Lie p-algebras of cohomological dimension one

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What is cohomological dimension?

A – an object in a category with "good" cohomology theory
(Lie algebras, associative algebras, groups, ...)

Cohomological dimension of A: the least number n such that

$$\mathrm{H}^{n+1}(A, \cdot\,) = 0$$

A natural question

Objects of "small" cohomological dimension (say, ≤ 1).

Example

Free objects have cohomological dimension 1.

Groups of cohomological dimension ≤ 1

Elementary fact

Groups of cohomological dimension 0 are trivial.

A classical theorem (Stallings, Swan, late 1960s)

Groups of cohomological dimension 1 are free.

Associative algebras of cohomological dimension ≤ 1

A classical result
(Eilenberg, Hochschild, Rosenberg, Zelinsky, ..., 1940-1950s)

Associative algebras of cohomological dimension 0 are finite-dimensional separable algebras (roughly, semisimple algebras).

Associative algebras of cohomological dimension 1: interesting, but messy (Cuntz–Quillen, 1995).

Lie algebras of cohomological dimension ≤ 1

Elementary fact

Lie algebras of cohomological dimension 0 are trivial.

Question (Bourbaki)

Is it true that Lie algebras of cohomological dimension 1 are free?

Theorem (G.L. Feldman, 1983)

Yes, for 2-generated algebras.

Theorem (A.A. Mikhalev–Umirbaev–Zolotykh, 1994)

No, generally. Example:

$$\langle\, x, y, z \mid x + [y, z] + (\operatorname{ad} x)^p z = 0 \,\rangle$$

over a field of characteristic $p > 2$.

Lie algebras of cohomological dimension 1

Remaining questions

What about characteristic 0? characteristic 2? Lie p-algebras?

Main fact about cohomological dimension

Cohomological dimension does not increase when passing to subalgebras. In particular, the class of Lie algebras of cohomological dimension 1 is closed with respect to subalgebras.

Proof

Shapiro's lemma about cohomology of coinduced modules.

Lie p-algebras of cohomological dimension 1

Example

Free Lie p-algebras have cohomological dimension ∞.

Proof

The free Lie p-algebra of rank 1, $\langle x, x^{[p]}, x^{[p]^2}, \ldots \rangle$, is infinite-dimensional and abelian, hence has cohomological dimension ∞.

Theorem

A Lie p-algebra of cohomological dimension 1 is 1-dimensional.

Proof

The condition

$$x^{[p]} = \lambda(x)x$$

$+$ Feldman's result.

On the condition $x^{[p]} = \lambda(x)x$

A famous question (Jacobson, 1960s)

Is a Lie p-algebra satisfying

$$x^{[p]^{n(x)}} = x,$$

abelian?

Theorem

A Lie p-algebra over an algebraically closed field satisfying

$$x^{[p]^{n(x)}} = \lambda(x)x,$$

with $n(x)$'s bounded, contains a nonzero p-nilpotent element.

Corollary

An alternative proof of the theorem about Lie p-algebras of cohomological dimension one.

Lie p-algebras of *restricted* cohomological dimension ≤ 1

Category of Lie p-algebras:
restricted cohomology $\rightsquigarrow$ *restricted* cohomological dimension (cd_*)

Theorem (Hochschild, 1950s)

Lie p-algebras of restricted cohomological dimension 0 are finite-dimensional tori.

Examples

Let

$$0 \to L_1 \to L \to L_2 \to 0.$$

Then

$$\mathrm{cd}_*(L_1) = \begin{matrix} 0 \\ 1 \end{matrix} \quad \text{and} \quad \mathrm{cd}_*(L_2) = \begin{matrix} 1 \\ 0 \end{matrix} \quad \Rightarrow \quad \mathrm{cd}_*(L) = 1.$$

(Follows from the Lyndon–Hochschild–Serre spectral sequence).

Lie p-algebras of restricted cohomological dimension 1

Conjecture

Any Lie p-algebra of restricted cohomological dimension 1 is of the form

$$(\dots((\mathscr{L} \bowtie T_1) \bowtie T_2) \dots) \bowtie T_n,$$

where $\mathscr{L}$ is a free Lie p-algebra, T_i's are finite-dimensional tori, and $\bowtie \in \{\rtimes, \ltimes\}$.

Some facts supporting the conjecture

For a Lie p-algebra of restricted cohomological dimension 1 the following holds:

- A p-subalgebra is either a torus, or is infinite-dimensional.
- An abelian p-subalgebra is either a torus, or (the free Lie p-algebra of rank 1) $\oplus$ (torus).
- It has (ordinary) cohomological dimension ∞.

That's all. Thank you.