

Continual Lie algebras determined by chain complexes

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Introduction

Continual Lie algebras are infinite-dimensional generalizations of Lie algebras with discrete root system by considering continual root systems. Here we establish a general relation between chain complexes and continual Lie algebras.


The natural orthogonality condition with respect to a product among elements of a chain complex \mathcal{C} spaces brings about to \mathcal{C} the structure of a graded algebra with differential relations.

We prove the main result: a chain complex endowed with an appropriate Leibniz-property product of elements of its spaces brings about the structure of a continual Lie algebra with the root space determined by parameters for the complex. That provides a new source of examples of continual Lie algebras.

Finally, as an example, we consider the case of Čech-de Rham complex associated to a foliation of a smooth manifold. In a particular case of this chain complex, we derive explicitly the commutation relations for the corresponding continual Lie algebra.


Continual Lie algebras introduced in [Saveliev] are generalizations of infinite-dimensional Lie algebras with systems of discrete roots [Kac]. Then the notion was generalized and developed in [Saveliev-Vershik]. These algebras are formulated in such a way that the space of roots is defined by continual sets of vectors. In commutation relations, the generators depend on kernels which are functionals of continual roots. Jacobi identity for continual Lie algebras result in non-trivial functional relations for kernels. Similar constructions appeared recently in the study of certain Hall algebras of coherent sheaves described in terms of certain continuum limits of Kac-Moody algebras [Appel-Sala-Schiffmann].

Though the general theory of continual Lie algebras and their representations is missing, there exist many applications, especially in the theory of completely integrable and exactly solvable models [Razumov-Saveliev]. Various applications can be also found in other fields of mathematics [Bakas, Devchand-Saveliev, Vershik].

In original papers [Saveliev-Vershik], in order to find new classes of examples of continual Lie algebras various approaches were studied. In a series of papers [Saveliev-Vershik] the authors have generalized first known examples and have found new non-trivial and fundamental ones 

arising from various branches of mathematics, in particular, functional analysis [Vershik], differential geometry algebraic geometry, non-commutative geometry, and mathematical physics [Bakas]. Some of them came from physical applications, e.g., equations related to the Ricci flow and cosmology [Bakas]. Though they appear in various corners of modern mathematical research, at some stage, the arsenal of ideas of generation of new examples of continual Lie algebras was exhausted. An intriguing question is how to construct a general way which would allow to create continual Lie algebras. In many cases, this problem reduces to the question of finding new appropriate solutions for relations (0.1) (which is a very interesting problem by itself) following from Jacobi identity.

The idea of this work is to use properties of general chain complexes, (with complex spaces depending on sets of parameters) in order to derive the structure of continual Lie algebras.

The plan is the following. We first recall the notion of a continual Lie algebra [Saveliev]. Then, starting from a general (infinite) chain complex, we first define a product acting of elements of various spaces of a complex. We require the Leibniz rule to be fulfilled for (co)boundary 

operators with respect to this product. In addition to that we assume its skew-symmetry properties. Thus we endow the complex with the structure of a graded differential algebra \mathcal{C} .

We then apply the natural condition of orthogonality with respect to a product among the chain complex spaces. For instance, in differential geometry, the orthogonality condition reduces to the integrability condition for differential forms. Further actions of (co)boundary operators, as well as consequences from the definition of the multiplication, generate a system of differential relations for elements of various chain complex spaces with extra compatibility conditions for indices. This determines the structure of a graded algebra with differential relations for \mathcal{C} . By choosing independent elements satisfying the above relations, we then show the main result: for a graded differential algebra associated to a chain complex, the independent generators, the system of differential relations, and Jacobi identity define the structure of a continual Lie algebra with the root space provided by the spaces parameters of a chain complex.

Then we consider the example of the Čech-de Rham cochain bicomplex [Crainic-Moerdijk] associated to a smooth manifold foliation 

[Dubrovin-Novikob-Fomenko, Bazaikin-Galaev, Bazaikin-Galaev-Gumenyuk]. This bicomplex has a deep geometric meaning [Galaev] and is defined for the spaces of differential forms and holonomy embeddings acting between sections of the transversal basis for a foliation.

The space of differential forms has already a structure of bigraded differential algebra with respect to the natural product [Crainic-Moerdijk]. For the bicomplex under consideration, we define the second product, satisfying properties required for the construction of a system of differential relations. According to the exposition we give, by involving the orthogonality conditions for an arbitrary pairs of the Čech-de Rham bicomplex, we apply the general scheme of construction of a bigraded differential algebra. Then we single out generators and commutation relations for the corresponding continual Lie algebra with the space of roots and kernels given by the sets of holonomy embeddings. In a particular geometric case associated to a codimension one foliation over a three-dimensional smooth manifold, we start from the integrability condition applied to elements of the same bicomplex space which leads to the appearance of a continual Lie algebra with generators and kernels.

Continual Lie algebras

First, we recall the notion of a continual Lie algebra introduced in [Saveliev]. It was then studied in [Saveliev-Vershik]. Suppose E is an associative algebra over \mathbb{R} or \mathbb{C} , and $K_0, K_{\pm 1}, K_{0,0} : E \times E \rightarrow E$, are bilinear mappings. The local part of a continual Lie algebra can be defined as $\widehat{\mathcal{G}} = \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{+1}$, where \mathcal{G}_i , $i = 0, \pm 1$, are isomorphic to E and parametrized by its elements. The subspaces \mathcal{G}_i consist of the elements $\{X_i(\phi), \phi \in E\}$, $i = 0, \pm 1$. The generators $X_i(\phi)$ are subject to commutation relations. For instance, in the principle grading they can have the form

$$\begin{aligned}[X_0(\phi), X_0(\psi)] &= X_0(K_{0,0}(\phi, \psi)), & [X_0(\phi), X_{\pm 1}(\psi)] &= X_{\pm 1}(K_{\pm 1}(\phi, \psi)), \\ [X_{+1}(\phi), X_{-1}(\psi)] &= X_0(K_0(\phi, \psi)),\end{aligned}$$

for all $\phi, \psi \in E$. It is also assumed that Jacobi identity is satisfied. Then the conditions on mappings $\widehat{K} = (K_{0,0}, K_0, K_{\pm 1})$ follow:

$$\begin{aligned}K_{\pm 1}(K_{0,0}(\phi, \psi), \chi) &= K_{\pm 1}(\phi, K_{\pm 1}(\psi, \chi)) - K_{\pm 1}(\psi, K_{\pm 1}(\phi, \chi)), \\ K_{0,0}(\psi, K_0(\phi, \chi)) &= K_0(K_{+1}(\psi, \phi), \chi) + K_0(\phi, K_{-1}(\psi, \chi)),\end{aligned}$$

for all $\phi, \psi, \chi \in E$.

An infinite dimensional algebra $\mathcal{G} (E; \widehat{K}) = \mathcal{G}' (E; \widehat{K}) / J$, is called a continual contragredient Lie algebra, where $\mathcal{G}' (E; \widehat{K})$ is a Lie algebra freely generated by $\widehat{\mathcal{G}}$, and J is the largest homogeneous ideal with trivial intersection with \mathcal{G}_0 (consideration of the quotient is equivalent to imposing the Serre relations in the case of finite-dimensional simple complex Lie algebras) [Saveliev-Vershik]. \mathcal{G} is endowed with a \mathbb{Z} -grading $\mathcal{G} = \bigoplus_{i \in \mathbb{Z}} \mathcal{G}_n$, where elements of subspaces \mathcal{G}_n satisfy the standard grading condition $[\mathcal{G}_n, \mathcal{G}_m] \subset \mathcal{G}_{n+m}$, where $X_n(\phi) \in \mathcal{G}_n$, and higher mapping $K_{n,m}(\phi, \psi)$ are present in commutation relations among $X_n(\phi)$ and $X_n(\psi)$. In general, Jacobi identity for generators that belong to all grading spaces has the form

$$\begin{aligned}
 [X_i(\phi), [X_j(\psi), X_k(\theta)]] &+ [X_j(\psi), [X_k(\theta), X_i(\phi)]] \\
 &+ [X_k(\theta), [X_i(\phi), X_j(\psi)]] = 0, \\
 K_{i,j+k}(\phi, K_{j,k}(\psi, \theta)) &+ K_{j,k+i}(\psi, K_{k,i}(\theta, \phi)) \\
 &+ K_{k,i+j}(\theta, K_{i,j}(\phi, \psi)) = 0. \quad (0.1)
 \end{aligned}$$


The algebra of differential relations from a chain complex Now we formulate the construction which allows us to generate an algebra of graded differential relations starting from a chain complex. Consider a complex of spaces with elements depending on sets of parameters Θ_i , $i \in \mathbb{Z}$, and given by

$$\dots \xrightarrow{\delta^{(i-1)}} C(i, \Theta_i) \xrightarrow{\delta^{(i)}} C(i+1, \Theta_{i+1}) \xrightarrow{\delta^{(i+1)}} \dots, \quad (0.2)$$

with a differential $\delta^{(i)}$ satisfying the chain property $\delta^{(i+1)} \circ \delta^{(i)} = 0$, for $i \in \mathbb{Z}$.

We assume also that there exists a (not necessary associative) product among element of the spaces $C(i, \Theta_i)$,

$$\cdot : C(i, \Theta_i) \cdot C(j, \Theta_j) \rightarrow C(f(i, j), \Theta_{g(i, j)}), \quad (0.3)$$

so that for any elements $\Phi_i \in C(i, \Theta_i)$ and $\Phi_j \in C(j, \Theta_j)$, $\Phi_i \cdot \Phi_j = \Phi_{f(i, j)}$, where $f(i, j)$ and $g(i, j)$ are some functions of indices i and j . Note that a function $f(i, j)$ defined the index for the product (0.3) resulting space. The function $g(i, j)$ defines the index for the resulting set of parameters. Let us assume that for all $i \in \mathbb{Z}$, the Leibniz rule 

formula for the operator $\delta(i)$ takes place, i.e.,

$$\delta(k+l)(\Phi_k \cdot \Phi_l) = (\delta(k)\Phi_k) \cdot \Phi_l + (-1)^{\deg(\Phi_k)} \Phi_k \cdot \delta(l)\Phi_l. \quad (0.4)$$

We assume also that the chain complex above has some properties of an algebra with respect to the product (0.3). Namely, consider the elements $\Phi_k \in C(k, \Theta_k)$, and $\Phi_l \in C(l, \Theta_l)$. Let us assume, in particular, that the product (0.3) satisfies the condition

$$\Phi_k \cdot \Phi_l = -\Phi_l \cdot \Phi_k. \quad (0.5)$$

for all $k, l \in \mathbb{Z}$, $\Phi_k \in C(k, \Theta_k)$ and $\Phi_l \in C(l, \Theta_l)$. Altogether, the complex (0.2) forms a graded differential algebra \mathcal{C} . In what follows, we skip the sets of parameters Θ_i in the notations for the spaces of complexes, i.e., we set $C(i) = C(i, \Theta_i)$. Though one has to keep in mind that a product of elements of two spaces of complex (0.2) depends on the resulting set of parameters $\Theta_{g(i,j)}$.

The algebra of graded differential relations

Let us assume that the product (0.3) is defined in such a way that common parameters of the sets Θ_i and Θ_j are present in the resulting set $\Theta_{g(i,j)}$ only once. For some i and $j \in \mathbb{Z}$, let us introduce the orthogonality condition for a pair $C(i, \Theta_i)$ and $C(j, \Theta_j)$, with respect to the product (0.3). In particular, let us require that for a pair $C(i, \Theta_i)$, $C(j, \Theta_j)$, there exist subspaces $C'(i) \subset C(i, \Theta_i)$ and $C'(j) \subset C(j, \Theta_j)$, such that, for any $\Phi_i \in C'(i)$ and $\Phi_j \in C'(j)$,

$$\Phi_i \cdot \delta(j)\Phi_j = 0, \quad (0.6)$$

namely, Φ_i is supposed to be orthogonal to $\delta(j)\Phi_j$ with respect to (0.3).

By applying further differentials to (0.6) (and further consequences of such action), and using properties of a particular function $f(i, j)$ we obtain relations among elements of spaces $C(i, \Theta_i)$, $i \in \mathbb{Z}$. In particular, taking into account that both sides of such relations belong to the same space, we obtain limitations (depending on the function f) on indices. In differential geometry, the orthogonality condition (0.6) provides the definition of integrability conditions for differential forms, and leads to the Frobenius theorem (see, e.g., [Ghys]).

Let us explain the notations we will use on few next pages. In (0.13) we obtain (infinite) sequences of pairs of differential relations of the form (0.8)–(0.10). The (infinite) sequence of pair of relations has a tree graph structure with two sequences outgoing from one point. At each point we call one branch "left" and another branch as "right" marking the corresponding pair by L or R . We denote such pairs by $(\alpha_i^{(K_i)})$, where $\alpha_i^{(K_i)}$ is an element of $C(n_i^{(K_i)})$ involved in differential relations, and (K_i) is a sequence of i entries each is either L or R for $i \geq 1$.

Let $\chi \in C(n_0)$, $\Phi \in C(n)$, for any $n_0, n \in \mathbb{Z}$. Due to the property (0.5) of the multiplication (0.3), the orthogonality condition (0.6) applied to Φ and χ , i.e.,

$$\Phi \cdot \delta(n_0)\chi = 0, \quad (0.7)$$

implies that there exists $\alpha_1^{(R)} \in C(n_1^{(R)})$, such that

$$\delta(n_0)\chi = \Phi \cdot \alpha_1^{(R)}. \quad (0.8)$$

Let $r_1^{(R)}$ be the number of common parameters among n - and $n_1^{(R)}$ -sets of parameters for Φ and $\alpha_1^{(R)}$. Since both sides of the last relation have

to belong to the same space of the complex, the compatibility condition

$$n_0 + 1 = n + n_1^{(R)} - r_1^{(R)}, \quad (0.9)$$

occurs. Acting by $\delta(n_0 + 1)$ on (0.8) we obtain

$$0 = (\delta(n)\Phi) \cdot \alpha_1^{(R)} + (-1)^{n_0+1} \Phi \cdot \delta\left(n_1^{(R)}\right) \alpha_1^{(R)}. \quad (0.10)$$

On the other hand, (0.7) implies that there exists $\alpha_1^{(L)} \in C\left(n_1^{(L)}\right)$, such that

$$\Phi = \alpha_1^{(L)} \cdot \delta(n_0)\chi, \quad (0.11)$$

with the condition

$$n = n_1^{(L)} + n_0 + 1 - r_1^{(L)}, \quad (0.12)$$

where $r_1^{(L)}$ is the number of common parameters among $n_1^{(L)}$ and $(n_0 + 1)$ for $\alpha_1^{(L)}$ and $\delta(n_0)\chi$.

Consequently applying the corresponding δ operators to (0.7), (0.8) and (0.11) we obtain the system of relations:

$$0 = \Phi \cdot \delta(n_0)\chi, \quad (1)$$

$$(1) \Rightarrow \Phi = \alpha_1^{(L)} \cdot \delta(n_0)\chi, \quad (2)$$

$$\delta.(2) \Rightarrow \delta(n)\Phi = \left(\delta \left(n_1^{(L)} \right) \alpha_1^{(L)} \right) \cdot \delta(n_0)\chi, \quad (2') \Rightarrow 0,$$

$$\delta.(1) \Rightarrow 0 = (\delta(n)\Phi) \cdot \delta(n_0)\chi, \quad (3)$$

$$(1) \Rightarrow \delta(n_0)\chi = \Phi \cdot \alpha_1^{(R)}.$$

$$0 = \delta(n)\Phi \cdot \alpha_1^{(R)} + (-1)^n \Phi \cdot \delta \left(n_1^{(R)} \right) \alpha_1^{(R)} \Rightarrow 0,$$

$$(3) \Rightarrow \delta(n_0)\chi = (\delta(n)\Phi) \cdot \alpha_2^{(RR)},$$

$$(4) : 0 = (\delta(n)\Phi) \cdot \delta\left(n_2^{(RR)}\right) \alpha_2^{(RR)} \Rightarrow \left(\alpha_3^{(RRR)}\right) \Rightarrow \dots$$

$$\Downarrow$$

$$\left(\alpha_3^{(RRL)}\right) \Rightarrow \dots$$

$$\Downarrow$$
$$\dots$$

$$(3) \Rightarrow \delta(n)\Phi = \alpha_2^{(LL)} \cdot \delta(n_0)\chi,$$

$$(5) : 0 = \delta\alpha_2^{(LL)} \cdot \delta(n_0)\chi \Rightarrow \left(\alpha_3^{(LLR)}\right) \Rightarrow \dots$$

$$\Downarrow$$

$$\left(\alpha_3^{(LLL)}\right) \Rightarrow \dots$$

$$\Downarrow$$
$$\dots$$

where we obtain (infinite) sequences (0.14) of pairs of relations for $\alpha_i^{(K_i)} \in C(n_i^{(K_i)})$, $i \geq 1$. Recall that we denote such relations as $(\alpha_i^{(K_i)})$. The corresponding indices $n_i^{(K_i)}$ satisfy (in addition to (0.9) and (0.12)) relations for the sequence starting from (4):

$$n_0 = n + n_i^{(RRK_{i+2})} - r_i^{(RRK_{i+2})}, \quad (0.15)$$

for the sequence starting from (5):

$$n = n_0 + n_j^{(LLK_{j+2})} - r_j^{(LLK_{j+2})}, \quad (0.16)$$

$i, j \geq 2$.

One can easily see that not all elements in (0.13) are independent. For instance, from (0.9) and (0.12) we obtain

$$\left(n_1^{(L)} - r_1^{(L)} \right) = - \left(n_1^{(R)} - r_1^{(R)} \right) = n - n_0 - 1.$$

From (0.8)-(0.11), and from (3')-(3'') of (0.13) we infer that $\alpha_1^{(L)}$, $\alpha_1^{(R)}$, and $\alpha_i^{(LLK_i)}$, $\alpha_i^{(RRK_i)}$ are related by a conjugation with respect to the product (0.3):

$$\Phi = \alpha_1^{(L)} \cdot \left(\Phi \cdot \alpha_1^{(R)} \right), \quad \delta(n_0)\chi = \left(\alpha_i^{(LLK_i)} \cdot \delta(n_0)\chi \right) \cdot \alpha_i^{(RRK_i)}.$$

Similar relations apply among other elements $\alpha_i^{(K_i)}$.

The sequence of relations (0.13) cancels when one of relations (0.9)–(0.12) or (0.15)–(0.16) for a sequence of pairs of equations is not fulfilled. The natural grading is given by the condition that both sides of differential relations in (0.13) belong to the same chain complex space. We consider the simplest form (0.2) of a complex. In general, for more complicated actions of δ , such that $\delta(i) : C(i) \rightarrow C(i + k(i))$, where $k(i)$ depends on $i \in \mathbb{Z}$ (see, e.g., [Huang] for non-trivial actions of certain δ operators among chain complex spaces for vertex algebras. Consideration of such complexes will be given by the author in another article). The corresponding differential relations as well as compatibility relations could be different from (0.13)

As an upshot, the orthogonality condition (0.6) for all choices of n_0 , $n \in \mathbb{Z}$, and the conditions (0.15)–(0.16), applied to the chain complex (0.2) bring about the structure of a graded algebra with differential relations (0.13) with respect to the multiplication (0.5). As we can see, the system of relations (0.13) has a tree structure. "Left" and "right" directions have a mixture of dependent elements. Let us denote $n(i) = n_i^{(K_i)}$. For $n_0, n \in \mathbb{Z}$, let $I(n_0, n)$ be the set of sequences of indices $(n(i))$, $i \geq 0$, marking all paths (K_i) in the tree structure of (0.13), describing "left" or "right" choice at each point. Let $I = \bigcup_{n_0, n \in \mathbb{Z}} I(n_0, n)$, be the space of all sequences over the tree graph for a complex (0.2). Denote by $I_0 \subset I$ the subset of such paths that include pairs of independent elements only. Then we are able to single out generators and commutation relations of a Lie algebra with a continual space of roots.

Construction of a continual Lie algebra from a chain complex

Let us further assume that the spaces $C(i)$, $i \in \mathbb{Z}$ admit also an ordinary (not necessary commutative) product $\Phi\Psi$ among elements for $\Phi_k \in C(k)$, and $\Phi_l \in C(l)$, for all $k, l \in \mathbb{Z}$. Then, as a product (0.3), satisfying conditions (0.5) and (0.4), one can take

$$\Phi_k \cdot \Phi_l = [\Phi_k, \Phi_l] = \Phi_k \Phi_l - \Phi_l \Phi_k, \quad (0.17)$$

where brackets mean the ordinary commutator. It is known that the introduction of the commutator with respect to the original multiplication of an algebra transfers it into a Lie algebra when Jacobi conditions are satisfied. In our case we show that the differential algebra \mathcal{C} defined above being supplied with the orthogonality conditions deliver the structure of a continual Lie algebra.

Our setup of combined with the previous discussion provides us with a proof of the main result:

Proposition

For the set I_0 of all pairs of independent elements $\alpha_i^{(K_i)}$, the orthogonality condition (0.6), the generators $\{\chi, \delta(n_0)\chi, \Phi, \delta(n)\Phi, \alpha_i^{(K_i)}, \delta(n(i))\alpha_i^{(K_i)}\}$, $n(i) = n_i^{(K_i)}$, the relations (0.13), and Jacobi identity (0.1) form a continual Lie algebra $\mathcal{G}(\Theta_{n(i)})$ with the root space depending on the set of parameters $\Theta_{n(i)}$, of spaces $C(n(i), \Theta_{n(i)})$.

Though the structure of the system (0.13) may seem to be not very complicated, actual properties of the corresponding continual Lie algebra depends on properties of the spaces of a specific the bicomplex (0.2) and the nature of parameters Θ_i . For a fixed choice of a path of independent functions/differential equations in the system (0.13), there exists a variety of choices on how to identify generators of a continual Lie algebra with generators of the differential algebra. Therefore, the actual form of commutation relations for the corresponding continual Lie algebra varies accordingly.

One can also choose various ways how to define a grading for each specific $\mathcal{G}(\Theta_{n(i)})$, for generators of a continual Lie algebra resulting from (0.13). The structure of the product (0.3) together with the condition (0.4), and the action of the differentials δ provide Jacobi identity for generators on the continual Lie algebra $\mathcal{G}(\Theta_{n(i)})$, and, simultaneously, apply conditions of the form (0.1) to elements of the parameter spaces $\Theta_{(n(i))}$, $i \geq 0$. Next we specify the above construction and Proposition (1) in the case of double cochain complex (0.27)–(0.28) associated with differential forms [Crainic-Moerdijk]. We derive explicitly the generators and commutation relations for the corresponding continual Lie algebras.

An example: double complex associated with foliations

Recall the formulation of the Čech-de Rham cohomology given in [Crainic-Moerdijk] for foliations on smooth manifolds. In this case the spaces in (0.2) are differential forms $C^{n,m}(\mathcal{F})$ defined on a foliation, and the coboundary operators is given by $\delta^{p,q} = (-1)^p d + \delta^q$, where d and δ are differentials.

The ordinary product for differential forms $\omega^{n,m}(h_1, \dots, h_m) \in C^{n,m}(\mathcal{F})$ is given by (0.29). The product (0.3) required for the formulation is provided by the commutator (0.17). As it was mentioned in [Crainic-Moerdijk], the bicomplex (0.27)–(0.28) has the structure of a bigraded differential algebra with respect to the ordinary product (0.29). According to the previous explanations, for the bicomplex (0.27)–(0.28) generate the graded differential algebra with relations (0.13) with respect to the product (0.17).

The general case

Let us recall that we assume in this case non-negative indices for all bicomplex spaces C^{n_i, m_i} . Let $\chi \in C^{n_0, m_0}$, $\Phi \in C^{n, m}$. The orthogonality condition (0.6) with respect to the product (0.17) leads to systems of the form (0.13) when applied to the double complex (0.2). In particular, for $\chi \in C^{n_0, m_0}$ and $\Phi \in C^{n_0, m_0}$, we obtain the system (0.13) of relations for elements χ , $\delta^{n_0, m_0} \chi$, Φ , $\delta^{n, m} \Phi$, $\alpha_i^{(K_i)} \in C^{n_i^{(K_i)}, m_i^{(K_i)}}$, $\delta^{n(i), m(i)} \alpha_i^{(K_i)}$, $i \geq 1$, for $n \geq 0$, $m \geq 0$.

Let $r_i^{(K_i)}$ and $t_i^{(K_i)}$ be numbers of common degrees and transversal sections of C^{n_i, m_i} for the forms $\chi \in C^{n_0, m_0}$ and $\Phi \in C^{n_0, m_0}$. Then the compatibility conditions (0.9)–(0.12) and (0.15)–(0.16) for indices n , m , n_0 , m_0 , $n_i^{(K_i)}$, $m_i^{(K_i)}$, satisfy the relations in vector form:

$$(n_0 + 1, m_0 + 1) = (n, m) + \left(n_1^{(R)}, m_1^{(R)} \right) - \left(r_1^{(R)}, t_1^{(R)} \right), \quad (0.18)$$

for (2) in (0.13):

$$(n, m) = \left(n_1^{(L)}, m_1^{(L)} \right) + (n_0 + 1, m_0 + 1) - \left(r_1^{(L)}, t_1^{(L)} \right), \quad (0.19)$$

For the sequence starting from (4) in (0.13) we have:

$$(n_0, m_0) = (n, m) + \left(n_i^{(RRK_i)}, m_j^{(RRK_j)} \right) - \left(r_i^{(RRK_i)}, t_j^{(RRK_j)} \right), \quad (0.20)$$

for the sequence starting from (5) in (0.13):

$$(n, m) = (n_0, m_0) + \left(n_i^{(LLK_i)}, m_j^{(LLK_j)} \right) - \left(r_i^{(LLK_i)}, t_j^{(LLK_j)} \right). \quad (0.21)$$

$i, j \geq 2$. Note that it is assumed that all the resulting indices in the compatibility conditions are non-negative. For the general complex (0.2), $i \in \mathbb{Z}$. For the bicomplex (0.27)–(0.28) we have p, q are non-negative. Since we assume that $0 \leq r_i^{(LLK_i)} \leq n_i^{(LLK_i)}$, and $0 \leq t_j^{(LLK_j)} \leq m_j^{(LLK_j)}$, from the compatibility conditions (0.18)–(0.19) and (0.20)–(0.21) we see that, depending on the signs of $n - n_0 - \delta_{i,1}$, $m - m_0 - \delta_{i,1}$, and only one branch of systems of the form (2)–(2') exists at each vertex of the three graph associated to the double complex (0.27)–(0.28).

Recall that according to Theorem 1 of [Crainic-Moerdijk], the definition of the spaces $C^{n,m}$ do not depend on the choice of the transversal basis \mathcal{U} , still it depends on parameters of foliation \mathcal{F} . Nevertheless, differential forms $\omega^{n,m}$ do depend on holonomy mappings h_j , $j \geq 0$, and play the role of extra parameters in the consideration.

As explained before, due to Proposition (1), a path (K_i) , $i \geq 1$, defining the generators $\left\{ \chi, \delta^{n_0, m_0} \chi, \Phi, \delta^{n, m} \Phi, \alpha_i^{(K_i)}, \delta^{n(i), m(i)} \alpha_i^{(K_i)} \right\}$, $n(i) = n_i^{(K_i)}$, $m(i) = m_i^{(K_j)}$, and relations for independent elements of $C^{n,m}$ in (0.13), form a continual Lie algebra $\mathcal{G}(\mathcal{F})$ with the space of roots provided by the holonomy mappings and of parameters of the foliation \mathcal{F} .

Double cochain complex: Godbillon-Vey type example

Here we provide the explicit example for Proposition (1) in the case of the orthogonality condition (0.6) applied to the particular case when, $\Phi = \chi$. In particular, in differential geometry, the case of a foliation \mathcal{F} of codimension one defined by a one-form on a three-dimensional manifold, and the formulation of the Godbillon-Vey cohomology class, are included in this consideration.

We require the orthogonality for $\chi \in C^{n,m}(\mathcal{F})$ within its own bicomplex space, i.e., to satisfy the condition

$$\chi \cdot \delta^{n,m} \chi = 0. \quad (0.22)$$

Thus, for $\alpha_1^{(R)} \in C^{n',m'}(\mathcal{F})$ one has

$$\delta^{n,m} \chi = \chi \cdot \alpha_1^{(R)}, \quad (0.23)$$

and $n + 1 = n + n' - r$, $m + 1 = m + m' - t$, and (0.23) is possible only when $n' = r + 1$, $m' = t + 1$, $0 \leq r \leq n$, $0 \leq t \leq n$, and $\alpha_1^{(R)} \in C^{r+1,t+1}(\mathcal{F})$. If we require from (0.22) that for $\alpha_1^{(L)} \in C^{\alpha,\beta}(\mathcal{F})$, $\chi = \alpha_1^{(L)} \cdot \delta^{n,m} \chi$, then $n = \alpha + n + 1 - r'$, $\alpha = r' - 1$, i.e., α is smaller

than the common degree which is not possible and thus such $\alpha_1^{(L)}$ does not exist.

Then, as a result of (0.13), we obtain the system of differential relations:

$$\begin{aligned} 0 &= \chi \cdot \delta^{n,m} \chi, & (1), & \quad 0 = (\delta^{n,m} \chi) \cdot \delta^{n,m} \chi, & (3) \\ \delta^{n,m} \chi &= \chi \cdot \alpha_1^{(R)}, & \quad 0 &= \delta^{n,m} \chi \cdot \alpha_1^{(R)} + (-1)^n \chi \cdot \delta^{r+1,t+1} \alpha_1^{(R)}, & (0.24) \end{aligned}$$

and the rest of the system (0.13) collapses since further its branches follow from (3) which is trivial.

Let us denote $h_n = (h_1, \dots, h_n)$, an n -tuple of holonomy mappings. In this case, for forms $\{\chi, \delta^{n,m} \chi, \alpha_1^{(R)}, \delta^{r+1,t+1} \alpha_1^{(L)}\}$, define the following continual Lie algebra by identifying the differential forms with generators of $\mathcal{G}(\mathcal{F})$ as

$$X_+ = \chi, \quad X_- = \delta^{n,m} \chi, \quad H = \alpha_1^{(R)}, \quad H^* = \delta^{r+1,t+1} \alpha_1^{(R)}, \quad (0.25)$$

and the commutation relations (in a non-principal grading):

$$\begin{aligned} [X_+(h_n), X_-(h'_n)] &= 0, & [X_+(h_n), H(h_{r+1})] &= X_-(K_{+1,0}(h_n, h_{r+1})), \\ [X_-(h_{n+1}), H(h_{r+1})] &+ (-1)^n [X_+(h_n), H^*(h_{r+2})] &= 0. & \Rightarrow \leftarrow \Rightarrow \quad (0.26) \end{aligned}$$

Note that an element of $C^{n,m}(\mathcal{F})$ is an n -form $\omega(h_1, \dots, h_n)$ depending on n holonomy maps. Thus the space of continual roots is provided by the space of holonomy embeddings. Taking into account (0.24), we find the kernels

$$\begin{aligned} K_{+1,-1}(h_n, h_{r+1}) &= 0, & K_{+1,0}(h_n, h_{r+1}) &= h_{n+1}, \\ K_{0,-1}(h_n, h_{r+1}) &= h_{n+2}, & K_{0,1}(h_n, h_{r+1}) &= h_{n+2}. \end{aligned}$$

It is easy to check that Jacobi identity for generators (0.25) are fulfilled. In the case of codimension one foliation defines by a one-form, $\chi, \alpha_1^{(R)}$, $\chi \in C^{1,m}(\mathcal{F})$. Recall [Ghys] that the Godbillon-Vey cohomology class is given by $\left[\alpha_1^{(R)} \wedge \delta^{1,m} \alpha_1^{(R)} \right]$. The construction above clarifies the Lie-algebraic meaning of this cohomology class.

Conclusions


In conclusion, we would like to mention a few directions of development and further applications of the material presented here. We propose a way to associate a continual Lie algebra to a chain complex. Thus the properties, in particular, Jacobi identity, kernels, and relations of resulting continual Lie algebras depend on the kind of chain complex spaces as well as on the set of their parameters.

One can think of introducing various types of products suitable for the construction of systems of differential relations more complicated than (0.13). In our particular case, in order to make connection with continual Lie algebras, we have chosen the commutator (0.17) (with respect to the original product defined on bicomplex spaces) as the simplest natural product. One could think of other possibilities which would be coherent with the orthogonality condition (0.6).

In our exposition, the standard form of chain complexes was involved. Nevertheless, one can consider more complicated setups, in particular, chain complexes where differentials act in non-trivial ways with respect to indices of spaces (c.f. examples in [Huang]).

That would lead to alternative forms of systems of differential relations as well as of compatibility conditions. What could be especially interesting, is to treat multiple chain complexes containing combinations of a few chain-cochains.

The example of the Čech-de Rham cohomology comes from the differential geometry of foliations. In classics, the orthogonality condition applied to elements and their differentials of one particular bicomplex space, boils down to the integrability condition, and leads to the Frobenius theorem. Then it delivers the Godbillon-Vey cohomological class whose geometric meaning is not yet completely studied [Galaev].

As for further applications in differential geometry, starting from the orthogonality condition, it would be interesting to find other cohomological invariants related to the Čech-de Rham bicomplex for foliations, and to understand their geometric meaning. The constructions considered here can be also used for the cohomology theory of smooth manifolds, in particular, in various approaches to the construction of cohomological classes (cf., in particular, [Losik]). Wide applications are awaiting for new examples of continual Lie algebras in the field of integrable models [Leznov-Saveliev, Razumov-Saveliev]. The cases of 

non-commutative fields used to define continual Lie algebras would also be useful for in non-commutative geometry.

Čech-de Rham complex for foliations

Here we recall [Crainic-Moerdijk] the notion of the basis of transversal sections, and Čech-de Rham complex for a foliation of a smooth manifold. Let M be a smooth manifold of dimension n , equipped with a foliation \mathcal{F} of co-dimension l [Crainic-Moerdijk]. A transversal section of \mathcal{F} is an embedded l -dimensional submanifold $U \subset M$ which is everywhere transverse to \mathcal{F} leaves. If α is a path between two points x and y on the same leaf, and if U and V are transversal sections through x and y , then α defines a transport along the leaves from a neighborhood of x in U to a neighborhood of y in V . Therefore, we define a germ of a diffeomorphism $hol(\alpha) : (U, x) \rightarrow (V, y)$, called the holonomy of the path α . If such a transport is defined in all of U and embeds U into V , this embedding $h : U \hookrightarrow V$ is sometimes also denoted by $hol(\alpha) : U \hookrightarrow V$. Embeddings of this form will be called holonomy embeddings.

Transversal sections U through a point x are neighborhoods of the leaf through x in the leaf space. One defines a transversal basis for (M, \mathcal{F}) as a family \mathcal{U} of transversal sections $U \subset M$ with the property that, if V is any transversal section through a given point $y \in M$, there exists a

holonomy embedding $h : U \hookrightarrow V$ with $U \in \mathcal{U}$ and $y \in h(U)$. A transversal section is a l -disk given by a chart for the foliation. One then constructs a transversal basis \mathcal{U} out of a basis $\tilde{\mathcal{U}}$ of M by domains of foliation charts $\phi_U : \tilde{U} \xrightarrow{\sim} \mathbb{R}^{n-l} \times U$, $\tilde{U} \in \tilde{\mathcal{U}}$, with $U = \mathbb{R}^l$.

Let us recall the construction of the Čech-de Rham cohomology in [Crainic-Moerdijk]. Let \mathcal{U} be a family of transversal sections of \mathcal{F} . Consider the double complex

$$C^{p,q}(\mathcal{F}) = \prod_{U_0 \xrightarrow{h_1} \dots \xrightarrow{h_p} U_p} \Omega^q(U_0), \quad (0.27)$$

where the product ranges over all p -tuples of holonomy embeddings h_i , $0 \leq i \leq p$, between transversal sections from a fixed transversal basis \mathcal{U} , and Ω^q is the space of differential forms of order q . The vertical differential is defined as $(-1)^p d : C^{p,q}(\mathcal{F}) \rightarrow C^{p,q+1}(\mathcal{F})$, where d is the usual de Rham differential. The horizontal differential

$\delta : C^{p,q}(\mathcal{F}) \rightarrow C^{p+1,q}(\mathcal{F})$, is given by $\delta = \sum_{i=0}^{k+1} (-1)^i \delta_i$, where







$$\delta_i \omega(h_1, \dots, h_{k+1}) = \begin{cases} h_1^* \omega(h_2, \dots, h_{k+1}), & \text{if } i = 0, \\ \omega(h_1, \dots, h_{i+1} h_i, \dots, h_{k+1}), & \text{if } 0 < i < k + 1, \\ \omega(h_1, \dots, h_k), & \text{if } i = k + 1. \end{cases} \quad (0.28)$$









The product is defined by

$$(\omega \eta)(h_1, \dots, h_{n+n'}) = (-1)^{nn'} \omega(h_1, \dots, h_n) (h_1^* \dots h_n^*) \cdot \eta(h_{n+1}, \dots, h_{n+n'}), \quad (0.29)$$






for $\omega \in C^{n,m}$ and $\eta \in C^{n',m'}$, and h_i^* being the dual to h_i . Thus $(\omega \eta)(h_1, \dots, h_{n+n'}) \in C^{n+n',m+m'}$, and the product (0.29) delivers the structure of a bigraded differential algebra. The cohomology of this complex is called the Čech-de Rham cohomology $\check{H}_{\mathcal{U}}^*(M/\mathcal{F})$ of the leaf space M/\mathcal{F} with respect to the transversal basis \mathcal{U} .

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