta_1^3}}, \kappa_j = \frac{\sqrt{\Delta_{j+1}\Delta_{j-1}}}{\Delta_j \sqrt{\Delta_1}}$$

General picture - Conformal manifolds



Invariant = function depending on the curve γ and **conformal** manifold $(M, [\mathbf{g}])$ that is independent on the choice of coordinates, **choice metric** $g \in [g]$ and parametrization of the curve, i.e., $Inv(\gamma, [g])(t) = Inv((\gamma_1(t), \dots, \gamma_n(t)), g_{ij})$

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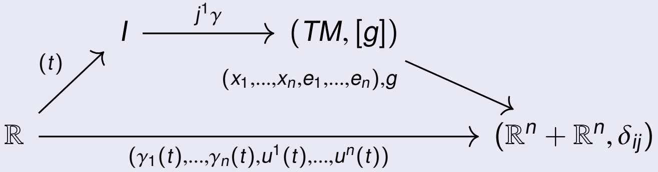

Coordinates on TM = choice of orthonormal frame $e_1, \dots, e_n$ for chosen $g \in [g]$

Change $\hat{g} = \exp(2\sigma)g$ => change of coordinates

$\hat{U} = (\exp(\sigma(\gamma(t)))u^1(t), \dots, \exp(\sigma(\gamma(t)))u^n(t))$

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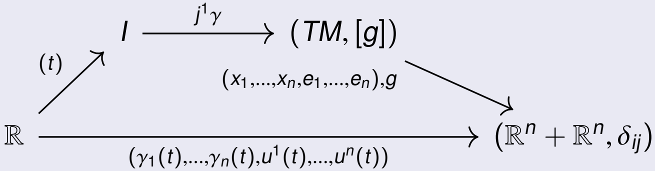

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Levi Civita connection

$\nabla^{\hat{g}}_U V = \nabla^{g}_U V + \Upsilon(U)V + \Upsilon(V)U - g(U, V)g^{-1}(\Upsilon)$, where

$\Upsilon(U) = \exp(-\sigma)\nabla^{g}_U \exp(\sigma)$

Second order frames and tractor bundle

$(e_1, \dots, e_n, \nabla^g)$ ... second order frames ... change of coordinates = Lie group $P = CO(n)\exp((\mathbb{R}^n)^*)$

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P is a (parabolic) subgroup of $SO(n+1,1)$ = stabilizer of a point in group of conformal isometries of round sphere

Second order frames and tractor bundle

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P is a (parabolic) subgroup of $SO(n+1,1)$ = stabilizer of a point in group of conformal isometries of round sphere

$=>$ P acts on $\mathbb{R}^{n+2}$, i.e., to a second order frame we assign coordinates of a (standard) tractor in $\mathbb{R}^{n+2}$:

$$\begin{pmatrix} \hat{\rho} \\ \hat{U} \\ \hat{\sigma} \end{pmatrix} = \begin{pmatrix} 1 & -\Upsilon & -\frac{1}{2}||\Upsilon||_g \\ 0 & E & g^{-1}(\Upsilon) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho \\ U \\ \sigma \end{pmatrix},$$

where $\rho, \hat{\rho}$ are conformal densities of weight -1, $U, \hat{U}$ are tangent vectors with conformal weight -1 and $\sigma, \hat{\sigma}$ are conformal densities of weight 1

Interpretation of tractor bundle and tractor connection

$\sigma := \sqrt{g(U,U)}$ is a conformal densities of weight 1, it defines a tractor

$$\begin{pmatrix} \frac{1}{n}(-\Delta^g \sigma + \frac{1}{2(n-1)} Scal^g \sigma) \\ g^{-1}(\nabla^g \sigma) \\ \sigma \end{pmatrix},$$

where Δ^g is Laplace–Beltrami operator.

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Tractor metric $\boldsymbol{g}(\begin{pmatrix} \rho_1 \\ U_1 \\ \sigma_1 \end{pmatrix}, \begin{pmatrix} \rho_2 \\ U_2 \\ \sigma_2 \end{pmatrix}) = \rho_1\sigma_2 + \sigma_1\rho_2 + g(U_1, U_2)$

Tractor connection $\boldsymbol{\nabla}_i \begin{pmatrix} \rho \\ V \\ \sigma \end{pmatrix} = \begin{pmatrix} \nabla^g_i\rho - \mathrm{P}(V, e_i) \\ \nabla^g_i V + \rho e_i + \sigma g^{-1}(\mathrm{P}(., e_i)) \\ \nabla^g_i\sigma - g(V, e_i) \end{pmatrix}$

(both independent on the second order frame), where $\mathrm{P} = \frac{1}{n-2}(Ric^g - \frac{Scal^g}{2n-2}g)$ is the Schouten tensor of g

Tractor Frenet frame - starting point

We have tractor bundle (instead of tangent bundle), with tractor connection $\boldsymbol{\nabla}$ and tractor metric $\boldsymbol{g}$ ($\boldsymbol{\nabla}\boldsymbol{g} = 0$)
$\rho := \frac{1}{\sqrt{g(U,U)}}$ is conformal density of weight -1 $=>$ starting tractor

$$\boldsymbol{T} = \begin{pmatrix} \rho \\ 0 \\ 0 \end{pmatrix}$$

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$\boldsymbol{U} := \boldsymbol{\nabla}_U \boldsymbol{T}, \boldsymbol{U}' := \boldsymbol{\nabla}_U \boldsymbol{U}, \dots$ conformally invariant objects

$Gram(\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}') = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & \alpha_1 \end{pmatrix}$ => $\Delta_3 = -1$ ... $\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}'$ lin. ind.

Δ_3 parametrization independent => $\boldsymbol{T} \wedge \boldsymbol{U} \wedge \boldsymbol{U}'$ is parametrization independent 3-tractor

α_1 conformally invariant, but depends on parametrization

$\hat{\alpha}_1 = \frac{\alpha_1 - 2S(f)}{f'(t)^2}$ ... Schwarzian derivative $S(f) = \frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2$

Conformal circles and conformal arc length

$$Gram(\boldsymbol{T},\boldsymbol{U},\boldsymbol{U}',\boldsymbol{U}'') = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\alpha_1 \\ -1 & 0 & \alpha_1 & \frac{1}{2}\alpha_1' \\ 0 & -\alpha_1 & \frac{1}{2}\alpha_1' & \alpha_2 \end{pmatrix} => \Delta_4 = \alpha_1^2 - \alpha_2 \ldots$$

if $\Delta_4 = 0$, then the curve is conformal circle ... circles and lines in the Euclidean space

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Proposition (Bailey, Eastwood)

The curve is a conformal circle if and only if there is a metric in the conformal class such that the curve is an affinely parametrized geodesic and $\mathrm{P}(U,.) = 0$.

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Depends on parametrization $\hat{\Delta}_4 = (f')^{-4}\Delta_4$ ... $\Delta_4 \leq 0 =>$ conformal arc length is parametrization for which $\Delta_4 = -1$

General Gram matrices and relative conformal invariants

Only, $\alpha_i := \boldsymbol{g}(\boldsymbol{U}^{(i)}, \boldsymbol{U}^{(i)})$ are "independent" in the Gram matrices.
Otherwise, $\boldsymbol{g}(\boldsymbol{U}^{(i)}, \boldsymbol{U}^{(i+j+1)}) = \boldsymbol{g}(\boldsymbol{U}^{(i)}, \boldsymbol{U}^{(i+j)})' - \boldsymbol{g}(\boldsymbol{U}^{(i+1)}, \boldsymbol{U}^{(i+j)})$

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Δ_i ... again related to linear independence of $\boldsymbol{T}, \boldsymbol{U}, \dots, \boldsymbol{U}^{(i-2)}$
With change of parametrization change $\hat{\Delta}_i = (f')^{-i(i-3)}\Delta_i$

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With change of parametrization change $\hat{\Delta}_i = (f')^{-i(i-3)}\Delta_i$

If $\Delta_i \neq 0$, then $\boldsymbol{V}_0 := \boldsymbol{T}, \boldsymbol{V}_1 := \boldsymbol{U}, \boldsymbol{V}_2 := -\boldsymbol{U}' - \frac{1}{2}\alpha_1 \boldsymbol{T}$ are subbundle with nondegenerate metric of signature $(2, 1)$ => it is possible to do Gram-Schmidt orthogonalization in the orthogonal complement $\boldsymbol{U}^{(2)}, \dots, \boldsymbol{U}^{(i-2)} \mapsto \boldsymbol{V}_3, \dots, \boldsymbol{V}_{i-1}$ ... tractor Frenet frame

$$Gram(\boldsymbol{V}_0, \dots, \boldsymbol{V}_{i-1}) = \left(\begin{array}{ccc|cc} & & 1 & & \\ & 1 & & & \\ 1 & & & & \\ \hline & & & 1 & \\ & & & & \ddots \end{array}\right)$$

Conformal curvatures

$$
\left(\begin{array}{c} \boldsymbol{V}_0' \\ \boldsymbol{V}_1' \\ \boldsymbol{V}_2' \\ \hline \boldsymbol{V}_3' \\ \boldsymbol{V}_4' \\ \dots \\ \boldsymbol{V}_{i-1}' \end{array}\right)
=
\left(\begin{array}{ccc|cccc}
 & 1 & & & & & \\
K_1 & & -1 & & & & \\
 & -K_1 & & -1 & & & \\
\hline
1 & & & & K_2 & & \\
 & & & -K_2 & & \ddots & \\
 & & & & \ddots & & K_{i-3} \\
 & & & & & -K_{i-3} &
\end{array}\right)
\left(\begin{array}{c} \boldsymbol{V}_0 \\ \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \\ \hline \boldsymbol{V}_3 \\ \boldsymbol{V}_4 \\ \dots \\ \boldsymbol{V}_{i-1} \end{array}\right)
$$

Conformal curvatures

$$\begin{pmatrix} V_0' \\ V_1' \\ V_2' \\ \hline V_3' \\ V_4' \\ \dots \\ V_{i-1}' \end{pmatrix} = \left(\begin{array}{ccc|ccccc} & 1 & & & & & & \\ K_1 & & -1 & & & & & \\ & -K_1 & & -1 & & & & \\ \hline 1 & & & & K_2 & & & \\ & & & -K_2 & & \ddots & & \\ & & & & \ddots & & & K_{i-3} \\ & & & & & -K_{i-3} & & \end{array}\right) \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ \hline V_3 \\ V_4 \\ \dots \\ V_{i-1} \end{pmatrix}$$

i up to $n+2 => n-1$ conformal curvatures for generic curve
$K_1 = -\frac{1}{2}(-\Delta_4)^{-\frac{5}{2}}(\alpha_1 \Delta_4^2 - \frac{1}{2}\Delta_4\Delta_4'' + \frac{9}{16}(\Delta_4')^2)$, $K_j = \frac{\sqrt{\Delta_{j+1}\Delta_{j+3}}}{-\Delta_{j+2}\sqrt[4]{-\Delta_4}}$
In conformal arc length parametrization $K_1 = -\frac{1}{2}\alpha_1$.

Conformal invariants of Euclidean space

g ... Euclidean metric on $\mathbb{R}^3$ ... $U = (\gamma_1', \gamma_2', \gamma_3')$, $\nabla^g_U = \frac{d}{dt}$, $\mathrm{P} = 0$

$$\boldsymbol{\nabla}_U \begin{pmatrix} \rho \\ V \\ \sigma \end{pmatrix} = \begin{pmatrix} \frac{d}{dt}\rho \\ \frac{d}{dt} V + \rho U \\ \frac{d}{dt}\sigma - V \cdot U \end{pmatrix}, \boldsymbol{T} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$ ... arc length parametrization

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$$\boldsymbol{U} = \begin{pmatrix} 0 \\ \gamma_1' \\ \gamma_2' \\ \gamma_3' \\ 0 \end{pmatrix}, \boldsymbol{U}' = \begin{pmatrix} 0 \\ \gamma_1'' \\ \gamma_2'' \\ \gamma_3'' \\ -1 \end{pmatrix}, \boldsymbol{U}'' = \begin{pmatrix} 0 \\ \gamma_1''' \\ \gamma_2''' \\ \gamma_3''' \\ 0 \end{pmatrix}, \boldsymbol{U}''' = \begin{pmatrix} 0 \\ \gamma_1'''' \\ \gamma_2'''' \\ \gamma_3'''' \\ (\gamma_1'')^2 + (\gamma_2'')^2 + (\gamma_3'')^2 \end{pmatrix}$$

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$$\boldsymbol{U} = \begin{pmatrix} 0 \\ \gamma_1' \\ \gamma_2' \\ \gamma_3' \\ 0 \end{pmatrix}, \boldsymbol{U}' = \begin{pmatrix} 0 \\ \gamma_1'' \\ \gamma_2'' \\ \gamma_3'' \\ -1 \end{pmatrix}, \boldsymbol{U}'' = \begin{pmatrix} 0 \\ \gamma_1''' \\ \gamma_2''' \\ \gamma_3''' \\ 0 \end{pmatrix}, \boldsymbol{U}''' = \begin{pmatrix} 0 \\ \gamma_1'''' \\ \gamma_2'''' \\ \gamma_3'''' \\ (\gamma_1'')^2 + (\gamma_2'')^2 + (\gamma_3'')^2 \end{pmatrix}$$

$\alpha_1 = (\gamma_1'')^2 + (\gamma_2'')^2 + (\gamma_3'')^2, \alpha_2 = (\gamma_1''')^2 + (\gamma_2''')^2 + (\gamma_3''')^2, \alpha_3 = (\gamma_1'''')^2 + (\gamma_2'''')^2 + (\gamma_3'''')^2$

Conformal invariants of Euclidean space

$$Gram(\boldsymbol{T},\ldots,\boldsymbol{U}''') = \begin{pmatrix} 0 & 0 & -1 & 0 & \alpha_1 \\ 0 & 1 & 0 & -\alpha_1 & -\frac{3}{2}\alpha_1' \\ -1 & 0 & \alpha_1 & \frac{1}{2}\alpha_1' & \frac{1}{2}\alpha_1'' - \alpha_2 \\ 0 & -\alpha_1 & \frac{1}{2}\alpha_1' & \alpha_2 & \frac{1}{2}\alpha_2' \\ \alpha_1 & -\frac{3}{2}\alpha_1' & \frac{1}{2}\alpha_1'' - \alpha_2 & \frac{1}{2}\alpha_2' & \alpha_3 \end{pmatrix}$$

$\Delta_4 = \alpha_1^2 - \alpha_2, \Delta_5 = -\alpha_2\alpha_3 + \frac{1}{4}(\alpha_2')^2 - \alpha_1'\alpha_1\alpha_2' - \alpha_1\alpha_2\alpha_1'' + 2\alpha_2^2\alpha_1 + \alpha_1^2\alpha_3 + \alpha_1^3\alpha_1'' - 3\alpha_1^3\alpha_2 + \frac{9}{4}(\alpha_1')^2\alpha_2 - \frac{5}{4}(\alpha_1')^2\alpha_1^2 + \alpha_1^5$
$K_1 = -\frac{1}{2}(-\Delta_4)^{-\frac{5}{2}}(\alpha_1\Delta_4^2 - \frac{1}{2}\Delta_4\Delta_4'' + \frac{9}{16}(\Delta_4')^2), K_2 = \frac{\sqrt{-\Delta_5}}{-\Delta_4\sqrt[4]{-\Delta_4}}$