Ranking nodes in signed networks: an algebraic perspective

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Complex networks

Complex networks are ubiquitous:

- Social networks
- Biological networks
- Computer networks
- Communication networks
- Electrocal networks
- et cetera ...

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Complex networks

Complex networks are closely related to *graphs*.

However, while graphs are *deterministic* objects, complex networks are mostly studied using *statistical* methods.

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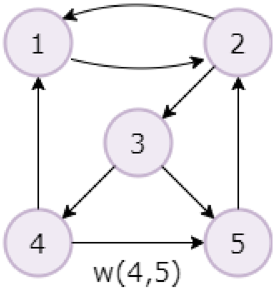
Graphs

A *weighted graph* is described by a tuple

$$\Gamma = (V, E, w),$$

where $V = \{1, \ldots, N\}$ is the set of vertices,
$E \subset V \times V$ the set of edges,
$w : E \to \mathbb{R} \setminus \{0\}$ the weighting function.

A weighted graph is said to be *unsigned* if the weighting function maps to $\mathbb{R}_+$ and *signed* if w maps to $\mathbb{R} \setminus \{0\}$.



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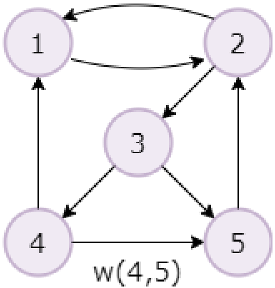

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Degree distribution in complex networks

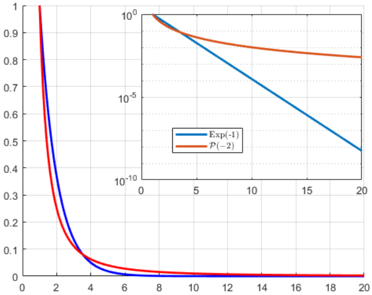
The degree distribution of most **real** complex networks is described by a *power law*:

$$X \sim \mathcal{P}(k): \quad f(x) = ax^{-k}, \;\; k > 1.$$

In most cases, $2 < k < 3$.

Some properties:

- The mean is defined for $k > 2$.
- Finite variance for $k > 3$.
- Heavy tail, finite probability of extreme events.



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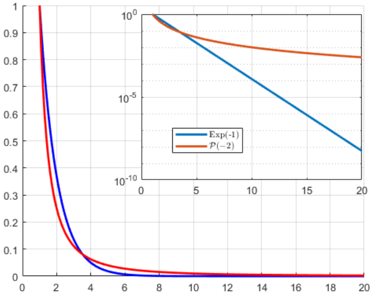

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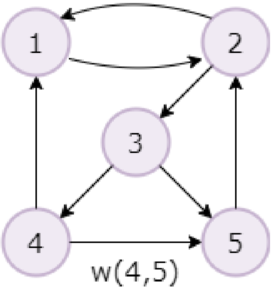
Algebraic description

A weighted graph Γ is uniquely described by its adjacency matrix A.

$$a_{ij} = \begin{cases} w(j,i), & (i,j) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

For the graph shown in Fig.:

$$\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$



Elementary algebraic operations

Define $\mathbf{1} = [1, 1, \ldots, 1]^T$ to be the column vector of all ones. Then,

$$\mathbf{d}^{out} = \mathbf{1}^T \cdot \mathrm{A}$$
$$\mathbf{d}^{in} = \mathrm{A} \cdot \mathbf{1}.$$

Furthermore, we define the square matrices $\mathrm{D}^{out} = \mathrm{diag}(\mathbf{d}^{out})$ and $\mathrm{D}^{in} = \mathrm{diag}(\mathbf{d}^{in})$.

Ranking of nodes in a graph

The problem of ranking consists in determining the relative weight or importance of each node in a network based upon the structure of node's connections.

We distinguish two characteristics of a vertex: a **score** and a *rank*.

The scoring function $s : V \to \mathbb{R}$ assigns to each node a real value that characterizes its importance.

The ranking function $r : V \to \mathbb{N}$ assigns to each node its position in the hierarchy (rank of the most important node equals 1).

Scoring function: properties

We will concentrate on considering scores. Any particular scoring scheme is required to fulfill some natural properties:

1. Be uniquely defined.
2. Generate non-negative scores.

Problem: there is no *ground truth*.

Possible directions:

- Axiomatic characterization of scoring (ranking).
- General properties: existence, uniqueness, convergence.
- Specific properties: behavior under structured perturbations, etc.

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Intuition behind the scores

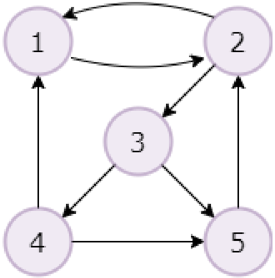
Let s_i be the score of the ith node. Suppose that each node "shares" its score with all nodes, to which it has outbound connections.

For instance, $s_1 = s_2 + s_4$, $s_2 = s_1 + s_5$, etc.

Using the previously defined adjacency matrix, these relations can be written succinctly as

$$\mathrm{A}\mathbf{s} = \mathbf{s}, \qquad (*)$$

where $\mathbf{s}$ is the column vector of individual scores.



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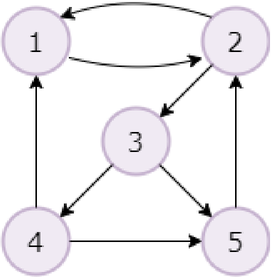

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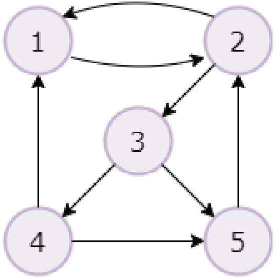
First remedy: normalization

To overcome this difficulty, we assume that the score is being split equally between all outbound nodes.

Thus, we modify the adjacency matrix by dividing the weights of outgoing arcs by the respective out-degrees:

$$A_{norm} = A\left(D^{out}\right)^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$



A_{norm} is a *column stochastic matrix*.

The modified problem

$$A_{norm}\mathbf{s} = \mathbf{s}$$

always has a solution (why?), which is referred to as the *eigenvector centrality score*.

Eigenvector centrality: pro and contra

The eigenvector centrality scheme is used in many applications.

However, it has several drawbacks:

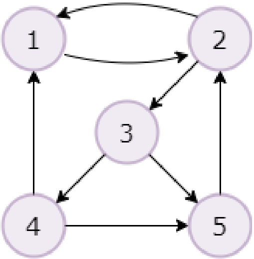
- Can yield multiple scores (= not unique)
- Is not defined if there are "dangling" nodes (i.e., nodes without outgoing connections).

We need a mathematical instrument to analyse such problems in a systematic way.

The main tool is the Perron-Frobenius th. and its friends, e.g., Gershgorin's circle th.

⋮

Eigenvector centrality has a unique and well defined solution if the underlying directed graph is connected.



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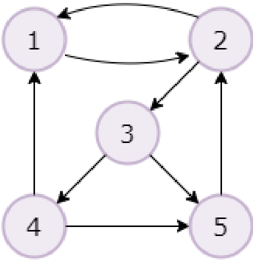

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Further improvement: PageRank

The problem with EC is that the matrix A_{norm} is only *non-negative*. Hence, it has to satisfy some additional properties (be irreducible $\Leftrightarrow$ graph be connected).

Let's approach this problem from a different side and make the matrix positive! Define

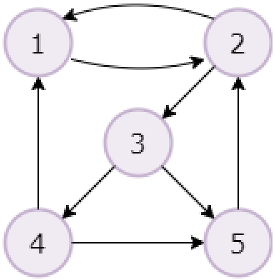
$$A_{PR} = \alpha A_{norm} + (1-\alpha)\frac{1}{n}J,$$

where $n = |V|$, $J = \mathbf{1} \cdot \mathbf{1}^\top$, and $\alpha \in (0,1)$ ($\alpha = 0.85$).

The PageRank score is defined as the solution to

$$A_{PR}\,\mathbf{s} = \mathbf{s}.$$

This solution always exists and is unique.



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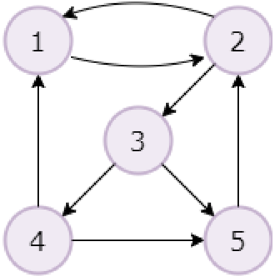
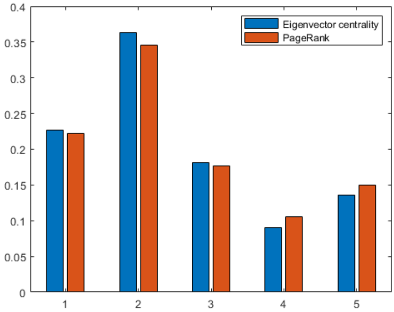
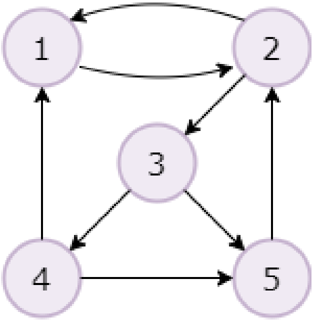


Comparison: PageRank vs. Eigenvector centrality





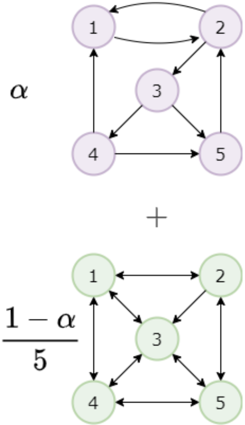
Interpretation: PageRank

The PageRank algorithm can be neatly interpreted in terms of a Markov chain. Consider the transition matrix

$$A_{\mathrm{PR}} = \alpha A_{norm} + (1-\alpha)\frac{1}{n}J,$$

The elements of A_{norm} describe the probability of moving from node i to node $j \in OUT(i)$, while the second term describes the probabilities of a random transition from node i to a random node with equal probability.

The PageRank score is thus the *stationary distribution* of the considered MC.



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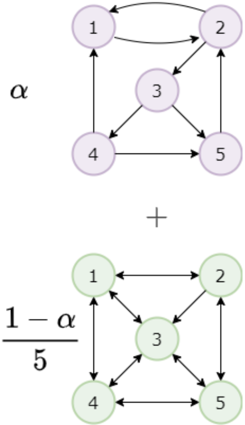

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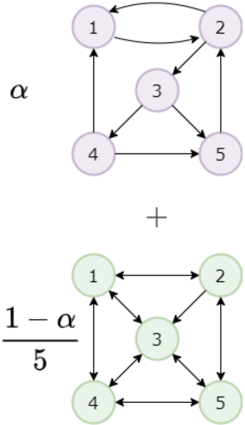

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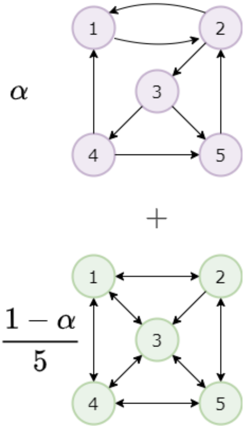

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An alternative approach: HITS

In the HITS algorithm, a node i is associated with two values: a non-negative *authority score* u_i, and a non-negative *hub score* h_i.

The hub score describes the property of a node to be connected to authoritative nodes.

The authority score indicates that the node is pointed at by many hubs.

Formally, this can be written as

$$u_i = \sum_{j \in IN(i)} w(j,i) h_j \quad \text{and} \quad h_i = \sum_{j \in OUT(i)} w(i,j) u_j$$

or, in a vector-matrix form, as

$$\mathbf{u} = \mathrm{A}\mathbf{h} \quad \text{and} \quad \mathbf{h} = \mathrm{A}^\top \mathbf{u}.$$

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HITS: the problem and the remedy

In $\mathbf{u} = AA^\top \mathbf{u}$, AA^T is a row Gramian matrix. It has many nice properties, but, in general, $1 \notin \Lambda(AA^T)$. Hence, there is no immediate solution to this problem.

In the original work, the vectors $\mathbf{u}$ and $\mathbf{h}$ were normalized during the computation:

$$\mathbf{u} = \frac{A\mathbf{h}}{\|A\mathbf{h}\|} \quad \text{and} \quad \mathbf{h} = \frac{A^\top \mathbf{u}}{\|A^\top \mathbf{u}\|}.$$

We consider a different scheme, in which the respective vectors are normalized to stochastic ones:

$$\mathbf{u} = A\mathbf{h}\,(\mathbf{1}^\top A\mathbf{h})^{-1} \quad \text{and} \quad \mathbf{h} = A^\top \mathbf{u}\,(\mathbf{1}^\top A^\top \mathbf{u})^{-1}.$$

By eliminating $\mathbf{h}$ from the above we get an expression for $\mathbf{u}$:

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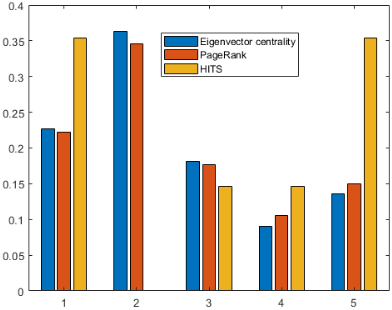
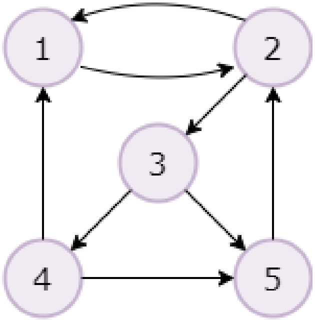
PageRank vs. Eigenvector centrality vs. HITS





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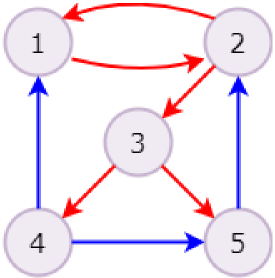
Signed graphs

In unsigned graphs, the adjacency matrix describes the structure of communications between the nodes.

In signed graphs, there is a two-level informational hierarchy: the communication structure + the information about the relation between the vertices: either positive or negative.



This implies that we cannot any longer consider the signed network as an inert medium, navigated by some "surfer", but rather as an interconnection of active actors.

Exponential ranking

The idea of ER consists in computing two score-like vectors:

- the stochastic vector of probabilities (trust probabilities), and
- the true scoring vector (reputation values).

The probability vector $\mathbf{p}_{\mathrm{EXP}}$ results from the fixed point equation

$$\mathbf{p} = \frac{\exp\left(\frac{1}{\mu}A\mathbf{p}\right)}{\left\|\exp\left(\frac{1}{\mu}A\mathbf{p}\right)\right\|_1} \tag{EXP}$$

The scoring vector is defined as $\mathbf{s}_{\mathrm{EXP}} = A\mathbf{p}_{\mathrm{EXP}}$.

Note that the r.h.s. of (EXP) is the *soft-max* function, which has several useful properties: 1) it is monotone; 2) it maps $\mathbb{R}$ to $(0, 1)$. The latter makes it ideal for computing probabilities from unstructured data.

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Theorem (structured perturbation)

Let $\mathrm{A} \in \mathbb{R}^{N \times N}$ be a signed adjacency matrix and $\mathbf{p}_{\mathrm{EXP}} \in \Sigma$ be the solution of (EXP). The solution of (EXP) remains invariant with respect to the following linear transformation:

$$\mathrm{A}' = \mathrm{A} + \mathrm{B} \tag{1}$$

where $\mathrm{B} = \mathbf{1}\mathbf{b}^\top$, and $\mathbf{b}$ is a non-zero vector, $\mathbf{b} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$. Furthermore, the transformation (1) preserves the ranking, i.e., $r_{\mathrm{A}}(i) = r_{\mathrm{A}'}(i)$ for all $i \in V$.

This result implies that one can uniformly shift each individual's evaluation of everyone else and the overall ranking is preserved.

Furthermore, this implies that there is no substantial difference between the negative and positive edges, given that all edges can be made either positive or negative by an appropriate choice of the values of $\mathbf{b}$.

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Quasi exponential ranking

The main drawback of exponential ranking is that it is non-linear.

Therefore, it was suggested[1] to use a (partially) linearized scheme (EXP), which we call *quasi exponential ranking*.

$$\mathbf{1}^\top \left[\mathbf{1} + \frac{1}{\mu}\mathrm{A}\mathbf{p}\right]\mathbf{p} = \mathbf{1} + \frac{1}{\mu}\mathrm{A}\mathbf{p}. \tag{qEXP}$$

This scheme has a unique solution $\mathbf{p}^*$ if $\mu > \max_{a_{ij}\le 0} |(a_{ij})|$ (which agrees with (EXP)). Furthermore, $\mathbf{p}^*$ is the eigenvector of $\frac{1}{\mu}\mathrm{A} + \mathrm{J}$ corresponding to the spectral radius $\rho\left(\frac{1}{\mu}\mathrm{A} + \mathrm{J}\right)$.

[1] Gromov D., Evmenova E. On the Exponential Ranking and Its Linear Counterpart (2022) Studies in Computational Intelligence, 1015, pp. 260 - 270.

Quasi exponential ranking: comparison

The following result remains valid for (EXP) and (qEXP). Recall that $\mathbf{s}^* = A\mathbf{p}^*$ is considered to be the *true* scoring vector.

Let A be a signed adjacency matrix and $\mathbf{p}^*$ be the vector of probabilities. Both $\mathbf{p}^*$ and $\mathbf{s}^* = A\mathbf{p}^*$ correspond to the same ranking of the vertices.

Comparison of ranking schemes

- Spearman's footrule.

$$\overline{\rho} = \sum_{i=1}^{n} |R(s_{1i}) - R(s_{2i})|$$

- Spearman's rank correlation coefficient.

$$\rho = \frac{\operatorname{cov}(\mathrm{R}(X), \mathrm{R}(Y))}{\sigma_{\mathrm{R}(X)}\sigma_{\mathrm{R}(Y)}},$$

- Kendall's τ: number of permutations needed to achieve the same order.
- Bar-Ilan measure: an extention of Spearman's footrule based upon the idea that top nodes are more important then the nodes in the bottom.

$$\sum_{i=1}^{n} \left(\frac{1}{R(s_{1i})} - \frac{1}{R(s_{2i})} \right)$$

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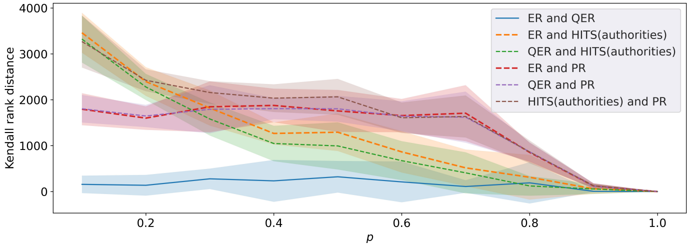
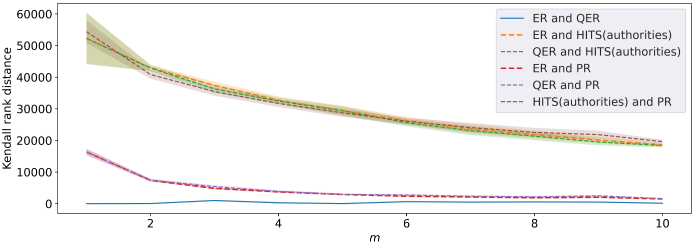
Quasi exponential ranking: numerical comparison

Erdős-Rényi graph with 500 nodes and parameter $p \in \{0.1, 0.2, ..., 1\}$:

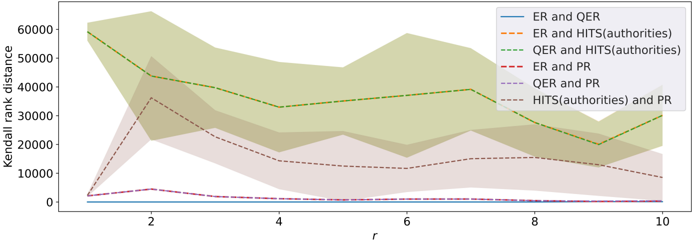


Barabási–Albert graph with 500 nodes and parameter $m \in \{1, 2, ..., 10\}$:



Quasi exponential ranking: numerical comparison

R-ary Tree graph with 500 nodes and parameter $r \in \{1, 2, ..., 10\}$:



Conclusion

- The problem of network ranking is a fascinating subject with many unsolved problems.
- It can be analysed using algebraic, computational, and statistical methods.
- Its results are of great practical importance.

Paldies!

Děkuji vám!