Transposed Poisson algebras

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Poisson type algebras

- $\mathbb{V}$ is a complex vector space (finite-dimensional or inf. dim).
- $-\cdot-$ — associative commutative multiplication.
  $x\cdot y = y\cdot x, \ \ (x\cdot y)\cdot z = x\cdot(y\cdot z)$.
- $[-,-]$ — Lie multiplication.
  $[x,y]=-[y,x], \ \ [[x,y],z]+[[y,z],x]+[[z,y],x]=0$.

- $[x, y\cdot z]=[x,y]\cdot z + y\cdot[x,z]$: Poisson algebras
- $[x, y\cdot z]=[x,y]\cdot z + y\cdot[x,z]+\mathfrak{D}(x)\cdot y\cdot z$: generalized Poisson algebras (=contact brackets; $\mathfrak{D}$ is a derivation of $(\mathbb{V},\cdot)$)
- $2x\cdot[y,z]=[x,y\cdot z]+[y,x\cdot z]$: transposed Poisson algebras[a]

[a] Bai C., Bai R., Guo L., Wu Y., Transposed Poisson algebras, Novikov-Poisson algebras, and 3-Lie algebras, arXiv:2005.01110

Examples

Poisson algebras

- Any associative algebra $(\mathfrak{L}, \cdot)$ is a Poisson algebra with respect to $[x, y] = x \cdot y - y \cdot x$.
- Any commutative associative algebra $(\mathfrak{L}, \cdot)$ admitting a pair of commuting (under $\circ$) derivations d_1 and d_2 is a Poisson algebra with respect to $[x, y] = d_1(x) \cdot d_2(y) - d_1(y) \cdot d_2(x)$.

Transposed Poisson algebras, generalized Poisson algebras.

- Any associative commutative algebra $(\mathfrak{L}, \cdot)$ with a derivation $\mathfrak{D}$ is a transposed Poisson algebra (and a generalized Poisson algebra) with respect to $[x, y] = x \cdot \mathfrak{D}(y) - y \cdot \mathfrak{D}(x)$.

Intersections

Poisson, transposed Poisson, generalized Poisson algebras

- Each Poisson algebra is a generalized Poisson algebra ($\mathfrak{D} = 0$)
- (Poisson algebras) $\cap$ (t-Poisson algebras) $=$ ($[x \cdot y, z] = [x, y] \cdot z = 0$)
- Each unital transposed Poisson algebra is a generalized Poisson algebra ($[x, y] = x \cdot \mathfrak{D}(y) - y \cdot \mathfrak{D}(y)$)

Novikov-Poisson algebras

Let $(\mathfrak{L}, \cdot)$ be an associative commutative algebra and $(\mathfrak{L}, \circ)$ be a Novikov algebra, i.e.

$$(x \circ y) \circ z = (x \circ z) \circ y, \ (x \circ y) \circ z - x \circ (y \circ z) = (y \circ x) \circ z - y \circ (x \circ z).$$

Then $(\mathfrak{L}, \cdot, \circ)$ is a Novikov-Poisson algebra iff

$$(x \cdot y) \circ z = x \cdot (y \circ z), \ (x \circ y) \cdot z - x \circ (y \cdot z) = (y \circ x) \cdot z - y \circ (x \cdot z).$$

Example

Let $(\mathfrak{L}, \cdot)$ be an associative commutative algebra with a derivation $\mathfrak{D}$.
Then $(\mathfrak{L}, \circ)$, where $x \circ y = x \cdot \mathfrak{D}(y)$ is a Novikov algebra;
and $(\mathfrak{L}, \cdot, \circ)$ is a Novikov-Poisson algebra.

Let $(\mathfrak{L}, \cdot, \circ)$ is a Novikov-Poisson algebra, then $(\mathfrak{L}, \cdot, [-,-])$, where $[x, y] = x \circ y - y \circ x$ is a transposed Poisson algebra.

Tensor product

Let $(\mathfrak{L}_1, \cdot_1, [-,-]_1)$ and $(\mathfrak{L}_2, \cdot_2, [-,-]_2)$ are Poisson algebras. Then $(\mathfrak{L}_1 \otimes \mathfrak{L}_2, \cdot, [-,-])$, where

$$(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) = (a_1 \cdot_1 b_1) \otimes (a_2 \cdot_2 b_2),$$
$$[a_1 \otimes a_2, b_1 \otimes b_2] = (a_1 \cdot_1 b_1) \otimes [a_2, b_2]_2 + [a_1, b_1]_1 \otimes (a_2 \cdot_2 b_2)$$

is a new Poisson algebra.

- Transposed Poisson algebras: Yes!
- Novikov-Poisson algebras: Yes![a]
- Generalized Poisson algebras: No![b]

[a]Xu X., On simple Novikov algebras and their irreducible modules, J. Algebra 905–934.

[b]Zusmanovich P., On contact brackets on the tensor product, Linear and Multilinear Algebra 70 (2022), 19, 4695-4706.

Extensions to fields of fractions

Denote by $\mathcal{Q}(\mathfrak{P})$ the field of fractions of the unital commutative associative algebra $\mathfrak{P}$.

- The Poisson bracket $\{-,-\}$ on $\mathfrak{P}$ can be extended to a Poisson bracket on $\mathcal{Q}(\mathfrak{P})$[a] and
$$\left\{\frac{a}{b},\frac{c}{d}\right\}=\frac{\{a,c\}\cdot b\cdot d-\{a,d\}\cdot b\cdot c-\{b,c\}\cdot a\cdot d+\{b,d\}\cdot a\cdot c}{b^2\cdot d^2}.$$
- The transposed Poisson bracket $[-,-]$ on $\mathfrak{P}$ can be extended to a transposed Poisson bracket on $\mathcal{Q}(\mathfrak{P})$[b] and
$$\left[\frac{a}{b},\frac{c}{d}\right]=\frac{[a,b]\cdot c\cdot d-a\cdot b\cdot[c,d]}{b^2\cdot d^2}.$$

[a] Makar-Limanov L., Shestakov I., Polynomial and Poisson dependence in free Poisson algebras and free Poisson fields, Journal of Algebra, 349 (2012), 372–379.

[b] Beites P., Ferreira B., Kaygorodov I., Transposed Poisson structures, arXiv:2207.00281

Relations with other structures

Jordan superalgebras [Kantor Double]

Let $(\mathfrak{L}, \cdot, [-,-])$ be a unital generalized Poisson algebra.
Then $(\mathfrak{L} \oplus \overline{\mathfrak{L}}, \bullet)$ is a Jordan superalgebra.

$$a \bullet b = a \cdot b,\ a \bullet \overline{b} = \overline{a \cdot b},\ \overline{a} \bullet b = \overline{a \cdot b},\ \overline{a} \bullet \overline{b} = [a, b].$$

$(\mathfrak{L}, \cdot, [-,-])$ is simple iff $(\mathfrak{L} \oplus \overline{\mathfrak{L}}, \bullet)$ is simple.

3-Lie algebras

Let $(\mathfrak{L}, \cdot, [-,-])$ be a transposed Poisson algebra with a derivation $\mathfrak{D}$.

$$[x, y, z] = \mathfrak{D}(x)[y, z] + \mathfrak{D}(y)[z, x] + \mathfrak{D}(z)[x, y].$$

Then $(\mathfrak{L}, [-,-,-])$ is a 3-Lie algebra, i.e. $[-,-,-]$ is anticommutative and

$$[[x, y, z], t, w] = [[x, t, w], y, z] + [x, [y, t, w], z] + [x, y, [z, t, w]].$$

Identities

Associative algebra case

The Amitsur–Levitsky theorem states that the algebra $\mathbb{M}_k$ of $k \times k$ matrices satisfies the identity $s_{2k} = 0$.

$$s_m(x_1, \ldots, x_m) = \sum_{\sigma \in \mathbb{S}_m} (-1)^\sigma x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(m)}.$$

Every associative PI algebra satisfies $(s_k)^t = 0$ by Amitsur's theorem.

Poisson algebra case

Every Poisson PI algebra satisfies some customary identity[a]

$$\sum_{\sigma \in \mathbb{S}_{2m}} c_\sigma \{x_{\sigma(1)}, x_{\sigma(2)}\} \cdot \ldots \cdot \{x_{\sigma(2m-1)}, x_{\sigma(2m)}\} = 0.$$

[a] Farkas D., Poisson polynomial identities, Communications in Algebra, 26 (1998), 2, 401–416.

Identities

Unital generalized Poisson algebra case

$$\sum_{i=0}^{[m/2]} \sum_{\sigma\in\mathbb{S}_m} c_{\sigma,i} \langle x_{\sigma(1)}, x_{\sigma(2)}\rangle \cdot \ldots \cdot \langle x_{\sigma(2i-1)}, x_{\sigma(2i)}\rangle \cdot \mathfrak{D}(x_{\sigma(2i+1)}) \cdot \ldots \cdot \mathfrak{D}(x_{\sigma(m)}) = 0,$$

$$\text{where } \langle x, y\rangle := \{x, y\} - \big(\mathfrak{D}(x)\cdot y - x\cdot\mathfrak{D}(y)\big).$$ [a]

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[a] Kaygorodov I., Algebras of Jordan brackets and generalized Poisson algebras, Linear and Multilinear Algebra, 65 (2017), 6, 1142–1157.

Unital transposed Poisson algebra case

$$\sum_{\sigma\in\mathbb{S}_m} c_\sigma \mathfrak{D}(x_{\sigma(1)}) \cdot \ldots \cdot \mathfrak{D}(x_{\sigma(m)}) = 0.$$

Associative commutative algebra case with derivations

$$\mathfrak{D}(x_1) \cdot \ldots \cdot \mathfrak{D}(x_m) = 0.$$ [a]

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[a] Dotsenko V., Ismailov N., Umirbaev U., Polynomial identities in Novikov algebras, Mathematische Zeitschrift, 303 (2023), 3, 60.

Transposed Poisson algebras: Relations

Let $(\mathfrak{L}, \cdot, [-,-])$ be a transposed Poisson algebra. Then

- the right multiplication $R_z : x \to x \cdot z$ in $(\mathfrak{L}, \cdot)$ gives a $\frac{1}{2}$-derivation of the Lie algebra $(\mathfrak{L}, [-,-])$. ϕ is a $\frac{1}{2}$-derivation of $(\mathfrak{L}, [-,-])$ if
$$\phi[x,y] = \frac{1}{2}([\phi(x), y] + [x, \phi(y)]).$$
- the right multiplication $R_z : x \to x \cdot z$ in $(\mathfrak{L}, \cdot)$ gives a quasi-automorphism of the Lie algebra $(\mathfrak{L}, [-,-])$. ϕ is a quasi-automorphism of $(\mathfrak{L}, [-,-])$ if there is a linear mapping ψ, such that
$$\psi[x,y] = [\phi(x), \phi(y)].$$
- the right multiplication $R_z : x \to x \cdot z$ in $(\mathfrak{L}, \cdot)$ gives a Hom-Lie structure on the Lie algebra $(\mathfrak{L}, [-,-])$. ϕ is a Hom-Lie structure on $(\mathfrak{L}, [-,-])$ if
$$[\phi(x), [y,z]] + [\phi(y), [z,x]] + [\phi(z), [x,y]] = 0.$$

Transposed Poisson [bla,bla,bla] algebras

Let $\mathfrak{L} = \mathfrak{L}_0 \oplus \mathfrak{L}_1$ be a $\mathbb{Z}_2$-graded vector space equipped with two nonzero bilinear super-operations $\cdot$ and $[-,-]$. The triple $(\mathfrak{L}, \cdot, [-,-])$ is called a transposed Poisson superalgebra if $(\mathfrak{L}, \cdot)$ is a supercommutative associative superalgebra and $(\mathfrak{L}, [-,-])$ is a Lie superalgebra that satisfies the condition: $2z \cdot [x, y] = [z \cdot x, y] + (-1)^{|x||z|}[x, z \cdot y],\ x, y, z \in \mathfrak{L}_0 \cup \mathfrak{L}_1$.

Let $n \geq 2$ be an integer. A transposed Poisson n-Lie algebra is a triple $(\mathfrak{L}, \cdot, [-, \dots, -])$ where $(\mathfrak{L}, \cdot)$ is a commutative associative algebra and $(\mathfrak{L}, [-, \dots, -])$ is an n-Lie (Filippov) algebra satisfying the following condition: $n\ w \cdot [x_1, \dots, x_n] = \sum_{i=1}^{n} [x_1, \dots, w \cdot x_i, \dots, x_n]$.

Hom[a]- and BiHom[b]-versions

[a]Laraiedh I., Silvestrov S., Transposed Hom-Poisson and Hom-pre-Lie Poisson algebras and bialgebras, arXiv:2106.03277

[b]Ma T., Li B., Transposed BiHom-Poisson algebras, Communications in Algebra, 51 (2023), 2, 528–551.

Poisson/t-Poisson algebra structures

Poisson algebra structures

Let $(\mathfrak{L}, \cdot)$ be an associative (or associative commutative) algebra (resp. $(\mathfrak{L}, [-,-])$ be a Lie algebra). A Poisson algebra structure on $\mathfrak{L}$ is a bilinear multiplication $[\cdot,\cdot]$ (resp. $\cdot$) on $\mathfrak{L}$ such that $(\mathfrak{L}, \cdot, [\cdot,\cdot])$ is a Poisson algebra.

- Prime noncommutative associative algebras[a]
- Canonical algebras[b]
- Finitary incidence algebras[c]

[a] Farkas D., Letzter G., Ring theory from symplectic geometry. J. Pure Appl. Algebra 125 (1998), no. 1-3, 155–190.

[b] Jaworska-Pastuszak A., Pogorzały Z., Poisson structures for canonical algebras. J. Geom. Phys. 148 (2020), 103564, 15 pp.

[c] Kaygorodov I., Khrypchenko M., Poisson structures on finitary incidence algebras, J. Algebra 578 (2021), 402–420.

Poisson/t-Poisson algebra structures

Transposed Poisson algebra structures

Let $(\mathfrak{L}, \cdot)$ be an associative (or associative commutative) algebra (resp. $(\mathfrak{L}, [-,-])$ be a Lie algebra). A transposed Poisson algebra structure on $\mathfrak{L}$ is a bilinear multiplication $[-,-]$ (resp. $\cdot$) on $\mathfrak{L}$ such that $(\mathfrak{L}, \cdot, [\cdot,\cdot])$ is a transposed Poisson algebra.

- Unital associative algebras[a]: $[x, y] = x \cdot \mathfrak{D}(y) - y \cdot \mathfrak{D}(x)$.

[a] Beites P., Ferreira B., Kaygorodov I., Transposed Poisson structures, arXiv:2207.00281

Transposed Poisson algebra structures on Lie algebras

First results

- Semisimple finite-dimensional Lie algebras: Trivial[a].
- Simple finite-dimensional Lie superalgebras: Trivial[a].
- Simple finite-dimensional n-Lie superalgebras: Trivial[a].
- Witt algebra: first non-trivial (interesting) example[a].

$$\{e_i\}_{i\in\mathbb{Z}} : [e_i, e_j] = (i-j)e_{i+j};\ e_i \cdot e_j = e_{i+j};$$
$$\text{(other t-Poisson structures } x \star_\omega y = \omega \cdot x \cdot y)$$

- Virasoro algebra: Trivial[a].

$$[e_m, e_n] = (m-n)e_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}c.$$

[a] Ferreira B., Kaygorodov I., Lopatkin V., 1/2-derivations of Lie algebras and transposed Poisson algebras,Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 115 (2021), 3, 142.

Transposed Poisson algebra structures on Lie algebras

Solvable and nilpotent cases

- Any nilpotent finite-dimensional Lie algebra has a non-trivial TPAS[a].
- Any solvable finite-dimensional Lie algebra has a non-trivial TPAS[b].

[a] Beites P., Ferreira B., Kaygorodov I., Transposed Poisson structures, arXiv:2207.00281

[b] Kaygorodov I., Lopatkin V., Zhang Z., Transposed Poisson structures on Galilean and solvable Lie algebras, Journal of Geometry and Physics, 187 (2023), 104781.

$Lie = (semisimple) \ltimes (solvable)$

- Galilean Lie algebras: trivial.
- $\mathcal{S}_n = (\mathfrak{sl}_2 \oplus \mathfrak{so}_n) \ltimes \mathfrak{h}_n$: non-trivial ($n = 2$, $\mathcal{S}_2^2 \neq \mathcal{S}_2$), trivial ($n \neq 2$)[a].
- $\mathfrak{sl}_2 \ltimes \mathbb{C}^3$: non-trivial ($(\mathfrak{sl}_2 \ltimes \mathbb{C}^3)^2 = \mathfrak{sl}_2 \ltimes \mathbb{C}^3$; Khudoyberdiyev A.).

[a] Yang Y., Tang X., Khudoyberdiyev A., Transposed Poisson structures on Schrödinger algebra in $(n+1)$-dimensional space-time, arXiv:2303.08180.

TPAS on Lie algebras $\mathcal{B}(q)$

Lie algebras $\mathcal{B}(q)$

Let $q \in \mathbb{C}$. Define $\mathcal{B}(q)$ to be the Lie algebra with a basis $\{e_{m,i} \mid m, i \in \mathbb{Z}\}$ and the following multiplication table

$$[e_{m,i}, e_{n,j}] = (n(i+q) - m(j+q))e_{m+n,i+j}.$$

Transposed Poisson structures

If $q \notin \mathbb{Z}$, then all the transposed Poisson structures on $(\mathcal{B}(q), [\cdot,\cdot])$ are trivial.
If $q \in \mathbb{Z}$, then, up to an isomorphism, there is only one non-trivial transposed Poisson structure $\cdot$ on $(\mathcal{B}(q), [\cdot,\cdot])$ given[a] by

$$e_{0,-2q} \cdot e_{0,-2q} = e_{0,-q}.$$

[a] Kaygorodov I., Khrypchenko M., Transposed Poisson structures on Block Lie algebras and superalgebras, Linear Algebra and Its Applications, 656 (2023), 167–197.

TPAS on Witt type Lie algebras $V(f)$

Witt type Lie algebra $V(f)$

Let Γ be an abelian group and $f : \Gamma \to \mathbb{C}$ be a map. The Witt type Lie algebra $V(f)$ is a vector space with basis $\{e_\alpha\}_{\alpha\in\Gamma}$ and multiplication

$$[e_\alpha, e_\beta] = (f(\beta) - f(\alpha))e_{\alpha+\beta}.$$

In particular, $G = \mathbb{Z}$ and $f(n) = n$ give the classical Witt algebra.

Transposed Poisson structures

The Witt type Lie algebra $V(f)$ has non-trivial transposed Poisson algebra structures.[a]

[a] Kaygorodov I., Khrypchenko M., Transposed Poisson structures on Witt type algebras, Linear Algebra and its Applications, 665 (2023), 196–210.

TPAS on generalized Witt algebras

Generalized Witt algebras $W(A, V, \langle \cdot, \cdot \rangle)$

Let $(A, +)$ be a non-trivial abelian group; $V \neq \{0\}$ a vector space; $\langle \cdot, \cdot \rangle : V \times A \to \mathbb{C}$ linear in the 1st variable, additive in the 2nd one; $W := \mathbb{C}A \otimes_{\mathbb{C}} V$ and

$$[a \otimes v, b \otimes w] = (a + b) \otimes (\langle v, b \rangle w - \langle w, a \rangle v).$$

Transposed Poisson structures

- If $\dim(V) > 1$: trivial.[a]
- If $\dim(V) = 1$: non-trivial.[a]

[a] Kaygorodov I., Khrypchenko M., Transposed Poisson structures on generalized Witt algebras and Block Lie algebras, arXiv:2302.00403

Low dimensional transposed Poisson algebras

2-dimensional transposed Poisson algebras[a]:

- $e_1 \cdot e_1 = e_1, \quad e_2 \cdot e_2 = e_2$
- $e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = e_2$
- $e_1 \cdot e_1 = e_1$
- $e_1 \cdot e_1 = e_2$
- $e_1 \cdot e_1 = e_2, \quad [e_1, e_2] = e_2$
- $e_1 \cdot e_2 = e_1, \quad e_2 \cdot e_2 = e_2, [e_1, e_2] = e_2$
- $e_1 \cdot e_1 = \lambda e_1, \quad e_1 \cdot e_2 = \lambda e_2, \quad [e_1, e_2] = e_2$

[a]Bai C., Bai R., Guo L., Wu Y., Transposed Poisson algebras, Novikov-Poisson algebras, and 3-Lie algebras, arXiv:2005.01110

3-dimensional transposed Poisson algebras[a]: 30 types
[12: non-isomorphic, trivial; 18: non-trivial (8 one-parameter families, 1 two-parameter family and 9 separated non-isomorphic algebras)]

[a]Beites P., Fernandez Ouaridi A., Kaygorodov I., The algebraic and geometric classification of transposed Poisson algebras, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 117 (2023), 2, 55

Special algebras

Lie algebras

$(A, \cdot)$ — associative $\to$ $(A, [[x, y]] = x \cdot y - y \cdot x)$ — Lie algebra.

Each Lie algebra can be embedded in an associative algebra under $[[-, -]]$.

Jordan algebras

$(A, \cdot)$ — associative $\to$ $(A, x \circ y = \frac{1}{2}(x \cdot y + y \cdot x))$ — Jordan algebra.

If a Jordan algebra can be embedded in associative algebra under $- \circ -$, then it is special. There are non-special Jordan algebras.

Special transposed Poisson algebras?

Special and $\mathfrak{D}$-special tranposed Poisson algebras

If a transposed Poisson algebra is isomorphic to a Novikov-Poisson algebra $(\mathfrak{L}, \cdot, \circ)$ under $[[x, y]] = x \circ y - y \circ x$, then it is special.

If a transposed Poisson algebra is isomorphic to an associative commutative algebra $(\mathfrak{L}, \cdot)$ with a derivation $\mathfrak{D}$ under $x \circ y = x \cdot \mathfrak{D}(y)$, then it is $\mathfrak{D}$-special.

- There is non-$\mathfrak{D}$-special 2-dimensional transposed Poisson algebra.
- All complex 2-dimensional transposed Poisson algebras are special.
- There is non-special 3-dimensional transposed Poisson algebra. [a]

[a] Beites P., Fernandez Ouaridi A., Kaygorodov I., The algebraic and geometric classification of transposed Poisson algebras, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 117 (2023), 2, 55

For open questions, see:
Beites P., Ferreira B., Kaygorodov I., Transposed Poisson structures, arXiv:2207.00281

Děkuji za Vaši pozornost!