

CD algebras

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“Non-associative algebras in Málaga”

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(Joint work with Ivan Kaygorodov)

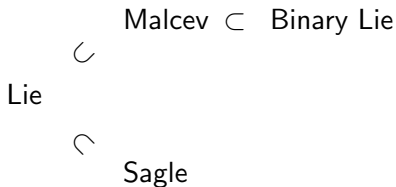
CD algebras

CD algebras are anticommutative algebras such that $[R_a, R_b]$ is a derivation for any two elements a, b of the algebra.

Such commutative algebras = Lie triple algebras
(Bertram, Dzhumadil'daev, Jordan–Rühaak, Osborn, Sidorov, ..., 1969–2018)

What about *anticommutative* ones?

Known classes of anticommutative algebras



binary Lie: $J([x, y], x, y) = 0$

Malcev: $J(x, y, [x, z]) = [J(x, y, z), x]$

Sagle: $[J(x, y, z), t] = J(t, z, [x, y]) + J(t, y, [z, x]) + J(t, x, [y, z])$

How CD fits into the picture?



$$\text{Malcev} \cap \text{CD} = \text{Sagle} \cap \text{CD} = \text{Malcev} \cap \text{Sagle}$$

$$\text{Binary Lie} \cap \text{Almost Lie} = \text{CD}$$

Almost Lie: $[J(x, y, z), t] = 0$

CD algebras are “central extensions” of Lie algebras

Corollary 1 (to inclusion at the previous slide)

For any CD algebra A , $A/Z(A)$ is a Lie algebra.

Corollary 2

Any simple CD algebra is a Lie algebra.

An old open question

Whether any simple binary Lie algebra is Malcev?

Yes in the class of finite-dimensional algebras over a field of characteristic 0 (Grishkov 1980).

Yes in the class of CD algebras.

Cohomology? Variant 1

Central extensions + derivations + deformations

$$0 \rightarrow LZ(M) \oplus (A \otimes M) \xrightarrow{d^0} C^1(A, M) \xrightarrow{d^1} C^2(A, M) \xrightarrow{d^2} C^4(A, M)$$

$LZ(M) = \{m \in M \mid xy \bullet m - x \bullet (y \bullet m) + y \bullet (x \bullet m) = 0 \forall x, y \in M\}$,
the "Lie center"

$C^n(A, M)$ - skew-symmetric multilinear maps

$$d^0(m)(b) = b \bullet m$$

$$d^0(a \otimes m)(b) = a \bullet (b \bullet m) - b \bullet (a \bullet m)$$

$$d^1(\varphi)(x, y) = \varphi(xy) - x \bullet \varphi(y) + y \bullet \varphi(x)$$

$$d^2(\varphi)(x, y, a, b) =$$

$$\begin{aligned} & \varphi((xy)a, b) - \varphi((xy)b, a) - \varphi((xa)b, y) + \varphi((xb)a, y) + \varphi((ya)b, x) - \\ & \varphi((yb)a, x) + a \bullet \varphi(xy, b) - b \bullet \varphi(xy, a) - x \bullet \varphi(ya, b) + x \bullet \varphi(yb, a) + \\ & y \bullet \varphi(xa, b) - y \bullet \varphi(xb, a) - a \bullet (b \bullet \varphi(x, y)) + b \bullet (a \bullet \varphi(x, y)) - x \bullet \\ & (a \bullet \varphi(y, b)) + x \bullet (b \bullet \varphi(y, a)) - y \bullet (b \bullet \varphi(x, a)) + y \bullet (a \bullet \varphi(x, b)) \end{aligned}$$

Cohomology? Variant 2

“CD derivations”

$$0 \rightarrow M \xrightarrow{d^0} C^1(A, M) \xrightarrow{d^1} C^3(A, M)$$

$$d^0(m)(x) = x \bullet m$$

$$d^1(\varphi)(x, y, a) =$$

$$\varphi((xy)a) - a \bullet \varphi(xy) - y \bullet \varphi(xa) + x \bullet \varphi(ya) + y \bullet (a \bullet \varphi(x)) - x \bullet (a \bullet \varphi(y))$$

Cohomology? Variant 3

By analogy with the Chevalley–Eilenberg

$$\begin{aligned}
 C^1(A, M) &\xrightarrow{d} C^3(A, M) \xrightarrow{d} C^5(A, M) \xrightarrow{d} \dots \\
 C^2(A, M) &\xrightarrow{d} C^4(A, M) \xrightarrow{d} C^6(A, M) \xrightarrow{d} \dots
 \end{aligned}$$

$$d(\varphi)(x, y, a_1, \dots, a_n)$$

$$\begin{aligned}
 &= \sum_{i=1}^n (-1)^i \left(\varphi((xy)a_i, a_1, \dots, \widehat{a}_i, \dots, a_n) + a_i \bullet \varphi(xy, a_1, \dots, \widehat{a}_i, \dots, a_n) \right. \\
 &\quad - x \bullet \varphi(ya_i, a_1, \dots, \widehat{a}_i, \dots, a_n) + y \bullet \varphi(xa_i, a_1, \dots, \widehat{a}_i, \dots, a_n) \\
 &\quad \left. - x \bullet (a_i \bullet \varphi(y, a_1, \dots, \widehat{a}_i, \dots, a_n)) + y \bullet (a_i \bullet \varphi(x, a_1, \dots, \widehat{a}_i, \dots, a_n)) \right) \\
 + \sum_{1 \leq i < j \leq n} &(-1)^{i+j+n+1} \left(\varphi((xa_j)a_j, y, a_1, \dots, \widehat{a}_i, \dots, \widehat{a}_j, \dots, a_n) - \varphi((xa_j)a_j, y, a_1, \dots, \widehat{a}_i, \dots, \widehat{a}_j, \dots, a_n) \right. \\
 &\quad - \varphi((ya_j)a_j, x, a_1, \dots, \widehat{a}_i, \dots, \widehat{a}_j, \dots, a_n) + \varphi((ya_j)a_j, x, a_1, \dots, \widehat{a}_i, \dots, \widehat{a}_j, \dots, a_n) \\
 &\quad \left. + a_i \bullet (a_j \bullet \varphi(x, y, a_1, \dots, \widehat{a}_i, \dots, \widehat{a}_j, \dots, a_n)) - a_j \bullet (a_i \bullet \varphi(x, y, a_1, \dots, \widehat{a}_i, \dots, \widehat{a}_j, \dots, a_n)) \right)
 \end{aligned}$$

Second CD cohomology

For any Lie algebra L and an L -module M ,

$$H^2(L, M) \subseteq H_{CD}^2(L, M).$$

Theorem (“Second CD Whitehead lemma”)

($p = 0$) For any simple finite-dimensional Lie algebra L , and any finite-dimensional L -module M ,

$$H_{CD}^2(L, M) = 0.$$

Conjecture

($p \neq 2, 3$) For any simple finite-dimensional Lie algebra L ,

$$H_{CD}^2(L, K) = H^2(L, K).$$

Supported by computations in GAP.

A new invariant in characteristic 2

For some simple Lie algebras in the list of simple Lie algebras of dimension ≤ 20 (Eick 2010), $H_{CD}^2(L, K) \neq H^2(L, K)$, and $H_{CD}^2(L, K)$ distinguishes some simple Lie algebras not distinguishable by the “classical” invariants.

Further questions

- 1) How “far” a CD algebra can be from Lie algebras? Describe CD algebras A with $LZ(A) = Z(A)$.
- 2) Study free CD algebras. Are they central extensions of free Lie algebras?
- 3) Study “CD speciality”.
- 4) Study representations of CD algebras. An analog of the Ado theorem?
- 5) CD algebras without conditions of commutativity or anticommutativity?

That's all. Thank you.