Non-semigroup gradings of associative algebras

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Based on:

- Lin. Algebra Appl. **523** (2017), 52–58 = arXiv:1609.03924
- J. Algebra **324** (2010), 3470–3486 = arXiv:0907.2034

These slides are at `http://www1.osu.cz/~zusmanovich/`

Do non-semigroup gradings exist?

$$A = \bigoplus_{\alpha \in G} A_\alpha$$

An algebra graded by a set G with a (partial) binary operation $*$:

$$A_\alpha A_\beta \subseteq A_{\alpha * \beta} \text{ if } A_\alpha A_\beta \neq 0$$

Question (vague)

How identities of A and G are related?

Fact

If A is commutative or anticommutative, then G can be embedded into a commutative magma.

Question (more concrete)

Is it true that if A is associative or Lie, then G can be embedded into a semigroup?

Do non-semigroup gradings exist? Part II

Answer (for Lie algebras)

Yes, it can (Patera and Zassenhaus 1989).

Do non-semigroup gradings exist? Part II

Answer (for Lie algebras)

Yes, it can (Patera and Zassenhaus 1989).

Turned out to be wrong $\sim$20 years later:
There are Lie algebras with a non-semigroup grading! (Elduque 2006,2009).

Example

$L = \langle a, u\rangle \oplus \langle v\rangle \oplus \langle w\rangle$
$[a, u\,] = u$
$[a, v\,] = w$
$[a, w] = v$

Question (remaining)

What about (classical) simple Lie algebras?

Do non-semigroup gradings exist? Part III

Answer (for associative algebras)

There are associative algebras with a non-semigroup grading (Zusmanovich 2017).

(The smallest known such algebra is 6-dimensional).

A method to construct non-semigroup gradings

Take δ-derivation D of an algebra A, i.e.

$$D(ab) = \delta\Big(D(a)b + aD(b)\Big)$$

and consider the root space decomposition

$$A = \bigoplus A_\lambda$$

with respect to D. Then

$$A_\lambda A_\mu \subseteq A_{\delta(\lambda+\mu)}.$$

The "multiplication"

$$\lambda * \mu = \delta(\lambda + \mu)$$

is nonassociative (subject to some conditions on δ).

All non-semigroup gradings constructed so far follow this scheme.

More questions

1. What is the smallest dimension of an associative algebra with a non-semigroup grading?
2. It is true that any grading of a full matrix algebra is a semigroup grading?
3. Given a grading of a Lie algebra, is it possible to construct a grading of its (restricted) universal enveloping algebra?
4. Does the presence/absence of non-semigroup gradings of algebras over a binary quadratic operad $\mathscr{P}$ entails the same for algebras over the operad Koszul dual to $\mathscr{P}$?

That's all. Thank you.