A variant of Baer's theorem

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AAA103, Tartu
June 9, 2023

"Baer's theorem", 1952

"If endomorphism rings of vector spaces are isomorphic, then the vector spaces themselves are isomorphic".

More precisely:

Theorem

Let V, W be (infinite-dimensional) vector spaces over a division ring D. If $\Phi : \mathrm{End}_D(V) \to \mathrm{End}_D(W)$ is an isomorphism, then there there is an isomorphism $\alpha : V \to W$ such that

$$\Phi(f) = \alpha \circ f \circ \alpha^{-1}$$

for any $f \in \mathrm{End}_D(V)$.

A bit of history

- Eidelheit, Mackey, et al. (end of 1930s-1940s): analytic setting (bounded operators on Banach spaces, continuous operators on normed spaces)
- Dieudonné, Jacobson (1940s): rings of finitary linear maps
- Baer (1952): using properties of idempotents
- Wolfson (1953, PhD thesis under Baer): using Jacobson's density theorem
- Racine (1998), Balaba (2005): super- and graded cases
- lot of authors: modules over abelian groups ("Baer-Kaplansky theorem")

Another variant of Baer's theorem

Instead of $\mathrm{End}_D(V)$, or its subring $\mathrm{FEnd}_D(V)$ of finitary linear maps, consider $\mathrm{FEnd}_D(V, \Pi)$, the ring generated by all "infinitesimal transvections"

$$t_{v,f} : u \mapsto vf(u)$$

where Π is a subspace of V^*, $v \in V$, $f \in \Pi$.

Another variant of Baer's theorem (cont.)

Theorem

Let V, W be right vector spaces over a division ring D, Π a nonzero finite-dimensional subspace of V^*, Γ a finite-dimensional subspace of W^*, and $\Phi : \mathrm{FEnd}_D(V, \Pi) \to \mathrm{FEnd}_D(W, \Gamma)$ an isomorphism of D-algebras. Then there is an isomorphism of D-vector spaces $\alpha : V \to W$ such that

$$\Phi(f) = \alpha \circ f \circ \alpha^{-1}$$

for any $f \in \mathrm{FEnd}_D(V, \Pi)$.

Idea of the proof

$\mathrm{FEnd}_D(V, \Pi)$ generally, does not have idempotents, and Jacobson's density theorem does not hold, so all previous methods will not work.

Instead, write $\mathrm{FEnd}_D(V, \Pi)$ as $V \otimes_D \Pi$ and use elementary linear algebra:

$$\mathrm{Hom}_D(V \otimes_D \Pi, W \otimes_D \Gamma) \simeq \mathrm{Hom}_D(V, W) \otimes_D \mathrm{Hom}_D(\Pi, \Gamma),$$

Hence Φ belonging to the left-hand side can be written as some element of the tensor product at the right-hand side of rank $\mathrm{rk}(\Phi)$.

Idea of the proof (cont.)

The crucial lemma

$$\operatorname{Tr}\Phi(\xi) = \operatorname{rk}(\Phi)\operatorname{Tr}(\xi)$$

for any $\xi \in V \otimes_D \Pi$.

Apply the crucial lemma for Φ and for Φ^{-1} to get $\operatorname{rk}(\Phi) = 1$, and the rest is trivial.

Open question

What about Lie and Jordan rings $\mathsf{FEnd}_D(V)^{(\pm)}$, $\mathsf{FEnd}_D(V, \Pi)^{(\pm)}$?

That's all. Thank you.