On extensions of multiary maps to superposition of binary ones

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The last question of Banach

Andrzej Alexiewicz's diary, December 29, 1944:

"There exists a nontrivial example of ternary multiplication, which is not generated by binary multiplication (Banach). Can any finite set with ternary commutative multiplication be extended so that ternary multiplication is generated by a binary multiplication?"

What does this question mean?

Interpretation of “multiplication” and “generated”?

Connection with:

- History
- Semigroups
- Theory of clones
- Hilbert’s 13th problem (superposition of functions)
- Logic
- Combinatorics
- Computer calculations
- Lie theory

First interpretation: semigroups

"Multiplication": ternary semigroup, i.e.

$$f : X \times X \times X \to X$$

$$f(f(x,y,z),u,v) = f(x,f(y,z,u),v) = f(x,y,f(z,u,v)).$$

Commutativity:

$$f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = f(x_1, x_2, x_3)$$

for any $\sigma \in S_3$.

"Generation": $f(x,y,z) = (x * y) * z$.

Theorem (Łoś 1955, Monk–Sioson 1966)

Every (commutative) ternary semigroup can be extended to a (commutative) ternary semigroup given by

$$f(x,y,z) = (x * y) * z,$$

where $*$ is a (commutative) binary semigroup.

Second interpretation: clones

"Multiplication": an arbitrary map $X \times X \times X \to X$.
"Generation": superposition.

Theorem (Sierpiński 1935, Banach 1935)

Any countable number of unary maps on an infinite set can be generated by two maps.
($\Leftrightarrow$ A countable transformation semigroup is 2-generated).

Generalizations:

- Other kinds of semigroups, generation "up to approximation" (Schreier–Ulam, Jarník–Knichal, et al.)
- Unary $\to$ multiary

Second interpretation: clones

Theorem

Any countable number of maps of arbitrary arity on a set X can be generated by one binary map on X.
($\Leftrightarrow$ The clone of all maps on X is generated by its binary fragment).

Webb 1935: finite X (generalization of Sheffer's stroke)

$$p \uparrow q = \neg(p \wedge q)$$

Łoś 1950, Goldstern 2012: infinite X

Conclusion

"Banach's claim" is false in this context.

Interlude

Erdös (on another occasion, from a preface to the *Scottish Book*):

"Now it frequently happens in problems of this sort that the infinite dimensional case is easier to settle than the finite dimensional analogues. This moved Ulam and me to paraphrase a well known maxim of the American armed forces in WWII: 'The difficult we do immediately, the impossible takes a little longer', viz: 'The infinite we do immediately, the finite takes a little longer'."

Stanisław Ulam
(PhD Lvov 1933)

Jerzy Łoś
(student at Lvov, 1937–1939)

Wacław Sierpiński
(professor at Lvov, 1908–1914)

Connection: Hilbert's 13th problem

Theorem (Pólya and Szegö, *Problems and Theorems in Analysis*, 1925, Problem 119 "Are there actually functions of 3 variables?", Sierpiński 1934,1945)

For any binary bijection $g : X \times X \to X$ on an (infinite) set X, and any ternary map $f : X \times X \times X \to X$, there is a binary map $h : X \times X \to X$ such that

$$f(x, y, z) = g(h(x, y), z).$$

Theorem (Kolmogorov 1956, Arnold 1957)

Any continuous real function in any number of variables can be represented as superposition of continuous real functions in 2 variables.

Questions (Mark Kac–Ulam 1960,1968)

Extensions of Hilbert's 13th problem: on $\mathbb{R}^n$, in various classes (smooth, analytic).

Connection: Logic

Łukasiewicz's 3-valued logic: "implication" $Ł$:

	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	0	$\frac{1}{2}$	1

and "negation" N:

$$0 \mapsto 1,\ \tfrac{1}{2} \mapsto \tfrac{1}{2},\ 1 \mapsto 0.$$

Theorem (Słupecki 1936)

There are multiary maps on $\{0, \frac{1}{2}, 1\}$ which are not superposition of $Ł$ and N.

$2\frac{1}{2}$th interpretation: clones with additional structure

"Multiplication": a map $X \times X \times X \to X$ preserving an additional structure on X.
"Generation": composition.

"Banach's claim" is true for smooth functions. For example:

$$F(x, y, z) = f(g(h(x, y), z), z)$$

satisfies the differential equation

$$\frac{\partial^2 F}{\partial x \partial z}\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\frac{\partial^2 F}{\partial y \partial z} = 0.$$

Not suitable for finite X.

Third interpretation: operadic-like

"Multiplication": an arbitrary map $f : X \times X \times X \to X$.
"Generation": $f(x, y, z) = (x * y) * z$ or $x * (y * z)$.

For X finite, "Banach's claim" is true:
number of ternary maps generated by binary ones $< 2|X|^{|X|^2}$
number of ternary maps $= |X|^{|X|^3}$.

Question(s): number of ternary maps generated by binary ones

$\lvert X \rvert = n$	number of binary maps $= n^{n^2}$	number of maps $(x * y) * z$	number of maps $(x * y) * z$ and $x * (y * z)$	number of commutative maps $(x * y) * z^{\dagger}$
1	1	1	1	1
2	16	14	21	5
3	19683	19292	38472	48

† the same number with and without assumption of commutativity of $*$

Absent in OEIS!

Jacobson answers Banach (sort of)

Theorem

Any ternary map $f : X \times X \times X \to X$ on a finite X can be extended to a ternary map generated by a binary map.

Proof (trivial)

$Y = X \cup (X \times X)$.
$x * y = (x, y); \quad (x, y) * z = f(x, y, z)$.

Modification of this: commutative version, result of Łoś and Monk-Sioson, generalization to n-ary maps, etc., etc. can be established using Jacobson's ideas about Lie and Jordan triple systems (1949).

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That's all. Thank you.