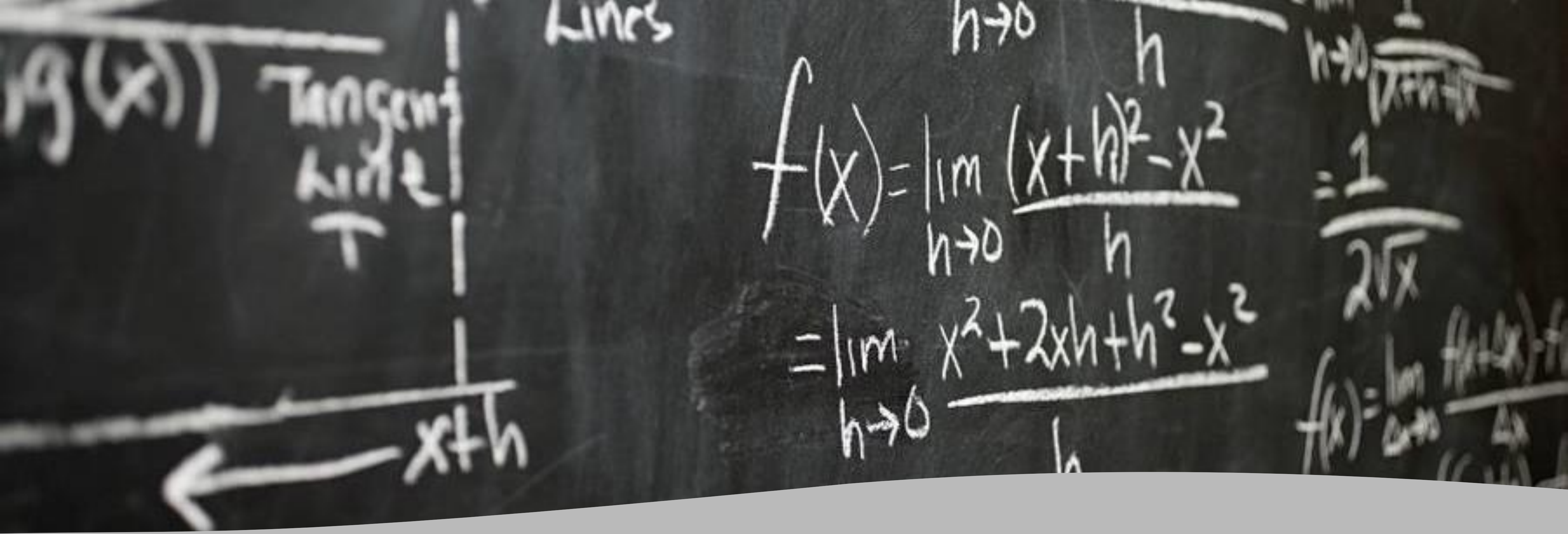


SDEs for Studying Species Competition

Introduction

Rapid Growth in Stochastic Modelling:

Applications of SDEs have cross-disciplinary reach used in biological modelling, addressing issues such as population dynamics, drug kinetics, and the spread of epidemics.

Modelling thermal noise in electrical circuits, and economical processes like stock prices.

Advantages Over Deterministic Models:

Due to the ability of SDEs to model systems with inherent randomness capturing random fluctuations over time, especially those that are unpredictable.

Impact in Specific Fields:

Stochastic modelling has significantly impacted biology, with valuable findings in neuroscience, environmental sciences, telecommunications, quantum field theory, and finance where understanding random fluctuations is crucial.

Stochastic Process	Stochastic Differential Equation	Euler Method	LV Model	Stochastic LV Model
Markov Chain Brownian Motion	Applications Definition Ito's Formula and Mathematical Framework	Theoretical form and Origin. Simulation by the Euler – Maruyama method.	Lotka-Volterra Predator Prey Model	Stochastic Model Lotka-Volterra Predator Prey Model

The factors that cause a species to go extinct or co-exit are crucial in population biology .Three different models considering avg. value of noise plus error to get to the best approximate comprehension in comparison to real life.

Process:

1. A process is an event that evolved over time intending to achieve a goal.
2. Generally the time is from 0 to T
3. During this time events may be happening at various points along the way that may have an effect on the eventual value of the process.

Stochastic Process:

1. A process that can be described by the change of some random variable over time, which maybe discrete or continuous.

Random Walk:

1. A stochastic process that starts off with a score of 0.
2. At each discrete event (fixed points in time) there is a probability chance p of increase score by (+1) and a (1-p) chance of decrease score by 1.
3. The event happens T times. 0+T(p+(1-p)(-1))= (2p-1) T, p=0.75 (20times) then the value = 10.

Markov Process:

1. Particular type of stochastic process where only the present value of a variable is relevant for predicting the future.
2. The history of the variable and the way that the present has emerged from the past is irrelevant.

Martingale Process:

1. A stochastic process where at any time t the expected value of final value is the current value.
2. Formula $E[X\_T|\ X\_t= x] = x$ (expected final value is the value at the current time).Example a random walk with p = 0.5.
3. All Martinagles are Markovian.

Brownian Motion

Significance of Wiener Process:

1. Wiener process, also known as Brownian Motion, is a crucial continuous-time stochastic process.
2. Originating from the observation of pollen grains' motion by the botanist Robert Brown, it has become a fundamental building block in complex models.
3. Introduced mathematically by Norbert Wiener in 1923, it is a key component in quantitative finance, especially in the Black-Scholes model.

Standard Brownian Motion Definition: A stochastic process, $\{W_t: 0 \leq t \leq \infty\}$ is a standard Brownian motion if :

1. $W_0 = 0$
2. It has continuous sample paths (UNLIKE random walk discrete-time Markov process)
3. It has independent normally distributed increments.

Brownian Motion

It is a Gaussian process, and any continuous-time stochastic process with independent increments and finite second moments is also a Gaussian process by the CLT.

A Wiener Process W_t has continuous sample paths and independent normally distributed increments with distribution $W_t - W_s \sim N(0, t-s)$. If $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2$ then $W_{t1} - W_{s1}$ and $W_{t2} - W_{s2}$ are independent random variables.

Simulation and Properties of Brownian Motion:

1. Simulation of Brownian motion involves discretization and random variables following a normal distribution.
2. Brownian motion is continuous everywhere but nowhere differentiable, exhibiting fractal characteristics.
3. Properties, such as the autocovariance function, continuous sample paths, and variance proportional to elapsed time, distinguish Brownian motion in continuous time.

Wiener Process with a drift

Consider **dx = a dt + b dW(t)**

dx the rate of change some variable x

adt models rate that grows with time (exact)

dW(t) is the derivative of Wiener process (implying randomness)

bdW(t) is a variable that might go up or down where b is the magnitude of this volatility

Where a and b (makes the randomness bigger or smaller) are constants

The dx= adt can be integrated to x = x0 + at , where x0 is the initial value and then if the time period is T , the variable increases by aT.

bdW(t) accounts for the noise or variability to the path followed by x. The amount of this noise or variablility is b times a Wiener process

Stochastic Differential Equations

Differential form and Integral form:

- The n-dimensional SDE has the form

  $dX_t = a(X_t,t)\, dt + b(X_t,t)\, dW_t$, $X_0 = x_0$.

- The integral form of an SDE is represented by $X_t = x_0 + \int a(X_s,s)ds + \int b(X_s,t)dW_s$ over the interval $0 \le t \le T$.
- Ito's Lemma supposes that the value of the variable x follows Ito's process dx=a(x,t)+b(x,t)dwt where a and b are functions of x and t where the drift rate is a and the variance is b^2.
- Then any function G of x and t follows the process of Ito's formula.

Ito's Formula and Mathematical Framework:

- $dG = (\partial G/ \partial x\, a + \partial G/ \partial t + 1/2\, \partial 2G/(dx)2\, b\hat{}2)dt + (\partial G/ \partial x)b\, dW$
- And thus G also follows an Ito process with Ito drift and variance $((\partial G/ \partial x)b\,)\hat{}2b\hat{}2$
- It involves F(t)-adapted integrable and square integrable processes, providing a differential expression for functions involving stochastic processes.
- SDE is a differential equation with one or more terms represented by stochastic processes, where the solution is also a stochastic process.

Geometric Brownian Motion

Application to Linear Stochastic Differential Equation (SDE):

- $dX(t) = \lambda X(t)\, dt + \mu X(t)\, dW(t)$,

- $\lambda X(t)$ drift coeff. term with λ *expected return rate (growth)*
- $\mu X(t)$ stochastic component with μ *volatility coeff. (standard deviation of the variables returns)*
- $dW(t)$ wp increment (random fluctuations)
- $\lambda X(t)dt$ = *deterministic growth*
- $\mu X(t)dW(t)$ = random fluctuations

- *Using Ito's Lemma we solve to get the solution:* $X(t) = X(0)\, \exp(\lambda - 1/2\mu\text{^}2)t + \mu W(t)$
- Stochastic process following geometric Brownian motion
- Effective drift with deterministic growth and volatility induced drift.
- The equation describes the dynamics of the variable over time under the Black Scholes framework

Euler Method

Euler–Maruyama Method for the Approximate Numerical Solution of a SDE:

- Numerical technique = is an extension of the Euler method for ODEs to SDEs

  $dX_t = a(X_t,t)\,dt + b(X_t,t)\,dW_t$, the with initial condition $X_0 = x_0$.

Suppose we want to solve the SDE on some interval of time $[0,T]$ using discrete time steps. Then the EM approx. to the true solution is the Markov chain Y defined as follows:

- Partition the interval $[0,T]$ into N equal subintervals of width $\Delta t > 0$
- $0 = \tau_0 < \tau_1 < \ldots\ldots\ldots < \tau_N = T$, $\Delta t = T/N$
- Set Y0 = X0
- Recursively define Y_n for $0 \leq n \leq N-1$ by

  $Y_{n+1} = Y_n + a(Y_n, \tau_N)\,\Delta t + b(Y_n, \tau_N)\,\Delta W_n$

  $\Delta W_n = W(\tau_{n+1}) - W(\tau_n)$.

The random variables ΔW_n are independent and identically distributed normal random variables with expected mean zero and variance Δt

Simulation by Euler-Maruyama Method

Implementation and Parameter Setting:

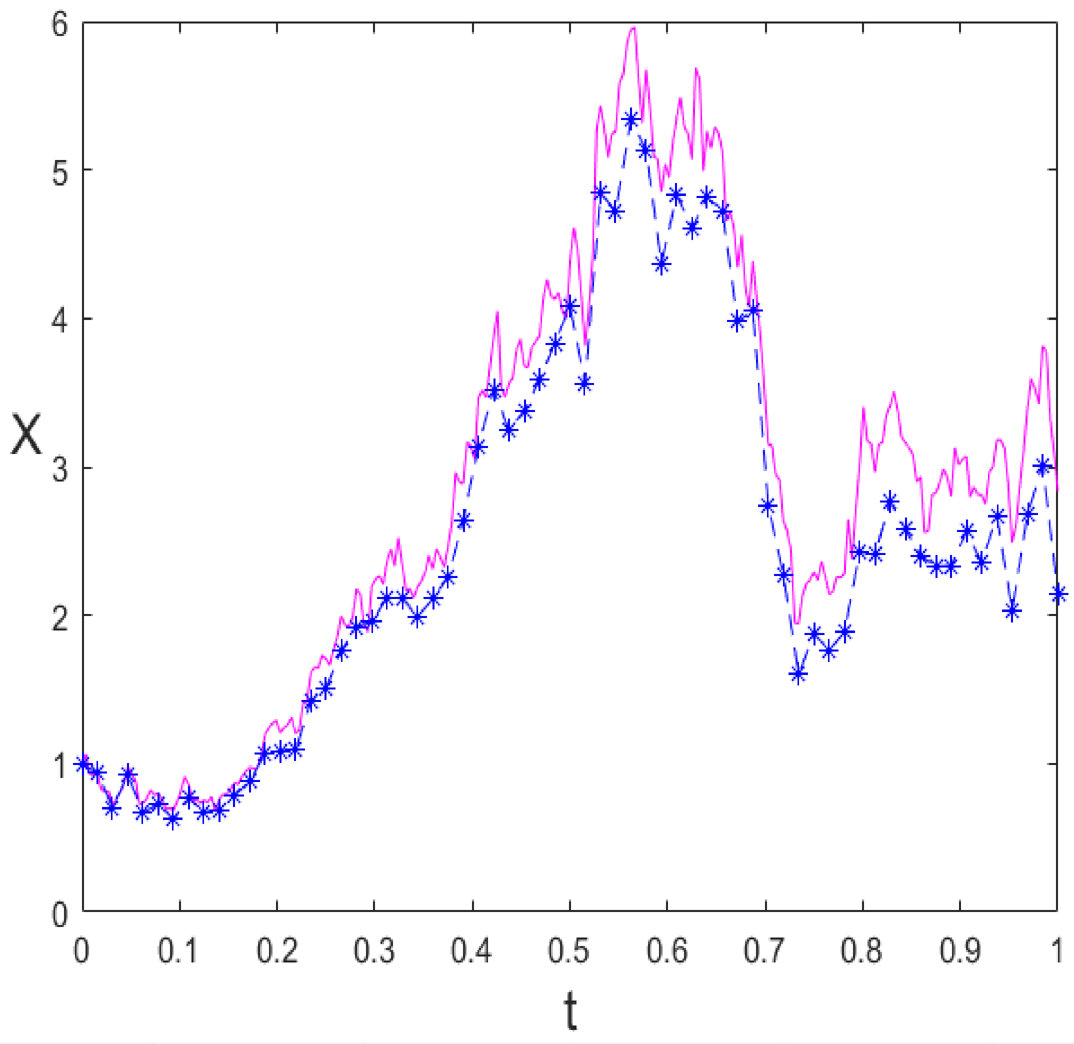
- Euler Method is applied with a step size $\Delta t=R\delta t$, where $R=4$.
- The SDE considered has parameters $\lambda=2$, $\mu=1$, and initial condition $X(0)=1$.
- The discretized Brownian path over the interval [0,1] is computed with a step size of $\delta=2^{-8}$, and the true solution *X*true is evaluated.

Comparison and Error Analysis

- Discrepancy between the exact solution (*X*true) and the Euler–Maruyama (EM) solution at the endpoint $t=T$ is computed as the error (err), yielding a value of 0.6907.

- Varying the step size $\Delta t=R\delta t$ with smaller R values of 2 and 1 results in reduced endpoint errors of 0.1595 and 0.0821, respectively.

- The comparison and error analysis highlight the sensitivity of the Euler–Maruyama method to the chosen step size, influencing the accuracy of the numerical approximation.



Lotka-Volterra Predator Prey Model

LV Deterministic

Illustrations depict a phase portrait and time series, showing cyclic dynamics of prey and predator populations over time.

Describes the dynamics between prey (x) and predator (y) populations with the system of equations :

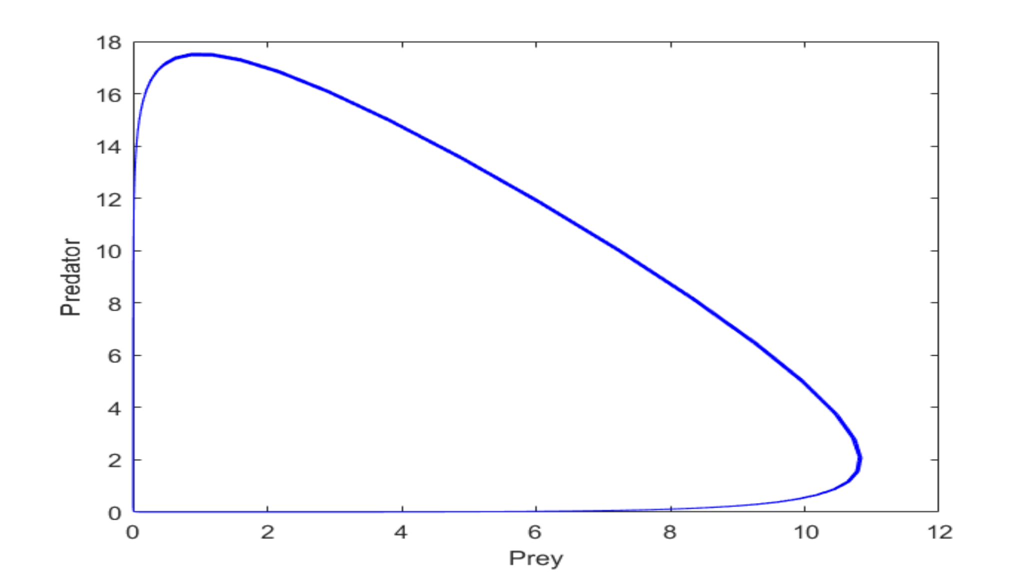
Prey : $dx/dt = ax(t) - bx(t)y(t),$

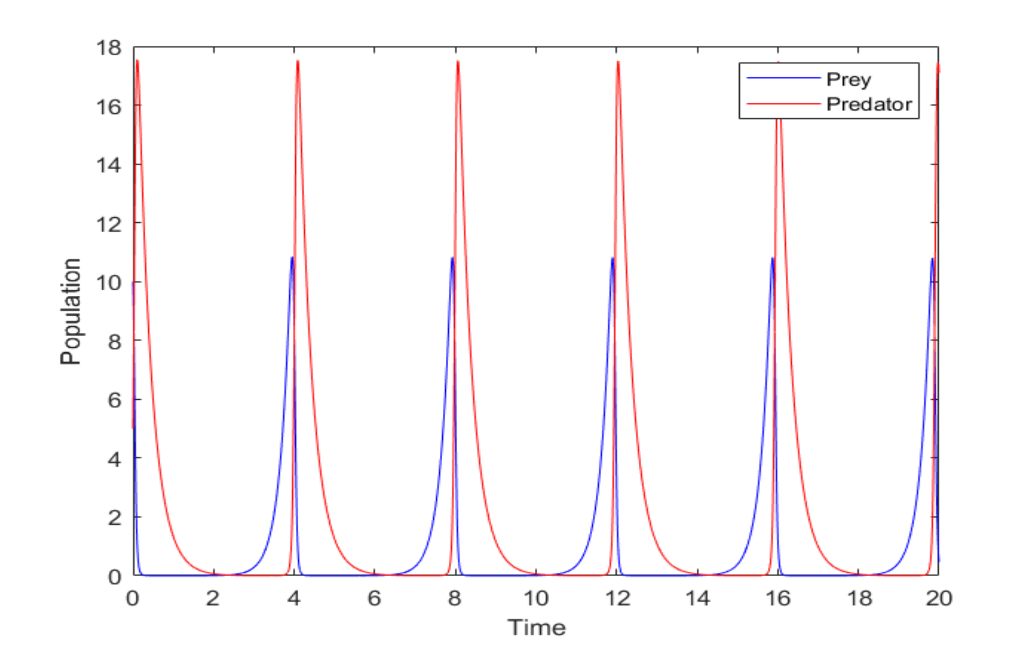
Predator: $dy/dt = -cy(t) + dx(t)y(t),$

- Key parameters : *a*,*b*,*c*,*d* are positive constants.
- Highlights the interdependence of the populations, with cycles of growth and decline.
- New young being born Prey being eaten
- Natural death Population growth from eating

Basic model with no environmental limit on the size of the population and any limit to the appetite of the predator. Set b=d=**α , c=1 , a=2, α = 0 (extinction)**

$$dR = R(2 - \alpha F)dt$$
$$dF = F(\alpha R - 1)dt$$





Stochastic Modeling Model 1
Langevin Approach:

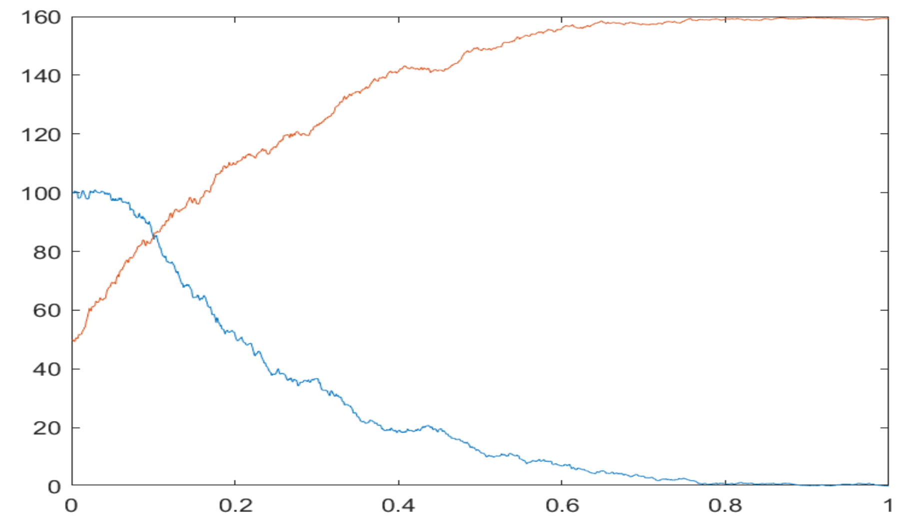
- The model considers changes in populations (ΔR and ΔF) or demographic variability over a small time interval Δt, represented by the system:

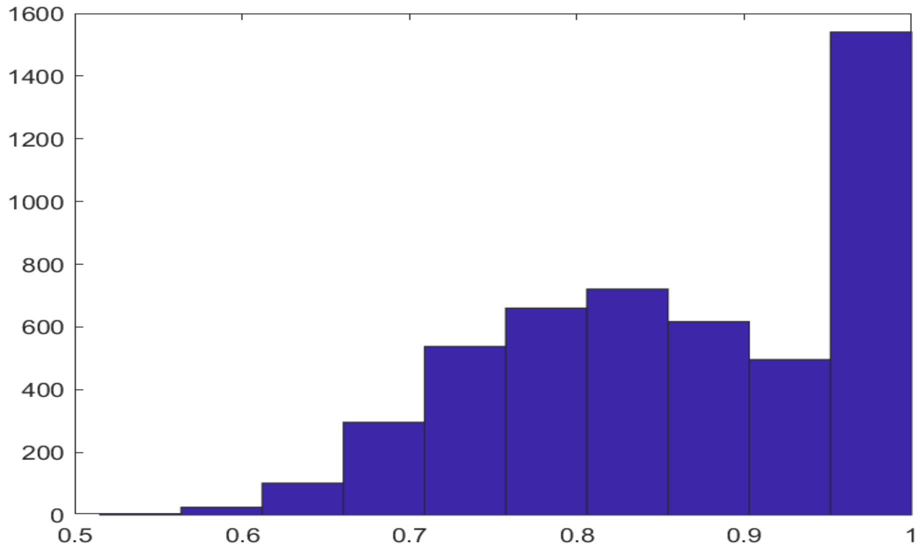
ΔR = R(t + Δt) − R(t),
ΔF = F(t + Δt) − F(t),

- Stochastic differential equations are derived based on expectations and covariances of the stochastic variables ΔR and ΔF, resulting in the system

dR = R(2 − αF)dt +sqrt[R(2 + αF)]dW1(t),
dF = F(αR − 1)dt +sqrt[F(αR + 1)]dW2(t),

The model, governed by a single parameter α, describes the dynamics of two populations (R and F) with two independent Wiener processes $dW1(t)$ and $dW2(t)$.





Stochastic Modeling Model 2:

As the environmental variability is introduced by modifying the parameters of the model, consider the following modification of the parameter α

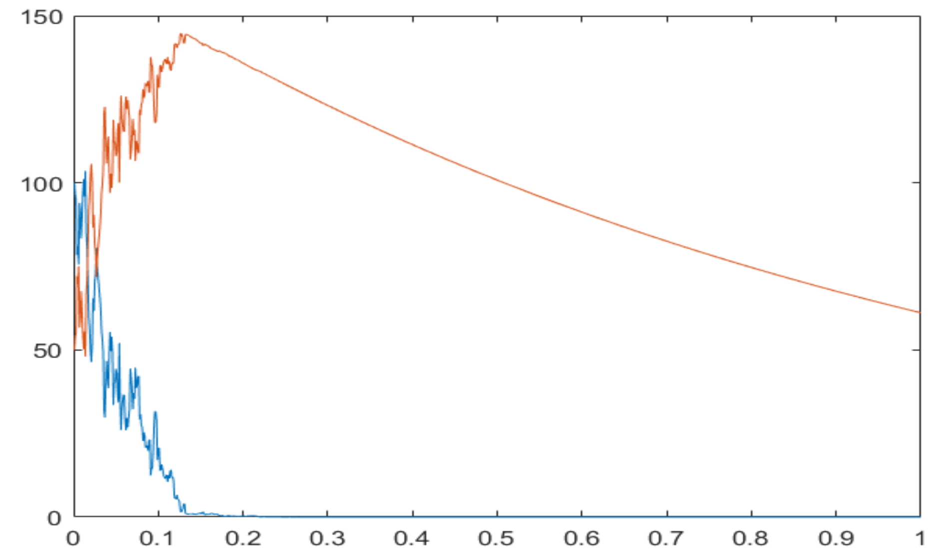
$\alpha \rightarrow \alpha + \sigma\, dW(t)/dt$

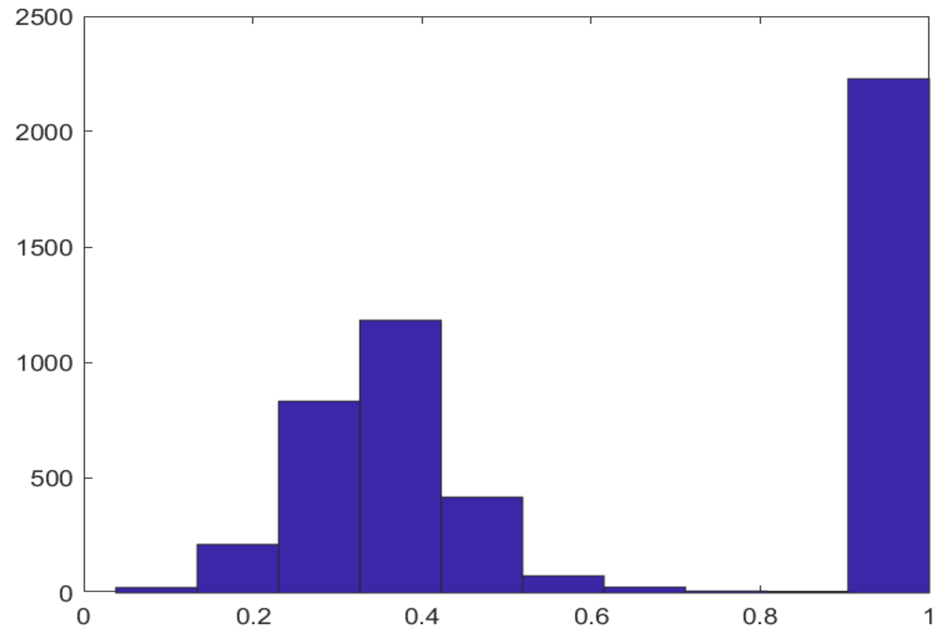
where W(t) is a Brownian motion and **σ** ≥ 0 is a constant.

R(t) and F(t) represent the size of the population of rabbits and foxes respectively, satisfying the stochastic differential system

$$dR = R(2 - \alpha f)dt - \sigma RFdW(t),$$
$$dF = F(\alpha R - 1)dt + \sigma RFdW(t),$$

where the term σRF represents the weighted contact term with noise intensity rate σ.





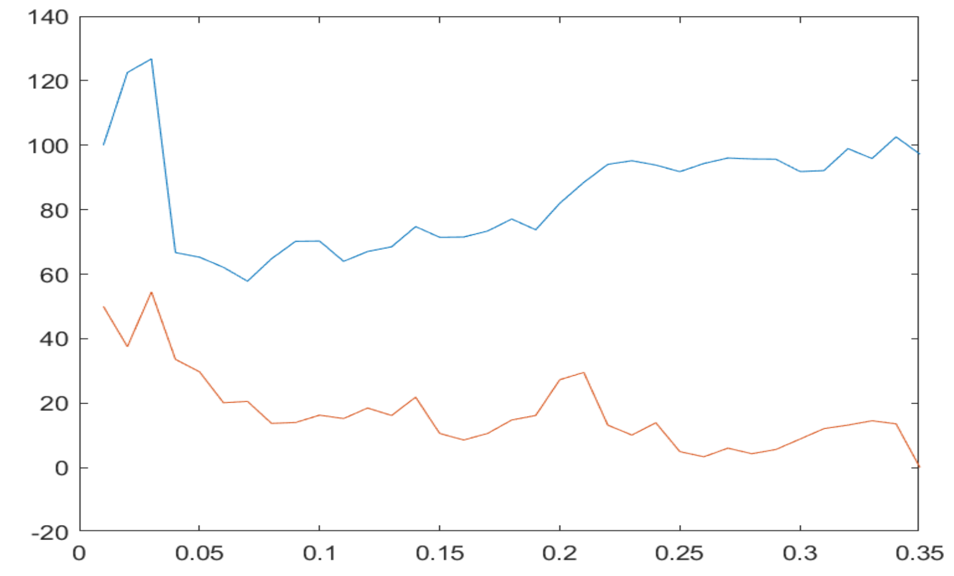
Stochastic Modeling Model 3:

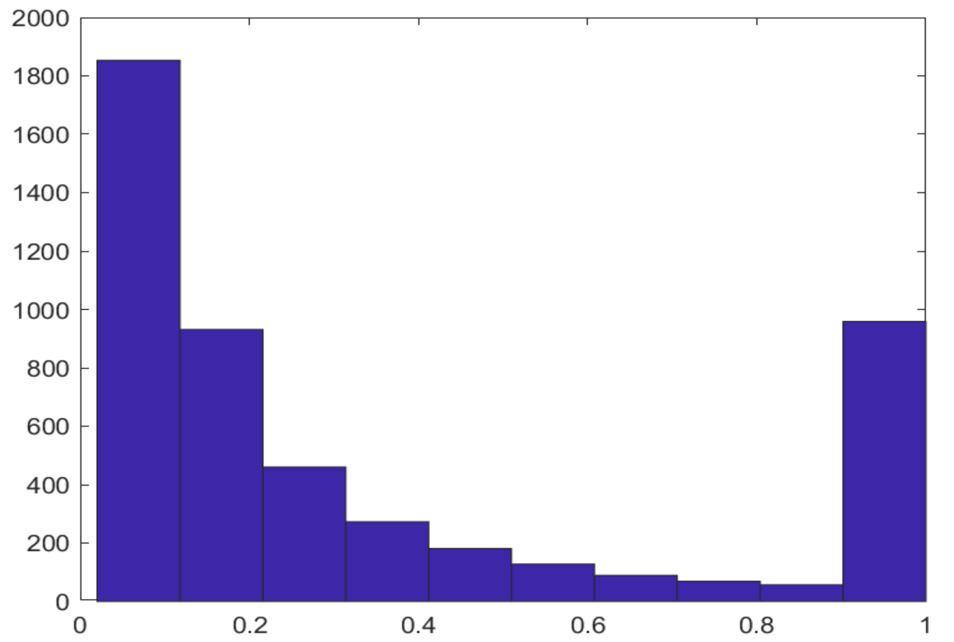
This model introduces distinct linear perturbations to the variations resulting in the following stochastic System:

$$dR = R(2 - \alpha F)dt + \sigma 1 R dW1(t),$$
$$dF = F(\alpha R - 1)dt + \sigma 2 F dW2(t),$$

where dW1(t),dW2(t) are two independent Wiener processes, the terms σ1R and σ2F represent the noise intensity rates for rabbits and foxes respectively.

$\sigma 1$ and $\sigma 2 > 0$





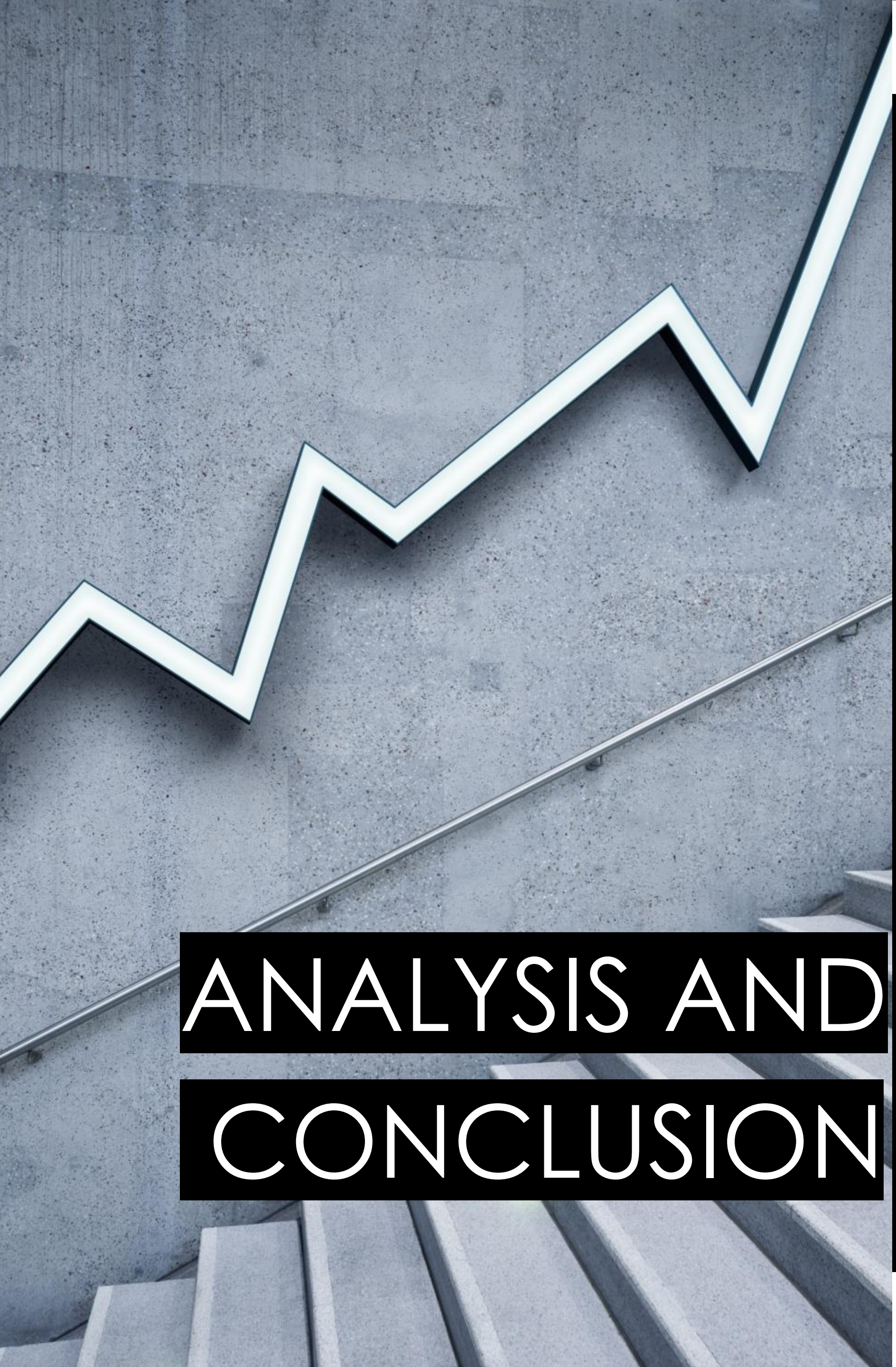
ANALYSIS

There are important differences in their behaviors regarding their extinctions

- Model 1 : both population extinguish in a reasonable time, obviously depending of the parameter α and the initial values of the populations.

- Model 2 : the mean extinction-time depends on new parameter σ, the white noise. For α =0.05 and a reasonable $\sigma \approx 0.001$ this mean large extinction-time.

- Model 3: great differences depending whether $\sigma_1 < 2$ or not.
  $0 < \sigma_1, \sigma_2 \ll 0.1$ very large extinction-time.



ANALYSIS AND CONCLUSION

Significance of Multi-Species Models:
Species in close proximity often compete for resources, highlighting the importance of multi-species models.

Focus on Lotka-Volterra Model:
Lotka-Volterra model, a key ecological tool, is extensively studied for its theoretical and practical relevance.

Recommendations Based on Results:
Model 1 is supported, while Models 2 and 3 show minimal validity, requiring further exploration.

SDEs for Studying Species Competition:
Stochastic differential equations (SDEs) provide valuable numerical solutions for studying species competition over time.

Permanence and Extinction Concepts:
Vital in population studies, with stochastic Lotka-Volterra systems driven to extinction under high white noise and persisting under low noise conditions.