On correlation matrices and versal deformations

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Matrix nearness problem

A problem: given a square (symmetric) matrix, find the nearest correlation matrix.

The problem is difficult (Higham, Boyd, ...)

… and its occurrence in "real life"

(Huge) "correlation" matrices, computed by bankers, are almost never positive-semidefinite (but they have to be).

... and its pedestrian "solution"

Let A be a symmetric matrix which need to be approximated.

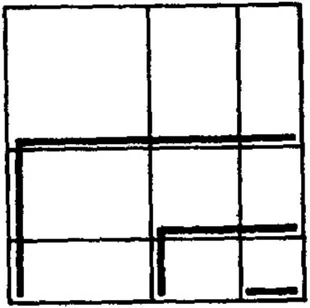
1. Do the spectral decomposition: $A = BJB^\top$.
2. Get a "corrected" diagonal matrix $\widehat{J}$ by replacing negative eigenvalues in J by "small" positive ones.
3. Go back to the whole matrix: $\widehat{A} = B\widehat{J}B^\top$.
4. Normalize: $\frac{\widehat{a}_{ij}}{\sqrt{\widehat{a}_{ii}\widehat{a}_{jj}}}$.

This is not a real solution of the problem, but it works surprisingly well. Why?

Arnold's theory of versal deformations of matrices

V.I. Arnold, *On matrices depending on parameters*, Russ. Math. Surv. **26** (1971), N2, 29–43.

"Jordan normal form" for a smooth family of matrices.



.. and its application

That means that for a diagonalizable matrix with a simple spectrum all matrices "sufficiently close" to it are obtained by "slightly perturbing" eigenvalues.

Hence, if we know *apriori* that a "close" correlation matrix exists, it can be obtained by the procedure described above.

Another problem: justification of CONCOR

Iterative correlation matrices:

$A \mapsto$ matrix of correlations between rows (or columns) of A;
iterate this procedure.

In practice, it “almost always” converges to a matrix consisting of 1 and -1, what can be used in (sort of) clustering of the initial data, but a rigorous proof of this is absent.

Iterative correlations can be chaotic

An example of “bad” matrix with chaotic convergence:

$$\begin{pmatrix} 1 & -0.73333333333 & -0.2 & 0.2 \\ -0.73333333333 & 1 & -0.2 & 0.2 \\ -0.2 & -0.2 & 1 & 0.33333333333 \\ 0.2 & 0.2 & 0.33333333333 & 1 \end{pmatrix}$$

Can Arnold's theory help?

Some works (e.g., J.B. Kruskal, *A theorem about CONCOR*, Technical Report, Bell Lab., 1978) suggest that if the initial matrix is "close enough" to a matrix consisting of ± 1, then its iterative correlations converge.

Can Arnold's theory of versal deformations help?

Partially based on:
J. Nonlin. Math. Phys. **20** (2013), N3, 431–439 = arXiv:1303.3226

Slides at `http://www1.osu.cz/~zusmanovich/math.html`

That's all. Thank you.