Value of Information from the Optimal Control Perspective

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Outline

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Value of Information: an informal description

Value of Information allows for evaluating the profit from obtaining new information against the current level of information. It builds upon:

- The information structure of the model, particularly the model uncertainties and their characterization.
- Data acquisition process aimed at reducing the model uncertainties and acquiring new information.
- Utility and cost functions, used to evaluate the profit, resp., expenses associated with procuring new information.

Hence, VoI is located at the intersection of information theory, statistical modeling and decision theory, data acquisition and optimization, resp., optimal control theory.

Value of Information: questions to answer

Some questions, the value of information theory seeks to answer:

- Does additional data provide additional information?
- What do we gain from the additional piece information?
- Which information is more valuable?
- Should we collect more information now, or it is better to wait?
- Which data should be collected for a given budget (time, resources)?
- What quality standards has the data to meet in order to be worth their evaluation?
- … etc.

Value of Information: a more formal description

Given the state $x \in \mathscr{X}$, the action (control) $a \in \mathscr{A}$, and the utility function $u : \mathscr{X} \times \mathscr{A}$, we may distinguish between two cases:

Perfect information: in this case, the decision-maker knows the exact value of the state and can implement an optimal response $a^*(x) = \text{argmax}_{a \in \mathscr{A}} u(x, a)$.

No information: the decision maker does not have information about the state and hence maximizes the average profit based upon their prior estimation of the probability distribution of s: $f(s)$.

Thus, the expected value of perfect information is defined as

$$EVPI = \int_{\mathscr{X}} \max_{a \in \mathscr{A}} u(x, a) f(x) \mathrm{d}x - \max_{a \in \mathscr{A}} \int_{\mathscr{X}} u(x, a) f(x) \mathrm{d}x.$$

Value of Information: information source

Now, assume that instead of exact information, the DM gets messages $y \in \mathscr{Y}$ from a certain information service (IS). The messages are related to the states by the joint probability distribution $f(x,y)$ (of which $f(x)$ and $f(y)$ are the marginal distribution).

Now, the expected value of the information service $\mathscr{Y}$ is

$$EVIS = \int_{\mathscr{Y}} \left[\max_{a \in \mathscr{A}} \int_{\mathscr{X}} u(x,a) f(x|y) \mathrm{d}x \right] f(y) \mathrm{d}y$$
$$- \max_{a \in \mathscr{A}} \int_{\mathscr{X}} u(x,a) f(x) \mathrm{d}x.$$

In contrast to the previous case, the estimation of the information structure of the IS plays a crucial role, as wrong assumptions may *distort* the information.

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Value of Information: Bayesian updating

Suppose that we carried out a preposterior analysis based upon some previously collected data. The decision can be improved by performing Bayesian updating of the probability of x based upon the collected information:

$$p(x|t) = \frac{f(x)g(t|x)}{h(t)},$$

where $g(t|x)$ is the likelihood function of observing t given a state x, and $h(t)$ is the predictive density of t:

$$h(t) = \int_{\mathscr{X}} f(x)g(t|x)dx.$$

The value of updating is thus

$$EVU = \int_{\mathscr{T}} \max_{a\in\mathscr{A}} \left[\int_{\mathscr{X}} u(x,a)p(x|t)\mathrm{d}x \right] h(t)\mathrm{d}t - \max_{a\in\mathscr{A}} \int_{\mathscr{X}} u(x,a)f(x)\mathrm{d}x.$$

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Initial value estimation in non-renew. resource extraction

Exploitation of a non-renewable resource is governed by[1]

$$\dot{x}(t) = -\sum_{i\in N} k_i u_i(t), \quad x(t_0) = x_0,$$

where $x(t)$ is the stock of the resource, $u_i(t)$ is the extraction effort of player i, and k_i converts the ith player's effort into the extraction intensity.

The objective function of the ith player is defined as

$$J_i(x_0, u_1, \dots, u_n) = \int_{t_0}^{T} e^{-rt} \frac{u_i^{1-\mu}(t)}{1-\mu} dt.$$

Let the initial stock x_0 be known only to be within $[x_1, x_2]$. How to find the optimal value of x_0?

[1] Tur, A.; Gromova, E.; Gromov, D. On the Estimation of the Initial Stock in the Problem of Resource Extraction. Mathematics 2021

Optimal solution

The optimal state trajectory is

$$\bar{x}(t) = x_0 + \frac{x_0 \left(e^{\frac{-rt}{\mu}} - e^{\frac{-rt_0}{\mu}} \right)}{\left(e^{\frac{-rt_0}{\mu}} - e^{\frac{-rT}{\mu}} \right)},$$

and the optimal value of the total payoff equals

$$J(x_0, \bar{u}) = \frac{x_0^{1-\mu}}{1-\mu} \left(\frac{K\mu}{r} \right)^{\mu} \left(e^{\frac{-rt_0}{\mu}} - e^{\frac{-rT}{\mu}} \right)^{\mu}.$$

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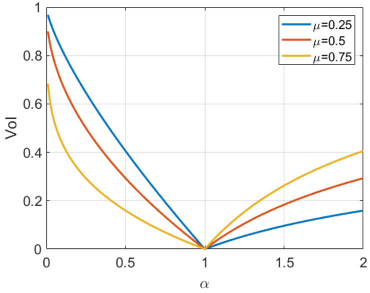
Optimal solution

Based upon the evaluation of the payoff function for different initial (estimated) conditions, the value of information about the initial resource stock is computed as

$$VoI^C = \frac{J(\widehat{x}_0) - J(x_0)}{J(x_0)} = \frac{\widehat{x}_0^\mu - x_0^\mu}{\widehat{x}_0^\mu} = 1 - \left(\frac{x_0}{\widehat{x}_0}\right)^\mu,$$

where $\widehat{x}_0$ is the estimation of the initial value.



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Optimal estimation

We wish to determine the optimal estimation for x_0, but we have no information beyond the interval $[x_1, x_2]$. So, solve the min max optimization problem

$$\min_{\hat{x}_0 \in [x_1, x_2]} \max_{x_0 \in [x_1, x_2]} \left(\sum_{i \in N} J(x_0, \bar{u}) - \sum_{i \in N} J(x_0, \widehat{u}) \right),$$

where $\hat{x}_0$ is the guess of the initial condition, and x_0 is the actual value thereof. Furthermore, $\bar{u}$ is the true optimal control, while $\hat{u}$ is the optimal control computed for an estimation.

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Fortunately, for this problem a nice analytic solution exists. The best estimate is

$$x_0^* = \operatorname{argmin}_{\widehat{x}_0} \left(\max\{(x_1^{1-\mu} - x_1 \widehat{x}_0^{-\mu}), (x_2^{1-\mu} - \widehat{x}_0^{1-\mu})\} \right).$$

Theorem

The optimal estimate of the initial stock x_0^* is given by the solution of

$$x_1^{1-\mu} - x_1 (x_0^*)^{-\mu} = x_2^{1-\mu} - (x_0^*)^{1-\mu}.$$

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Extracting money from probability

Let X be a random variable defined on $[0,1]$ with PDF $m(x)$. This variable enumerates the continuum of investment options. Let $v(x)$ be the investment profile. The return on investments is defined as

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Assume that the we know "real" PDF of the market $g(x) \neq m(x)$ and we want to make money on that knowledge. Also assume that we can invest the amount of money corresponding to $I = 1$. So, we have the following optimization problem:

$$\text{maximize } J = \int_0^1 g(x)v(x)dx$$
$$s.t.\ I = \int_0^1 m(x)v(x)dx = 1.$$

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Extracting money from probability

The straightforward application of variational calculus does not work, and a detailed analysis bring us outside of the realm of real functions. Also, we encounter a problem similar to the St. Petersburg paradox.
So, we reformulate the problem as follows:

$$\text{maximize } J = \int_0^1 g(x)U(v(x))dx$$
$$s.t.\ I = \int_0^1 m(x)v(x)dx = 1,$$

where $U(\cdot)$ is the utility function.

Extracting money from probability: logarithmic utility function

For $U(v) = \ln(v)$, the optimal investment is

$$v^*(x) = \frac{g(x)}{m(x)}$$

and

$$J = \int_0^1 g(x) \ln\left(\frac{g(x)}{m(x)}\right) dx = D_{KL}(g \| m),$$

which is the Kullback-Leibler relative entropy of $g(x)$ with respect to $m(x)$.

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Extracting money from probability: general isoelastic utility function

In the general case

$$U(v)=\frac{v^{1-\eta}-1}{1-\eta},\quad \eta\in[0,1), \tag{1}$$

the optimal investment is

$$v^*(x)=\left[\int_0^1 m(x)\left(\frac{g(x)}{m(x)}\right)^{\frac{1}{\eta}}dx\right]^{-1}\left(\frac{g(x)}{m(x)}\right)^{\frac{1}{\eta}}$$

and

$$\begin{aligned}J&=\frac{1}{1-\eta}\int_0^1 g(x)\left\{\left[\int_0^1 m(x)\left(\frac{g(x)}{m(x)}\right)^{\frac{1}{\eta}}dx\right]^{-1}\left(\frac{m(x)}{g(x)}\right)^{-\frac{1}{\eta}}\right\}^{1-\eta}dx-\frac{1}{1-\eta}\\&=\frac{1}{1-\eta}\left[\int_0^1 m(x)\left(\frac{g(x)}{m(x)}\right)^{\frac{1}{\eta}}dx\right]^{\eta}-\frac{1}{1-\eta}.\end{aligned}$$

Note that for $g(x)\equiv m(x)$ we have $J=0$.

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Paldies!

Děkuji vám!

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